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Inelastic Scattering of Neutrons by Rotational Excitation

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(Received May 30, 1955)

The collective nuclear model of A. Bohr is applied to the interpretation of inelastic scattering of neutrons, as was briefly described in our previous report⁵⁾. The present paper deals with the detailed theory of the inelastic scattering of neutrons by a direct excitation of the nuclear surface rotation, as well as by the resonance scattering via compound nuclei. The cross section for the former process is proportional to the square of the intrinsic quadrupole moment and depends strongly upon the amplitudes of neutron waves at the nuclear surface. This strong dependence makes it difficult to compare our theory with experiments. Conversely the strong dependence may be employed to know the optical potential which distorts neutron waves. It is pointed out that the contributions from the direct and compound nucleus forming processes can be distinguished by observing the angular distribution.

§ 1. Introduction

The inelastic scattering of neutrons has thus far been analyzed mostly in terms of the statistical theory of nuclear reactions. However, for the reaction in which only a few energy levels are excited in a residual nucleus, the statistical assumption is no longer valid for the final nucleus, so that the properties of the final levels make a significant influence on the scattering cross section. This fact is taken into account by Hauser and Feshbach¹⁾ who gave the cross sections for given initial and final nuclear spins and parities. In their theory an impinging neutron is assumed always to form a compound nucleus whose level density is high enough to permit the application of the statistical assumption for the intermediate nucleus. The assumptions are justified, provided that the interaction of a neutron with a nucleus is so strong that the neutron is always absorbed once it hits the nuclear surface, and provided that the level distance of a compound nucleus is small compared with the level width and with the energy spread in the neutron beam.

The latter is not satisfied in a recent experiment by Kiehn and Goodman²⁾ who were able to observe the resonances which are thought to show the contributions from the individual levels of compound nuclei. The strong absorption may not strictly hold if one takes into account the rather weak interaction shown by Feshbach, Porter and Weisskopf³⁾ through the analysis of elastic scattering of neutrons. As the resonance reaction may be accounted for in ordinary terms of the resonance theory, we are mainly concerned with the reaction in the case of the weak interaction between a neutron and a nucleus.

What is important in this case, is the cause of transition between initial and final

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states. The transition may be caused by the excitation of the single particle motion, as discussed by Hayakawa and Sasakawa⁴⁾ or by that of the collective motion, as reported by the present authors⁵⁾. The latter mode of excitation plays a significant role in the nuclear reactions taking place at the nuclear surface, since the interaction assumed by A. Bohr⁶⁾ takes place only at the surface and its strength increases with the nuclear quadrupole moment. As this generally becomes larger, the heavier the nucleus, the relative importance of the surface interaction is expected to be constant, or rather to increase with mass number of a nucleus, whereas the relative number of nucleons at the surface decreases as $A^{-1/3}$. This behaviour seems to have been observed through various experiments and the role of the collective excitation has been suspected in some occasions. The validity of this idea must, however, be examined by means of the quantitative calculation of the cross section.

In the previous report⁵⁾ we presented formulas for the cross section in Born approximation, as well as in the method of distorted waves. The Born approximation formula was compared with the inelastic scattering of 17 Mev protons by Fe⁷⁾. The qualitative agreement between the experiment and our theory, though there was a discrepancy of factor two or so, seems to assure the role of the collective (rotational) excitation. The distorted wave method is found very sensitive to the potential which distorts the neutron waves, so that the quantitative prediction of cross sections could hardly be made, unless, of course, the behaviour of neutron waves at the nuclear surface were known better than at present. Nevertheless, one may be able to see from our calculation how much important the collective excitation is.

In the present paper we describe the derivation of the formulas given in the previous report. In § 2 the formal derivation of the cross section is presented, following the method described by one of us (S.H.)⁸⁾ The potential which causes the distortion of neutron waves is not restricted to the real one, as employed in reference 5, but is allowed to be the complex optical potential. Its imaginary part is considered to be due to the formation of compound states. This naturally leads us to the resonance formula that may account for the experiment of reference 2. Needless to say, only a part of the compound states are the collective states to which the theory developed by one of us (S.Y.)⁹⁾ is applicable. Leaving the detailed work for the resonance reaction for the future, we are mainly concerned to the direct reaction.

§ 3 is devoted to the calculation of the cross section in reference to the theory of Bohr and Mottelson¹⁰⁾. Three possible cases are considered: (i) the target nucleus forms a core by itself; (ii) the weak coupling approximation; and (iii) the strong coupling approximation.

In § 4 our theory is compared with experiments and the contributions from the direct and the resonance parts are discussed. The resonance reaction formulated by Bohr and Mottelson¹⁰⁾ is related to our theory in the Appendix.

§ 2. Derivation of cross section

Following the method of reference 8, the Hamiltonian, H , of the total system, i.e. a target nucleus plus an impinging neutron, is separated into three parts,

$$H=H_0+H_t+V. \quad (2.1)$$

H_0 is a function only of the impinging neutron coordinate, \mathbf{r} , and consists of its kinetic energy K and the smeared spin-independent potential $U(\mathbf{r})$:

$$H_0=K+U. \quad (2.2)$$

H_t is the Hamiltonian for the target nucleus and defines energy levels ε_τ and eigen functions Φ_τ by

$$(H_t(x) - \varepsilon_\tau) \Phi_\tau(x) = 0, \quad (2.3)$$

where x means all the coordinates of individual nucleons of the target nucleus. V represents the interaction between a neutron and a nucleus, which is not represented by the smeared potential U , i.e. the total interaction between the neutron and the nucleus minus U .

At infinite separation of the neutron from the nucleus U and V vanish, so that the initial state is described by

$$F_a(\mathbf{r}, x) = \exp(i\mathbf{k}_a \cdot \mathbf{r}) \Phi_a(x). \quad (2.4a)$$

$\hbar \mathbf{k}_a$ is the momentum of the incident neutron and Φ_a the initial (ground) state of the target nucleus. The energy of the initial state is given by

$$E_a = (\hbar^2 k_a^2 / 2M_a) + \varepsilon_a, \quad (2.5)$$

where M_a is the reduced mass of the incident neutron.

With this initial state the scattered wave is written as

$$\Psi_a^{(+)} = F_a + \frac{1}{E_a - K - H_t + i\eta} (U + V) \Psi_a^{(+)}, \quad (2.6)$$

where η is a small positive quantity indicating the outgoing wave. Owing to the interaction, V , the nucleus is excited to a state Φ_b and the scattered neutron has a momentum $\hbar \mathbf{k}_b$. Then the final state, in analogy with (2.4a), is described by

$$F_b(\mathbf{r}, x) = \exp(i\mathbf{k}_b \cdot \mathbf{r}) \Phi_b(x), \quad (2.4b)$$

with energy

$$E_b = (\hbar^2 k_b^2 / 2M_b) + \varepsilon_b = E_a. \quad (2.7)$$

The reduced mass M_b is practically equal to M_a , so that they are denoted collectively as M .

The matrix element for the transition from states a to b is given by

$$T_{ba} = \langle F_b | U + V | \Psi_a^{(+)} \rangle. \quad (2.8)$$

This can be reduced to more convenient forms in the following way. We introduce the ingoing wave distorted by U .

$$G_b^{(-)}(\mathbf{r}, x) \equiv g_b^{(-)}(\mathbf{r}) \Phi_b(x) = F_b + \frac{1}{E_b - K - H_t - i\eta} U^\dagger G_b^{(-)}. \quad (2.9)$$

Now, U may contain an imaginary part and may be regarded as an optical potential. Eliminating F_b from (2.8) with the aid of (2.9), we have

$$T_{ba} = \langle G_b^{(-)} | V | \Psi_a^{(+)} \rangle + \langle g_b^{(-)} | U | \exp(i\mathbf{k}_a \cdot \mathbf{r}) \rangle \delta_{ba}. \quad (2.10)$$

The last term is concerned only with the elastic scattering due to the potential U . In the inelastic scattering we have only to consider the term $\langle G_b^{(-)} | V | \Psi_a^{(+)} \rangle$, but the explicit form $\Psi_a^{(+)}$ can hardly be obtained. Its formal solution is written as

$$\Psi_a^{(+)} = G_a^{(+)} + \frac{1}{E_a - H + i\eta} V G_a^{(+)}, \quad (2.11)$$

where $G_a^{(+)}$ is the outgoing wave distorted by U analogous to $G_b^{(-)}$ in (2.9). Thus the amplitude for the inelastic scattering, $a \neq b$, can be expressed as

$$T_{ba} = \langle G_b^{(-)} | V | G_a^{(+)} \rangle + \langle G_b^{(-)} | V \frac{1}{E_a - H + i\eta} V | G_a^{(+)} \rangle. \quad (2.12)$$

This is the expansion of the amplitude with respect to V . For V being small, the first term of the right hand side may provide a good approximation,

$$T_{ba}^{(1)} \equiv \langle G_b^{(-)} | V | G_a^{(+)} \rangle. \quad (2.13)$$

This means the direct interaction of the neutron with the nuclear surface and the cross section derived in reference 5 is based merely on this term.

The second term consists of two parts, the one being the multiple interactions at the surface and the other the formation of compound states. On account of the smallness of V compared with U , the former effect is considered to be of little importance. The latter is supposed to explain the excitation curve observed in reference 2. Leaving the detailed discussions on this effect, in connection with the treatment of Bohr and Mottelson¹⁰⁾, to the Appendix, only a brief account of the resonance is given here.

The compound states are described by an orthogonal set of wave functions, Ψ_j , denoted as $|j\rangle$ for brevity. A compound state is not stationary, but decays through the interaction V . Thus the eigen value of H is complex, i.e.

$$\langle j | H | j \rangle \equiv W_j = E_j + (i/2) \gamma_j. \quad (2.14)$$

E_j is the energy level in the absence of V and a complex quantity, γ_j , is given, to the second order in V , by

$$\frac{i}{2} \gamma_j = \sum_t \frac{\langle j | V | t \rangle \langle t | V | j \rangle}{E - E_t + i\eta}, \quad (2.15)$$

where $|t\rangle$ means a state of an entrance channel. The real part of γ_j represents the level width. With this approximation the last term of (2.12) is reduced to

$$T_{ba}^{(2)} \equiv \sum \frac{\langle G_b^{(-)} | V | j \rangle \langle j | V | G_a^{(+)} \rangle}{E_a - W_j}. \quad (2.16)$$

In evaluating the cross section we carry out the partial wave analysis, following the method developed by Blatt and Biedenharn¹¹⁾. Since the spin-orbit coupling is neglected,

their channel spin can be replaced by the spin, I , of the target nucleus, its Z component being M . We choose an eigen state in which the orbital angular momentum, l , and nuclear spin, I , form a total angular momentum J with Z component N , these being diagonal. The coordinates of a neutron, \mathbf{r} , are expressed by the radial part r and the polar coordinate ω . The eigen state for the angular part is then expressed as

$$|l I J N\rangle = \sum_{m, M} (l I J N | l I m M) i^l Y_l^m(\omega) |I M\rangle, \quad (2.17)$$

where m is the Z component of l . i^l should be introduced for the correct time reversibility¹²⁾

With this eigen state, abbreviated as $|A\rangle$ or $|B\rangle$, $G_a^{(+)}$ is expanded as¹³⁾

$$G_a^{(+)} = \sum_i |A\rangle \langle A|_i. \quad (2.18a)$$

Each factor in the right hand side can be obtained in reference to the partial wave analysis

$$g_a^{(+)}(\mathbf{r}) = 2\pi(k_a \mathbf{r})^{-1} \sum i^{l_a+1} u_{l_a}(r) Y_{l_a}^0(\omega_a) Y_{l_a}^0(\omega), \quad (2.19)$$

where ω_a is the polar coordinate of \mathbf{k}_a and is taken to be zero. $u_{l_a}(r)$ is the radial wave distorted by U and is expressed, with the notations employed by Blatt and Weisskopf¹¹⁾ as

$$u_{l_a}(r) = u_{l_a}^{(-)}(r) - \eta_{l_a}(k_a) u_{l_a}^{(+)}(r), \quad (r > R). \quad (2.20)$$

Thus we have

$$|A\rangle = \sqrt{M/b^2 k_a} i r^{-1} u_{l_a}(r) |l_a I_a J N\rangle \quad (2.21a)$$

and

$$\langle a|A\rangle = (2\pi\hbar/\sqrt{Mk_a}) (l_a I_a 0 M_a | J N) Y_{l_a}^{0*}(\omega_a), \quad (2.22)$$

where $(l_a I_a 0 M_a | J N)$ is the abbreviated expression of the Clebsch-Gordan coefficient appeared in (2.17). The normalization of $\langle a|A\rangle$ is slightly at variance with reference 13. That is, the number of states per unit energy is 2π . This choice results in the unitarity of the S -matrix and the correspondence to reference 11 becomes manifest.

Corresponding quantities of the final state are obtained as follows. The eigen state is

$$|B\rangle = \sqrt{M/b^2 k_b} i r^{-1} v_{l_b}(r) |l_b I_b J N\rangle, \quad (2.21b)$$

where $v_{l_b}(r)$ is the radial part of the ingoing wave, expressed as*

$$\begin{aligned} v_{l_b}(r) &= -u_{l_b}^{(+)}(r) + \eta_{l_b}^*(k_b) u_{l_b}^{(-)}(r) \\ &= -u_{l_b}^*(r), \quad (r > R). \end{aligned} \quad (2.20b)$$

In terms of $|B\rangle$, $G_b^{(-)}$ is expanded as

$$G_b^{(-)} = \sum_B |B\rangle \langle B|_b, \quad (2.18b)$$

with

* This solution can readily be seen from (2.9), if one compares it with the equation for $G_a^{(+)}$ which leads to $u_l(r)$.

$$\langle B|b\rangle = (2\pi\hbar/\sqrt{Mk_b}) (I_b I_b m_b M_b | JN) Y_{I_b}^{m_b*}(\omega_b), \quad (2.22b)$$

where ω_b indicates the polar coordinate of the scattered direction.

With the aid of (2.21) and (2.22), we are able to express the transition matrix as

$$\begin{aligned} T_{ba} &= \sum_{AB} \langle B|b\rangle^* \langle B|R|A\rangle \langle A|a\rangle \\ &= \frac{(2\pi\hbar)^2}{M\sqrt{k_a k_b}} \sum_{AB} \sqrt{\frac{2I_a+1}{4\pi}} (I_a I_a 0 M_a | JN) (I_b I_b m_b M_b | JN) Y_{I_b}^{m_b}(\omega_b) (B R A), \end{aligned} \quad (2.23)$$

where $Y_{I_a}^0(0) = \sqrt{(2I_a+1)/4\pi}$ is utilized. In this expression $S=1+R$ has the unitary character and is diagonal with respect to J , but nondiagonal with respect to l and I .

Evaluating the square sum of (2.23), we obtain the differential cross section, as represented in reference 11,

$$d\sigma/d\omega_b = (2I_a+1)^{-1} k_a^{-2} \sum B_L P_L(\cos \theta_b) \quad (2.24)$$

where θ_b is the zenith angle of k_b , and B_L is given by

$$\begin{aligned} B_L &= ((-)^{I_b-I_a}/4) \sum Z(I_a J I_a' J'; I_a L) Z(I_b J I_b' J'; I_b L) \\ &\quad \times R[\langle B|R'|A\rangle \langle B'|R''|A'\rangle^*]. \end{aligned} \quad (2.25)$$

Here Z is the Z -coefficient defined in reference 11, but with the difference in its phase as remarked by Huby¹²⁾.

The theory of Hauser and Feshbach¹ consists in taking the average over such an energy range, that the interference between different J 's and I 's disappears and the intermediate states are described in terms of the statistical theory. The neglect of the interference results in the angular distribution symmetric about $\theta_b = \pi/2$, because odd L 's do not appear. One can, therefore, reach the formula given by Hauser and Feshbach*.

In the present work we do not impose the statistical assumption and maintain the interference. Hence the angular distribution is not always symmetric about $\pi/2$, but the total cross section has the same form in both theories.** This difference in the angular distribution will provide a criterion, for deciding which of the two interactions, direct or compound nucleus formation, is more appropriate.

§ 3. Rotational excitation by the direct interaction

In evaluating the matrix element (2.13), we have to know the interaction energy. This should come out as a result of very complicated interactions which can hardly be estimated.

* In the formula given in (3.17) of reference 3, the two Z^2 are identical, so that the expression becomes simpler than that given there.

** By the same form we mean that the cross sections expressed as (2.24) with (2.25) have the same looking. Needless to say, however, the contents of matrix elements are different, depending on the interactions assumed.

If, however, the core of a nucleus is replaced by an irrotational, incompressible fluid as done by Bohr and Mottelson¹⁰⁾, the interaction can be simplified as

$$V = -WR\delta(r-R)\sum_{\lambda\mu}\sqrt{\hbar\omega_\lambda/2C_\lambda}Y_\lambda^\mu(\omega)(b_{\lambda\mu}+(-)^{\mu}b_{\lambda-\mu}^*). \quad (3.1)$$

This creates a phonon of energy $\hbar\omega_\lambda$ with angular momentum λ and its Z component μ by an operator $b_{\lambda-\mu}^*$, and destroys such a phonon by $b_{\lambda\mu}$. W may be equated to the depth of a square well potential, $U(r)$, with range R and the hydrodynamical constant C_λ may be taken from B-M.

As long as the interaction (3.1) is adopted, we do not have to consider the spin of neutrons, because the spin-orbit interaction is neglected. Moreover, the states of such nucleons in the nucleus that do not form the core remain unchanged in the course of reactions. These facts, implied in assuming (3.1), considerably simplify our calculations.

The nuclear state can be described in the following three ways. In the first case the target nucleus itself consists of a core. In the other cases a nucleus is considered to consist of a core with nucleons outside the core, and the interaction between them is described by the weak and the strong coupling approximations.

(i) The target is a core. A state of the core, $\psi_\lambda^\mu(N)$, is characterized by the number of phonons N , their angular momentum λ and μ the Z component thereof. In the initial state the target or the core is in the ground state, $N=\lambda=\mu=0$. In the final state the lowest order of excitation ($N=1$, $\lambda=2$, $\mu=0, \pm 2$) is taken into consideration. Hence

$$\phi_a = \phi_0^0(0), \quad \phi_b = \phi_2^{\mu_b}(1). \quad (3.2)$$

Now $I_a=0$ and $I_b=2$, so that only $\lambda=2$ in (3.1) remains. Hence we may drop the suffices of ω_2 and C_2 without confusion.

On account of (2.21) the R -matrix can be calculated as

$$\begin{aligned} \langle B|R|A \rangle &= \langle B|V|A \rangle = -WR\sqrt{\frac{\hbar\omega}{2C}}\sqrt{\frac{M^2}{\hbar^4 k_a k_b}}i^{l_a-l_b}(-)^{l_b+I_a-J} \\ &\times \left[\frac{5(2l_a+1)(2I_b+1)}{4\pi} \right]^{1/2} (l_a 200 | l_b 0) W(I_a l_a I_b l_b; J 2) u_{l_a}(R) v_{l_b}^*(R), \end{aligned} \quad (3.3)$$

where $W(I_a l_a I_b l_b; J 2)$ is a Racah coefficient. The substitution of this into (2.25) yields*)

$$\begin{aligned} B_L &= \frac{(-)^{I_a-I_b}}{4} \sum (WR)^2 \frac{\hbar\omega}{2C} \frac{M^2}{\hbar^4 k_a k_b} \frac{2I_b+1}{4\pi} \\ &\times (l_a 200 | l_b 0) (l'_a 200 | l'_b 0) Z(l_a l_a l'_a l'_a; 0L) \\ &\times Z(l_b l_a l'_b l'_a; 2L) u_{l_a}(R) u_{l'_a}^*(R) v_{l_b}^*(R, v_{l'_b}(R). \end{aligned} \quad (3.4)$$

* In our previous report⁵⁾ a phase factor $(-)^{l_b+l_{b'}}$ is erroneously introduced. However, this is insensitive to the total cross section.

(ii) Weak coupling approximation. For a target nucleus composed of a core and some nucleons outside, one may treat the nucleus by the weak coupling approximation. Since this approximation seems in many cases unable to give a satisfactory agreement with observed nuclear properties, we shall present a method only for the case of one nucleon outside the core.

The target nucleus with spin I_a whose Z component is M_a , is described by the superposition of core states as

$$\phi_a = \sum_{\lambda, \mu_a, N_a} a_{j\lambda, \mu_a} (N_a) (j\lambda_a m_j \mu_a I_a M_a) \phi_j^{mj}(i) \psi_{\lambda_a}^{\mu_a}(N_a). \quad (3.5)$$

$\phi_j^{mj}(i)$ is the wave function of the i -th nucleon with total angular momentum j and its Z component m_j . $a_{j\lambda}(N)$ is the probability amplitude for the assigned nucleon and core states. The corresponding expression holds for the final nuclear state. It is permissible to restrict the core states to $N=0, 1$ and $\lambda=0, 2$. A similar treatment can be applied to the case of many nucleons outside, provided that the mixture of single nucleon levels due to the interaction between nucleons is neglected.

Since no interaction with the impinging neutron is assumed to exist for the i -th nucleon, the integral of $|\phi_j^{mj}|^2$ over the i -th coordinate yields simply unity. The core part results in those elements of V which give rise to the transitions from $\lambda_a=0$ to $\lambda_b=2$ and from $\lambda_a=2$ to $\lambda_b=0$. The existence of the core state with $\lambda_a=0$ implies $j=I_a$ and $m_j=M_a$. Since j and m_j are conserved, the final state of $\lambda_b=0$ is allowed only for $I_b=I_a$ and $M_b=M_a$.

The fact that the nuclear spin is composed of the spins of a core and a nucleon forbids the expression for the cross section as simple as in (i). However, the conservation of j allows us to discard the presence of the nucleon outside the core, just as one can neglect the spin of an impinging neutron in the absence of the spin-orbit coupling. Thus $\psi_{\lambda_a}^{\mu_a}(N)$ can be regarded as a nuclear state instead of ϕ , but a weight factor $a_{j\lambda}(N)$. Thus we have only to attach the following factor to the cross section in case (i):

$$|a_{I_a}^{*2}(1_b) a_{I_a 0}(0_a) + a_{I_a}^{*0}(0_b) a_{I_a 2}(1_a) \delta_{I_a I_b}|^2. \quad (3.6)$$

(iii) Strong coupling approximation. Following B-M, it is appropriate to express V in (3.1) as

$$V = -WR\delta(r-R) \sum_{\lambda\mu} Y_{\lambda}^{\mu}(\omega) \sum_{\nu} \mathfrak{D}_{\mu\nu}^{\lambda*}(\tau) q_{\nu}(\sigma). \quad (3.7)$$

Now the interaction is not represented by the creation and the annihilation of phonons, but by the rotation of the nucleus through $\mathfrak{D}_{\mu\nu}^{\lambda*}(\tau)$ and the vibration of the nuclear surface through $q_{\nu}(\sigma)$. τ and σ represent the coordinates indicating the nuclear orientation and the surface deformation respectively.

Correspondingly the nuclear state is characterized by the combination of the following eigen functions.

$$\begin{aligned} \phi_a = & (\sqrt{2I_a+1}/4\pi) \{\chi_{\Omega_a}(j) \mathfrak{D}_{M_a K_a}^{I_a}(\tau) \\ & + (-)^{I_a-j} \chi_{-\Omega_a}(j) \mathfrak{D}_{M_a -K_a}^{I_a}(\tau)\} \varphi_{n_a}(\sigma). \end{aligned} \quad (3.8)$$

\mathfrak{D}_{MK}^I is the eigen function of a symmetrical top with angular momentum I whose components to the space axis and the nuclear symmetry axis are denoted as M and K respectively. The angular momentum I , that is the nuclear spin, is composed of the rotational angular momentum of a core, λ , and the total angular momentum of an outer nucleon, j . The outer nucleons are described by $\chi_{\Omega}(j)$, where Ω is the component of j to the nuclear axis. The cases of $j=1/2$ and $3/2$ are exceptional and will not be concerned here. If there are more than one nucleon outside, j in $(-)^{-j}$ is replaced by the simple sum of individual j 's and Ω by that of individual Ω 's. Two terms in the curly bracket are required for the symmetrization. For $\Omega=0$ and $K=0$, however, this is not needed and only the first term remains, but with a factor $\sqrt{2}$ for the correct normalization. $\varphi_n(\sigma)$ indicates the eigen function for the surface vibration and n represents the vibrational quantum numbers. In the present work we are not concerned with the vibrational excitation because of its secondary importance, so that n is conserved in the course of scattering.

As the initial nucleus lies in the ground state, in which the rotation is not excited, we can put $I_a=K_a=\Omega_a$, except for $K_a=\Omega_a=1/2$. Since there is no interaction with outer nucleons in V , we have $\Omega_b=\Omega_a$ as well as $K_b=K_a$ and the spin change of the nucleus is due entirely to the addition of the rotational angular momentum, whose magnitude is now restricted to $\lambda=2$. This situation can be seen from the integral over orientations

$$\int \mathfrak{D}_{M_b K_b}^{I_b*} \mathfrak{D}_{\mu\nu}^{\lambda*} \mathfrak{D}_{M_a K_a}^{I_a} d\tau = (8\pi^2 / (2I_b + 1)) (-)^{\mu-\nu} (\lambda I_a - \nu K_a | I_b K_b) \times (\lambda I_a - \mu M_a | I_b M_b). \quad (3.9)$$

The last factor is the same as that which arises in the decomposition of \mathcal{Q}_b in case (ii), corresponding to (3.5). Because no vibrational motion is assumed to be excited, the matrix element of q_ν is expressed in terms of the deformation parameter, Q_ν , of order ν , as

$$\int \varphi_{n_b}^* q_\nu \varphi_{n_a} d\sigma = (\sqrt{5\pi} / 3ZR^2) Q_\nu \delta_{n_b n_a}, \quad (3.10)$$

where Z is the nuclear charge number. ν is restricted to zero, because of $K_b=K_a$ and $(\lambda I_a - \nu K_a | I_b K_b)$ in (3.9). Hence there appears only Q_0 , the intrinsic quadrupole moment, which is intimately related to the static quadrupole moment.

Now we are ready to give the matrix element of R , by comparison with case (i), as

$$\begin{aligned} \langle B | R | A \rangle = & -WR \frac{\sqrt{5\pi} Q_0}{3ZR^2} \frac{M}{\hbar^2 \sqrt{k_a k_b}} i^{I_a - I_b} (-)^{I_b + I_a - J} \\ & \times \sqrt{\frac{5(2I_a + 1)(2I_b + 1)}{4\pi}} (I_a 200 | I_b 0) W(I_a I_a I_b I_b; J2) \\ & \times (2I_a 0 K_a | I_b K_b) v_{I_b}^*(R) u_{I_a}(R). \end{aligned} \quad (3.11)$$

For odd-odd nuclei a special consideration is needed, because of the degeneracy in Ω , but we are not concerned with this case, since there are few such cases.

Substituting (3.11) into (2.24) with (2.25), we have the differential cross section. As the strong coupling approximation is of the most practical use, we give the explicit expression,

$$\begin{aligned}
\frac{d\sigma}{d\omega_b} &= \frac{(-)^{I_b - I_a}}{4k_a^2} \sum Z(l_a l_a' l_b' l_b'; 0L) Z(l_b l_a' l_b' l_a'; 2L) \\
&\quad \times P_L(\cos \theta_b) (WR)^2 \frac{1}{4\pi} \left(\frac{\sqrt{5\pi} Q_0}{3ZR^2} \right)^2 \frac{M^2}{\hbar^4 k_a k_b} (l_a 200 l_b 0) (l_a' 200 l_b' 0) \\
&\quad \times (2I_a 0 K_a | I_b K_a)^2 v_{l_b}^*(R) v_{l_a}(R) v_{l_b}'(R) v_{l_a}'^*(R). \quad (3.12)
\end{aligned}$$

The total cross section is obtained, by taking terms of $L=0$, as

$$\begin{aligned}
\sigma &= \frac{\pi}{k_a^2} \frac{M^2}{\hbar^4 k_a k_b} W^2 R^2 \frac{5}{4} \left(\frac{Q_0}{3ZR^2} \right)^2 (2I_a 0 K_a | I_b K_a)^2 \\
&\quad \sum (2l_b + 1) (l_b 200 | l_a 0)^2 |v_{l_a}(R)|^2 |v_{l_b}(R)|^2. \quad (3.13)
\end{aligned}$$

§ 4. Comparison with experiments

In comparing our theory with experiments, we are particularly interested in even-even nuclei as targets. The ground state of these nuclei has zero spin, $I_a=0$, and the first excited state is assigned to be a rotational state of spin two, $I_b=2$. The cross section is, therefore, simplified on account of $(2I_a 0 K_a | I_b K_a) = 1$. As an example of this case we have considered the inelastic scattering of 17 Mev protons by using Born approximation⁷⁾. The agreement with the experiment of reference 7 is not satisfactory, but is good enough to account for the qualitative feature of the inelastic scattering in terms of the rotational excitation.

As we gave in reference 5 only the result of the strong coupling approximation, here we compare the three approximations discussed in § 3 with each other.

If the target itself is a core, there arises a factor

$$\hbar\omega/2C \simeq 3A^{-7/6} (1 - Z^2/50A)^{-1/2} \quad (4.1, i)$$

in the cross section, as given by B-M on the basis of the hydrodynamical model. In comparison with this, the weak and the strong coupling approximations are considered for $I_a=0$ and $I_b=2$. The corresponding factors in the cross section are

$$|a_{02}^{\pi}(1) a_{10}|^2 \hbar\omega/2C \quad (4.1, ii)$$

and

$$\pi(Q_0/3ZR^2)^2 \simeq 0.9 \times 10^3 Q_0^2 / Z^2 A^{1/3} \quad (Q_0 \text{ in barn}) \quad (4.1, iii)$$

respectively.

Numerical values will be discussed for an example of ^{56}Fe . As $A=56$ and $Z=26$, we get from (4.1, i) $\hbar\omega/2C \simeq 3 \times 10^{-2}$. According to the shell structure, the orbits of outer nucleons are supposed to be $(f_{7/2})^{-2}$ for protons and $(p_{3/2}, f_{5/2})^2$ for neutrons. In the weak coupling approximation one has to know the magnitudes of the a 's. These are estimated by B-M as giving a factor of $1/10 \sim 1/5$ multiplied by $\hbar\omega/2C$.

In the strong coupling approximation, the value of Q_0 can be obtained from the static quadrupole moment, Q , through the formula

$$Q/Q_0 = (2I-1)I/(I+1)(2I+3). \quad (4.3)$$

As this vanishes for $I=0$, Q_0 for ^{56}Fe is estimated from the Q 's of ^{55}Mn ($I=5/2$) and ^{59}Co ($I=7/2$). This is justified on account of the smooth variation of Q_0 against mass number. In both cases we have

$$Q_0 = 1.1 \text{ b}, \quad (4.4)$$

with errors due to the Q 's, $\pm 0.6\text{b}$ and $\pm 0.4\text{b}$, respectively. Hence we may regard the value of Q_0 in (4.4) as the Q_0 of ^{56}Fe . Then (4.1, iii) gives us 0.8×10^{-2} , which is a little larger than (4.1, ii). If the intermediate coupling nature is taken into account, however, it seems better to take

$$Q_0 = 0.7 \pm 0.3 \text{ b}. \quad (4.4')$$

This choice of Q_0 gives 0.3×10^{-2} to (4.1, iii), in rough agreement with (4.1, ii). This should be expected, because the values of a 's are estimated on the basis of the intermediate strength of the coupling. Thus we may adopt 3×10^{-3} for (4.1), which appears in the cross section as a factor.

With this numerical value of (4.1), the total cross section (3.13) is written as

$$\sigma = \frac{5}{16} R^2 \frac{W^2}{E_a E_b} \sqrt{\frac{E_b}{E_a}} (4.1) \sum (2l_b + 1) (l_b 200 | l_a 0)^2 | u_{l_a}(R) |^2 | v_{l_b}(R) |^2, \quad (4.5)$$

where $E_a = \hbar^2 k_a^2 / 2M$ and $E_b = \hbar^2 k_b^2 / 2M$ are the initial and final energies of neutrons respectively.

In our previous report⁵⁾, we adopted the following numerical values

$$R = 1.4 A^{1/3} \times 10^{-13} \text{ cm}, \quad W = 33 \text{ Mev} \quad (4.6)$$

and compared the total inelastic cross section with experiments at $E_a = 2.50 \text{ Mev}$ and $E_b = 1.65 \text{ Mev}$. Consequently σ is obtained as $5 \times 10^{-2} \sum$ barn, where \sum stands for the summation in (4.5). This is to be compared with experimental values, about $1.5 \text{ barn}^{2), 15)}$. In order to explain the experimental cross section in terms of the direct rotational excitation, \sum has to be as large as 30. This requires considerable magnitudes of resonances of neutron waves due to the potential U .

In our previous report⁵⁾, the potential U was taken as a square well of depth 33 Mev. The cross section obtained at $E_a = 2.5 \text{ Mev}$ was about 1 barn, which was due chiefly to the sharp resonance of a G -wave. The contribution from non-resonant waves gives the order of unity to \sum in (4.5). Indeed, for the depth of U equal to 30 Mev the resonance of the G -wave disappears and the cross section turns out to be about 20 mb at the same energy. The situation is similar if the potential of depth 33 Mev is added to an imaginary part of depth 1 Mev. Even such a small imaginary part is great enough to eliminate the sharp G -wave resonance and the cross section is reduced by about two orders. The main contribution is found to come from $l_a = l_b = 1$.

The analysis of neutron total cross sections by Feshbach et al³⁾ has led to the optical potential of depth and radius

$$W(1+i\zeta) = 42(1+0.03i) \text{ MeV}, \quad R = 1.45 \times A^{1/3} \times 10^{-13} \text{ cm} \quad (4.7)$$

respectively. This gives rise to a strong S -wave resonance at $A=56$. Assuming the same potential both for the entrance and the exit channels, we obtain a medium value of \sum (2.5 at 2.5 MeV) due to the S -wave resonance. The contribution to the cross section comes almost exclusively from $l_a=2$ and $l_b=0$. Since $|v_a(R)|^2$ has a resonance maximum, the cross section also shows a broad maximum. This resonance character governs the main behaviour of the cross section even for $\zeta=0.1$.

The numerical examples discussed above are considered to show that the cross section is quite sensitive to the potential which distorts neutron waves. Therefore, this calculation can not predict experimental cross sections unless the potential is known with a good accuracy. This sensitivity to the potential may be employed to know the detailed shape of the potential if the description in terms of the optical potential holds also in inelastic scattering. Strictly speaking, however, the optical potential should depend on neutron energies and be different for elastic and inelastic processes, as shown in the Appendix.

Thus far we have confined ourselves to the discussion of the direct rotational excitation. Looking at the excitation curve observed by Kielln and Goodman², however, the resonance scattering through compound nuclear states seems largely responsible for the inelastic cross section. In that case the cross section is expressed by the resonance formula, or the formula given by Hauser and Feshbach¹ if compound nucleus levels are closely spaced. In the former case, if there is no interference between resonance levels, the angular distribution is given by (2.24) , but with

$$B_L = 4^{-1} (-)^{l_b-l_a} \sum_a Z(l_a, l_a, J; l_a, L) Z(l_b, l_b, J; l_b, L) \times |\langle B|R|A \rangle|^2. \quad (4.8)$$

The matrix element $\langle B|R|A \rangle$ can be obtained from the knowledge of reduced widths. But we do not attempt to evaluate them in this paper and only discuss the case, in which an interaction such as (3.1) takes part in the resonance reaction.

The absolute magnitude of the cross section is now not far from that predicted by Hauser and Feshbach. The resonance effect as well as the detailed knowledge of reduced widths may alter the cross section by a certain factor. If the direct process competes with the compound nucleus formation, an interference between these two should arise. In order to see which of the contributions from these two causes is more important, the observation of angular distributions is awaited for.

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Appendix

Some possible effects of compound nuclear states

Bohr and Mottelson formulated the resonance elastic scattering problem by taking ac-

count of the interaction given in (3.1) in Appendix V of their paper¹⁰⁾. Their method is here extended to the case of inelastic scattering by taking account of the direct interaction. For simplicity and in order to preserve the similarity to Bohr-Mottelson, we confine ourselves to the case of two open channels only; one is an entrance *S*-wave neutron channel and the other is the channel for inelastically scattered neutrons.

The whole of space is divided into two parts by a sphere of radius R , inside of which an orthogonal set of compound nucleus states, $\Psi_j(\mathbf{r}, x)$, are defined. They should be chosen so that all Ψ_j are orthogonal to the product of a target state, Φ_i , and a single particle neutron state, $\chi_t r^{-1}\varphi_t(r)$, where t indicates a or b according to the initial or final state, and χ_t is the angular part of single particle wave function; for the case of $t=a$, $\chi_a = (4\pi)^{-1/2}$. In analogy with Appendix V.1 in reference 10, the wave function for the total system is expanded as

$$\Psi(\mathbf{r}, x) = (4\pi)^{-1/2} r^{-1} \varphi_a(r) \Phi_a(x) + c_b \chi_b r^{-1} \varphi_b(r) \Phi_b(x) + \sum_j c_j \Psi_j(\mathbf{r}, x). \quad (\text{A} \cdot 1)$$

$\varphi_t(r)$ obeys the following equation:

$$(K + U + \varepsilon_t - E) \varphi_t(r) = 0 \quad (r \geq R) \quad (\text{A} \cdot 2)$$

except at $r=R$. Now U is a real potential. For the entrance channel, $t=a$, $\varphi_a(r)$ consists of a partial wave of an incident plane wave and an elastically scattered wave, so that it corresponds to $g_a^{(+)}$ in § 2. For the exit channel, $t=b$, only the divergent wave is taken into account. In this case we have $\varphi_b(r) = u_{lb}^{(+)}(r)$ together with the scattering amplitude c_b , where $u_{lb}^{(+)}(r)$ is the same as in (2.20a). Because of the singular interaction, V , the logarithmic derivative of $\varphi_t(r)$ is not continuous at $r=R$, but

$$R[(d/dr) \varphi_t]/\varphi_t|_{r=R+} \equiv f_+(E_t) \text{ and } R[(d/dr) \varphi_t]/\varphi_t|_{r=R-} \equiv f_-(E_t) \quad (\text{A} \cdot 3)$$

are different. $f_+(E_b)$ can be expressed by Δ and s introduced in reference 14 as

$$f_+(E_b) = \Delta_b + is_b. \quad (\text{A} \cdot 4)$$

$f_-(E_t)$ is obtained from interior wave functions, as in reference 3.

The energy eigenvalue of the compound state is given by

$$\int \int \Psi_j^* H \Psi \, dx d\mathbf{r} = W. \quad (\text{A} \cdot 5)$$

The coupling energy between the open channel and compound states is defined by

$$V_{ij} = \sqrt{2} R^{-3/2} \int \chi_i^*(\omega) \Phi_i^*(x) V(\mathbf{r}, x) \Psi_j(\mathbf{r}, x) d\mathbf{r} dx. \quad (\text{A} \cdot 6)$$

Furthermore the coupling between the entrance and exit channels also causes scattering. Its energy is defined, analogously to (A.6), as

$$V_{ba} = (\pi R^6)^{-1/2} \int \chi_b^*(\omega) \Psi_b^*(x) V(\mathbf{r}, x) \Phi_a(x) d\mathbf{r} dx. \quad (\text{A} \cdot 7)$$

Now we are ready to reduce the Schrödinger equation for Ψ to a set of coupled equations. Analogously to Bohr-Mottelson, we have

$$-\sqrt{\frac{R}{2}} \frac{\hbar^2}{MR^2} [f_+(E_a) - f_-(E_a)] \varphi_a(R) + c_b \sqrt{\frac{R}{2}} V_{ba}^* \varphi_b(R) + \sum_j c_j V_{aj} = 0, \quad (\text{A} \cdot 8a)$$

$$\sqrt{\frac{R}{2}} V_{aj}^* \varphi_a(R) - c_b \sqrt{\frac{R}{2}} \frac{\hbar^2}{MR^2} [A_b + is_b - f_-(E_b)] \varphi_b(R) + \sum_j c_j V_{bj} = 0, \quad (\text{A} \cdot 8b)$$

$$\sqrt{\frac{R}{2}} V_{aj}^* \varphi_a(R) + c_b \sqrt{\frac{R}{2}} V_{bj}^* \varphi_b(R) + c_j (W_j - E) = 0. \quad (\text{A} \cdot 8c)$$

Eliminating c_b and c_j from (A·8), we obtain

$$f_+(E_a) - f_-(E_a) = \frac{MR^2}{\hbar^2} \sum_j \frac{|V_{aj}|^2}{E - W_j} + \left(\frac{MR^2}{\hbar^2} \right)^2 \times \frac{|V_{ba}^* + \sum_j V_{aj} V_{bj}^* / (E - W_j)|^2}{A_b + is_b - f_-(E_b) - (MR^2/\hbar^2) \sum_j |V_{bj}|^2 / (E - W_j)}. \quad (\text{A} \cdot 9)$$

By using this logarithmic derivative at $r=R_+$, the radial wave function for the entrance channel is expressed as

$$\varphi_a(r) = \exp(ik_a r) - \eta_a \exp(ik_a r), \quad r > R,$$

where

$$\eta_a = \frac{f_+(E_a) + ik_a R}{f_+(E_a) - ik_a R}.$$

This η_a gives us the S -matrix for the elastic scattering. If we take an energy average of η_a over energy range which is very broad compared to the average distance of the compound levels but very narrow compared to the energy distance of the single particle level, the energy average of η_a , $\langle \eta_a \rangle$, may be approximately interpreted as due to the complex potential which is given by U plus an imaginary part which is considered to come from the energy average of the effect of compound states. This complex potential is nothing but the so-called optical potential. If the channel b is neglected, the modification to $f_-(E_a)$ is only $(MR^2/\hbar^2) \sum_j |V_{aj}|^2 / (E - W_j)$ which is the same as the result given by reference 10, and is valid for the pure elastic scattering.

From (A·8b) and (A·8c) the scattering amplitude is solved as

$$c_b = \frac{MR}{\hbar^2 k_b} \varphi_b^*(R) \frac{s_b}{A_b + is_b - f_-(E_b) - (MR^2/\hbar^2) \sum_j [|V_{bj}|^2 / (E - W_j)]} \times \left[V_{ba} + \sum_j \frac{V_{bj} V_{aj}^*}{E - W_j} \right] \varphi_a(R), \quad (\text{A} \cdot 10)$$

where $\varphi_b^*(R) \varphi_b(R) = s_b / (k_b R)$ is taken into account. If the potential is modified to the optical potential to take into account the $(MR^2/\hbar^2) \sum_j [|V_{bj}|^2 / (E - W_j)]$ as in the case of the pure elastic scattering, one can see the correspondence to the ingoing wave, i.e.

$$g_b^{(-)*}(R) = \chi_b R^{-1} \langle \varphi_b^*(R) \frac{2is_b}{\Delta_b + is_b - f_-(E_b) - (MR^2/\hbar^2) \sum_j [|V_{bj}|^2 / (E - W)]} \rangle. \quad (\text{A} \cdot 11)$$

If $\varphi_a(R)$ is solved on account of the logarithmic derivative given in (A·9), (A·10) is nothing but (2·12) in the text. In (A·10) the first term in the square bracket represents the direct process, while the second term the resonance process.

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Nonadiabatic Treatment of Nuclear Forces, II

— Nonstatic Corrections —

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The nonstatic corrections to the second plus fourth order potentials are studied in the symmetrical $P_S(p\bar{p})$ meson theory. The potentials are constructed by means of canonical transformation, and the nonstatic corrections are obtained by the use of the expansion in powers of $(p/2M) \sim (k/2M) \sim (\mu/2M)$, the validity of which is examined for the special case of the no-pair contributions.

The nonstatic corrections are negligible for the large separation $r > 1/\mu$ between two nucleons, and are large in the interior region but would not alter the main features of the static potentials except for the central force in triplet odd state and the one-pair contributions. The results for the small separation should be accepted only qualitatively because of the inadequacy of the power series expansion in this region.

It is shown that the separation of the no-pair contributions from the one- and the two-pair contributions leads to the unreasonable results for the velocity dependent potentials, and that the more careful treatment is needed for the inclusion of the effects of the pair suppression. The spin-orbit coupling in the two-pair contributions obtained by Klein is found to be cancelled by the inclusion of the one-pair contributions.

Finally, it is remarked that the potentials determined by this canonical transformation give the correct results for the scattering problem up to g^4 .

§ 1. Introduction

The low energy properties of two nucleon system are found to be explicable in terms of the second plus fourth order static potentials in the symmetrical $P_S(p\bar{p})$ meson theory or the second plus fourth order no-pair potentials in the $P_S(p\bar{p})$ theory.^{1,2,3,4)} Therefore, it is important to study higher order corrections and nonstatic corrections systematically, the former of which has been evaluated by many authors.^{2,5)} It has been shown that the higher order corrections are effective when the separation between two nucleons is small, but are negligible for the large separation $r > 0.6/\mu$ and that in this region ($r > 0.6/\mu$) the potentials can be approximated by the second plus fourth order terms at least as regards the no-pair contributions.

In this paper we investigate the nonstatic corrections to the second plus fourth order terms in the symmetrical $P_S(p\bar{p})$ meson theory. We consider the exchange of mesons between two nucleons, and neglect the processes in which the meson emitted by one nucleon is re-absorbed by the same one. The latter must be cancelled by the appropriate renormalization. We construct the potentials by means of the canonical transformation

method which was presented in the previous article^{6)*}, and evaluate the nonstatic corrections by expanding the expression for the potentials in powers of $p/2M \sim k/2M \sim \mu/2M$, where p is the momentum of the nucleon. The validity of this expansion is discussed for some special cases.

§ 2. Evaluation of the nonstatic contributions

We start with the Hamiltonian for meson-nucleon system with the symmetrical $P_S(p_S)$ coupling which is invariant under nucleon-antinucleon conjugation, instead of the hole-theoretical one as in I.

$$H = 1/2 \cdot \int \left[\bar{\psi}(\mathbf{r}) \left(\gamma_i \frac{\partial}{\partial r_i} + M \right) \psi(\mathbf{r}) + \bar{\psi}'(\mathbf{r}) \left(\gamma_i \frac{\partial}{\partial r_i} + M \right) \psi'(\mathbf{r}) \right] d\mathbf{r} \\ + 1/2 \cdot \int \sum_i [\phi_i(\mathbf{r}) (\mu^2 - \nabla^2) \phi_i(\mathbf{r}) + \pi_i^2(\mathbf{r})] d\mathbf{r} + ig/2 \int [\bar{\psi}(\mathbf{r}) \gamma_5 \tau_i \psi(\mathbf{r}) \phi_i(\mathbf{r}) \\ + \bar{\psi}'(\mathbf{r}) \gamma_5 \tau_i' \psi'(\mathbf{r}) \phi_i(\mathbf{r})] d\mathbf{r}$$

— (zero point energy of the free nucleon, antinucleon and meson fields)

$$\text{where} \quad \psi' = C \bar{\psi}^T, \quad \bar{\psi}' = \psi^T (C^{-1})^T, \quad \gamma_\mu^T = \gamma_\mu^* = -C^{-1} \gamma_\mu C, \quad C^T = -C \quad (1)$$

$$\text{and} \quad C + C = 1. \quad (2)$$

We expand the field operators in (1) as follows:

$$\psi_\alpha(\mathbf{r}) = (2\pi)^{-3/2} \left\{ \sum_p u_\alpha^{+p}(\mathbf{p}) b_+^p(\mathbf{p}) e^{i\mathbf{p}\mathbf{r}} + \sum_\mu u_\alpha^{-p}(\mathbf{p}) b_-^{p*}(\mathbf{p}) e^{i\mathbf{p}\mathbf{r}} \right\} \\ \bar{\psi}_\alpha(\mathbf{r}) = (2\pi)^{-3/2} \left\{ \sum_p u_\alpha^{+p*} b_+^{p*}(\mathbf{p}) e^{-i\mathbf{p}\mathbf{r}} + \sum_\mu u_\alpha^{-p*}(\mathbf{p}) b_-^p(\mathbf{p}) e^{-i\mathbf{p}\mathbf{r}} \right\} (\beta)_{\beta\alpha} \\ \phi_i(\mathbf{r}) = (2\pi)^{-3/2} \int (a_i(\mathbf{k}) + a_i^*(-\mathbf{k})) e^{i\mathbf{k}\mathbf{r}} (2\omega_k)^{-1/2} d\mathbf{k} \quad (3) \\ \pi_i(\mathbf{r}) = (2\pi)^{-3/2} \int (a_i^*(-\mathbf{k}) - a_i(\mathbf{k})) e^{i\mathbf{k}\mathbf{r}} i\omega_k (2\omega_k)^{-1/2} d\mathbf{k}.$$

Here $b_\pm^{p*}(\mathbf{p})$ and $b_\pm^p(\mathbf{p})$ are the creation and destruction operator for the free states of nucleon and antinucleon with momentum $\pm\mathbf{p}$ and σ - and τ -spin ρ , and $a_i^*(\mathbf{k})$ and $a_i(\mathbf{k})$ are the creation and destruction operator for the meson with momentum \mathbf{k} and isotopic variable i ($i=1, 2, 3$). The spinor $u^{\pm p}(\mathbf{p})$ is constructed from $u^{\pm p}(0)$ which corresponds to the zero momentum state:

$$u^{\pm p}(\mathbf{p}) = (E_p + M/2E_p)^{1/2} \Lambda_\pm(\mathbf{p}) u^{\pm p}(0), \quad (4)$$

and

$$\Lambda_\pm(\mathbf{p}) = \{E_p \pm (\alpha \cdot \mathbf{p} + \beta M)\} / 2E_p.$$

* Hereafter we refer to this paper as I.

Then the Hamiltonian (1) can be written as

$$\begin{aligned}
 H = & \sum_{\pm} \int b_{\pm}^*(\mathbf{p}) E_p b_{\pm}(\mathbf{p}) d\mathbf{p} + \sum_i \int a_i^*(\mathbf{k}) \omega_k a_i(\mathbf{k}) d\mathbf{k} \\
 & + i f (2\pi)^{-3/2} \sum_i \int [b_+^*(\mathbf{p}_1) \tau_i \{ \mathbf{p}_1/p_1 \cdot X_-(\mathbf{p}_1) X_+(\mathbf{p}_2) - \mathbf{p}_2/p_2 \cdot X_-(\mathbf{p}_2) X_+(\mathbf{p}_1) \} b_+(\mathbf{p}_2) \\
 & - b_-^*(\mathbf{p}_2) \tau_i^T \{ \mathbf{p}_1/p_1 \cdot X_-(\mathbf{p}_1) X_+(\mathbf{p}_2) - \mathbf{p}_2/p_2 \cdot X_-(\mathbf{p}_2) X_+(\mathbf{p}_1) \} b_-(\mathbf{p}_1) \\
 & + f b_+^*(\mathbf{p}_1) \tau_i \{ X_+(\mathbf{p}_1) X_+(\mathbf{p}_2) + (\mathbf{p}_1 \sigma) (\mathbf{p}_2 \sigma) / p_1 p_2 \cdot X_-(\mathbf{p}_1) X_-(\mathbf{p}_2) \} b_-^*(\mathbf{p}_2) \\
 & + f b_-(\mathbf{p}_1) \tau_i \{ X_+(\mathbf{p}_1) X_+(\mathbf{p}_2) + (\mathbf{p}_1 \sigma) (\mathbf{p}_2 \sigma) / p_1 p_2 \cdot X_-(\mathbf{p}_1) X_-(\mathbf{p}_2) \} b_+(\mathbf{p}_2)] \quad (5) \\
 & \cdot (a_i^*(-\mathbf{k}) + a_i(\mathbf{k})) (2\omega_k)^{-1/2} \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}) d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k},
 \end{aligned}$$

where $E_p = (p^2 + M^2)^{1/2}$, $\omega_k = (k^2 + \mu^2)^{1/2}$, $X_{\pm}(p) = (E_p \pm M/2E_p)^{1/2}$.

Here a parameter f is introduced to specify the interaction through pair creation or annihilation. The indices for the spin and isotopic spin components are dropped and the summation over these indices is implied. Of course, σ and τ are the usual σ - and τ -spin 2-2 matrices. (5) is same with (I.2.8) except that the order of b_+^* and b_- is changed in consequence of the invariance under nucleon-antinucleon conjugation. The decomposition into the four free nucleon-antinucleon states is used instead of the Tani-Foldy transformation in I.

To the Hamiltonian (5) we apply the canonical transformations, the detailed description of which was given in I. We eliminate by the successive transformations 1) the terms of first order in f , 2) the terms corresponding to the double production or absorption of mesons, and 3) the terms which represent the second order interaction through the pair creation or annihilation, and obtain the potentials 1) $V_2 + V_4^b$, 2) V_4^a and 3) V_4^c respectively. Then expanding these potentials in powers of $p/2M \sim k/2M \sim \mu/2M$, all the terms up to $\sim (\mu/2M)^4$ (and up to $\sim (\mu/2M)^5$ in the no-pair terms for the sake of comparison with the $P_5(pv)$ coupling theory) are retained.

The results are as follows: (V^{st} 's are given in I)

$$\begin{aligned}
 V \cong & V_2^{st} + V_4^{ast} + V_4^{bst} + V_4^{cst} + V_2^{nost} + V_4^{anost} + V_4^{bnost} + V_4^{cnost}, \\
 V_2^{nost} = & -1/2 \cdot g^2 / (2\pi)^3 \cdot \int b_+^{*(1)}(\mathbf{p}_1') b_+^{*(2)}(\mathbf{p}_2') (\tau^{(1)} \cdot \tau^{(2)}) (1/2M)^4 \\
 & [(i\sigma^{(1)} \cdot \mathbf{k}) (i\sigma^{(2)} \cdot \mathbf{k}) \{ (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{k})^2 / 2\omega_k^4 + (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k})^2 / 2\omega_k^4 \\
 & - 5(\mathbf{p}_1^2 + \mathbf{p}_1'^2 + \mathbf{p}_2^2 + \mathbf{p}_2'^2) / 4\omega_k^2 \} \\
 & - 3/4 \cdot \{ (i\sigma^{(1)} \cdot \mathbf{k}) (i\sigma^{(2)} \cdot \mathbf{p}_2 + \mathbf{p}_2') (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}) / \omega_k^2 \\
 & + (i\sigma^{(1)} \cdot \mathbf{p}_1 + \mathbf{p}_1') (i\sigma^{(2)} \cdot -\mathbf{k}) (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{k}) / \omega_k^2 \}] \\
 & b_+^{(2)}(\mathbf{p}_2) b_+^{(1)}(\mathbf{p}_1) \delta(\mathbf{p}_1' - \mathbf{p}_1 - \mathbf{k}) \delta(\mathbf{p}_2' - \mathbf{p}_2 + \mathbf{k}) d\mathbf{p}_1' d\mathbf{p}_2' d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k}, \quad (6)
 \end{aligned}$$

$$\begin{aligned}
V_4^{a\text{ nonst}} \cong & 1/2 \cdot g^4 / (2\pi)^6 \cdot \int b_+^{*(1)}(\mathbf{p}_1') b_+^{*(2)}(\mathbf{p}_2') \\
& \times [6(1/2M)^4 f^4 \{ \mathbf{p}_1^2 + \mathbf{p}_1'^2 + \mathbf{p}_2^2 + \mathbf{p}_2'^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2) \} / \omega_1 \omega_2 (\omega_1 + \omega_2) \\
& + 3/2 \cdot (1/2M)^4 f^4 \{ (-i\sigma^{(1)} \cdot (\mathbf{k}_1 + \mathbf{k}_2) \times (\mathbf{p}_1 + \mathbf{p}_1')) + (i\sigma^{(2)} \cdot (\mathbf{k}_1 + \mathbf{k}_2) \times (\mathbf{p}_2 + \mathbf{p}_2')) \\
& - (\mathbf{p}_1 + \mathbf{p}_1')^2 - (\mathbf{p}_2 + \mathbf{p}_2')^2 \} / \omega_1 \omega_2 (\omega_1 + \omega_2) \\
& + 3/2 \cdot (1/2M)^4 f^4 \{ (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{k}_1 + \mathbf{k}_2)^2 + (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}_1 + \mathbf{k}_2)^2 \} / \omega_1 \omega_2 (\omega_1 + \omega_2)^3 \\
& - 3/2 \cdot (1/2M)^4 f^2 \{ (i\sigma^{(1)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{k}_1) - (i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}_1) \\
& + 2(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \} / \omega_1^3 \omega_2 (\omega_1 + \omega_2) \\
& + 3/2 \cdot (1/2M)^4 f^2 \{ (i\sigma^{(1)} - i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_1 + \mathbf{p}_1' + \mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}_1 + \mathbf{k}_2) \} / \omega_1^2 \omega_2 (\omega_1 + \omega_2)^2] \\
& b_+^{(2)}(\mathbf{p}_2) b_+^{(1)}(\mathbf{p}_1) \hat{\partial}(\mathbf{p}_1' - \mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2) \hat{\partial}(\mathbf{p}_2' - \mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{p}_1' d\mathbf{p}_2' d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k}_1 d\mathbf{k}_2,
\end{aligned} \tag{7}$$

$$\begin{aligned}
V_4^{b\text{ nonst}} \cong & 1/2 \cdot g^4 / (2\pi)^6 \cdot \int b_+^{*(1)}(\mathbf{p}_1') b_+^{*(2)}(\mathbf{p}_2') \\
& [6(1/2M)^4 f^2 \{ (i\sigma^{(1)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{k}_2) - (i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}_2) \\
& - 4(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \} / \omega_1^2 \omega_2^3 \\
& + (1/2M)^5 \{ 3 + 2(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \} \{ (i\sigma^{(1)} - i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{p}_1 + \mathbf{p}_1' + \mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{k}_2) (\mathbf{k}_1 \cdot \mathbf{k}_2) \\
& - 2(\mathbf{k}_1 \cdot \mathbf{k}_2)^3 \} / \omega_1^2 \omega_2^4] \\
& b_+^{(2)}(\mathbf{p}_2) b_+^{(1)}(\mathbf{p}_1) \hat{\partial}(\mathbf{p}_1' - \mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2) \hat{\partial}(\mathbf{p}_2' - \mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{p}_1' d\mathbf{p}_2' d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k}_1 d\mathbf{k}_2,
\end{aligned} \tag{8}$$

$$V_4^{c\text{ nonst}} \cong 0, \tag{9}$$

or in configuration space

$$\begin{aligned}
V_2^{\text{nonst}} \cong & -\mu/2 \cdot (g^2/4\pi) \int \psi_+^{*(1)}(\mathbf{r}_1) \psi_+^{*(2)}(\mathbf{r}_2) (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\mu/2M)^4 (1/12\mu^2) \\
& [5(\mathbf{p}_1^2 + \mathbf{p}_1'^2 + \mathbf{p}_2^2 + \mathbf{p}_2'^2) \{ \mathcal{S}_{12}(1/x + 3/x^2 + 3/x^3) + (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})/x \} \\
& + \mathcal{S}_{12} \{ (3(\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{x})^2/x^2 + 3(\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{x})^2/x^2) (1/3 + 2/x + 5/x^2 + 5/x^3) \\
& - ((\mathbf{p}_1 + \mathbf{p}_1')^2 + (\mathbf{p}_2 + \mathbf{p}_2')^2) (1/x + 3/x^2 + 3/x^3) \} \\
& + (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \{ (3(\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{x})^2/x^2 + 3(\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{x})^2/x^2) (1/3 + 1/x + 2/x^2 + 2/x^3) \\
& - ((\mathbf{p}_1 + \mathbf{p}_1')^2 + (\mathbf{p}_2 + \mathbf{p}_2')^2)/x \} \\
& + 3 \{ (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}_1 + \mathbf{p}_1') (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x}) (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{x}) + (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}_2 + \mathbf{p}_2') (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{x}) \\
& - 2(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}_2 + \mathbf{p}_2') (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x}) (\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{x}) - 2(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}_1 + \mathbf{p}_1') (\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{x}) \} \\
& \times (1/x^3 + 3/x^4 + 3/x^5) \\
& - \{ 3(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}_1 + \mathbf{p}_1') (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}_1 + \mathbf{p}_1') + 3(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}_2 + \mathbf{p}_2') (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}_2 + \mathbf{p}_2') \} (1/x^2 + 1/x^3)] \\
& \times e^{-x} \psi_+^{(2)}(\mathbf{r}_2) \psi_+^{(1)}(\mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_2
\end{aligned} \tag{10}$$

$$\begin{aligned}
V_4^{nonst} &\cong \mu/2 \cdot (g^2/4\pi)^2 \int \phi_+^{*(1)}(\mathbf{r}_1) \phi_+^{*(2)}(\mathbf{r}_2) (\mu/2M)^4 \\
&[1/\pi \cdot \{((30f^4 - 138f^2)/x^4 + (12f^4 - 72f^2)/x^2) K_1(2x) \\
&+ ((30f^4 - 138f^2)/x^3 - 24f^2/x) K_0(2x)\} \quad (11, a) \\
&+ (\mu/2M) (3 + 2(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})) \{ (1/x + 3/x^2 + 12/x^3 + 30/x^4 + 36/x^5 + 18/x^6) \\
&- S_{12}/3 \cdot (2/x^2 + 11/x^3 + 28/x^4 + 36/x^5 + 18/x^6) \\
&+ 2(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})/3 \cdot (2/x^2 + 8/x^3 + 16/x^4 + 18/x^5 + 9/x^6) \} e^{-2x} \quad (11, b) \\
&+ 3/\pi \mu \cdot \{(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{x} \times (\mathbf{p}_1 + \mathbf{p}_1')) - (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x} \times (\mathbf{p}_1 + \mathbf{p}_2'))\} (2K_0(2x)/x^3 + 3K_1(2x)/x^4) \\
&\quad \times (f^4 - f^2) \quad (11, c) \\
&+ (1/2M) (3 + 2(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})) ((\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \cdot \mathbf{x} \times (\mathbf{p}_1 + \mathbf{p}_1' + \mathbf{p}_2 + \mathbf{p}_2'))/2) \\
&\quad \times (1/x^3 + 4/x^4 + 6/x^5 + 3/x^6) e^{-2x} \quad (11, d) \\
&+ f^4 (1/2\pi \mu^2) \{2(\mathbf{p}_1^2 + \mathbf{p}_1'^2 + \mathbf{p}_2^2 + \mathbf{p}_2'^2) (3K_0(2x)/x + 9K_1(2x)/x^2) \\
&\quad + 12(\mathbf{p}_1 \cdot \mathbf{p}_1' + \mathbf{p}_2 \cdot \mathbf{p}_2') (K_0(2x)/x + K_1(2x)/x^2) \\
&\quad + 3(3(\mathbf{p}_1 + \mathbf{p}_1' \cdot \mathbf{x})^2/x^2 - (\mathbf{p}_1 + \mathbf{p}_1')^2 + 3(\mathbf{p}_2 + \mathbf{p}_2' \cdot \mathbf{x})^2/x^2 - (\mathbf{p}_2 + \mathbf{p}_2')^2) \\
&\quad \cdot (-2K_0(2x)/x - K_1(2x)/x^2 + 2/x^2 \cdot \int_0^{2x} dy K_0(y) \\
&\quad - 1/x^3 \int_0^{2x} dy \int_0^y dz K_0(z)) \}] \\
&\cdot \phi_+^{*(2)}(\mathbf{r}_2) \phi_+^{(1)}(\mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_2, \quad (11, e)
\end{aligned}$$

where

$$\phi_+(\mathbf{r}) = (2\pi)^{-3/2} \int b_+(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{r}} d\mathbf{p} \quad (12)$$

$$\mathbf{x} = \mu(\mathbf{r}_1 - \mathbf{r}_2) \quad (13)$$

and

$$S_{12} = (3(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{x})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x})/x^2 - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})), \quad (14)$$

\mathbf{p} and \mathbf{p}' in (10) and (11) should be understood as the differential operators standing at the right-hand and the left-hand end respectively.

The spin-orbit coupling term (11. d) shows the peculiar \mathbf{p} -dependence which is not Galilei invariant,* and will disappear by the inclusion of the one- and the two-pair terms to this order $\sim (\mu/2M)^6$. This fact suggests that the careless treatment of the suppression of the pair formation may lead to absurdity. The same thing can be said concerning the spin-orbit

* The presence of this term has been pointed out by S. Otsuki and by S. Sato for the $P_5(p\nu)$ coupling theory. S. Sato has shown that this term disappears by the inclusion of the $\rho_1 \pi$ coupling in addition to the $\sigma \cdot \mathbf{p} \phi$ coupling, and that there remains the usual $\mathbf{L} \cdot \mathbf{S}$ -coupling with the sign favorable to the shell model. The author thanks Mr. S. Sato for communicating his results in advance of the publication.

terms (11. c).

After tedious calculations, it can be shown that the potentials defined in this manner in terms of the canonical transformation give the same results with that of the covariant perturbation theory for the scattering problems up to g^4 .

§ 3. Qualitative discussion on the nonstatic contributions*

(10) and (11) contain the terms which show the complicated dependence on the momenta of nucleons. However, to our regret we have no method at hand to treat such interactions other than the Born approximation. Hence, we shall discuss only the main features of the velocity independent terms (11.a, b) and the spin-orbit coupling terms (11.c, d).

While the two-pair terms are small, the one-pair terms are large enough to suppress the repulsive potential which arises from the leading one- and two-pair terms obtained by Klein.⁶ (see Fig. 2) We obtain two spin-orbit terms in addition to the peculiar terms above mentioned. The one arising from the two-pair terms cancels the other which arises from the one-pair terms, provided we do not take into account the suppression of the pair formation ($f^2=1$). If the coupling through the pair formation is damped ($f^2 < 1$), there will remain a spin-orbit coupling term with the sign favorable to the shell model together with the term which violates the Galilei invariance. But we have no simple prescription to treat this damping effect. The magnitude of the resultant spin-orbit term, if it existed, would be at most a few tenths of the central potentials. The no-pair terms do not change the qualitative features of the T. M. O. (Taketani, Machida and Onuma) potentials¹⁾ except for the case of the central force in triplet odd state, where the T. M. O. potential is

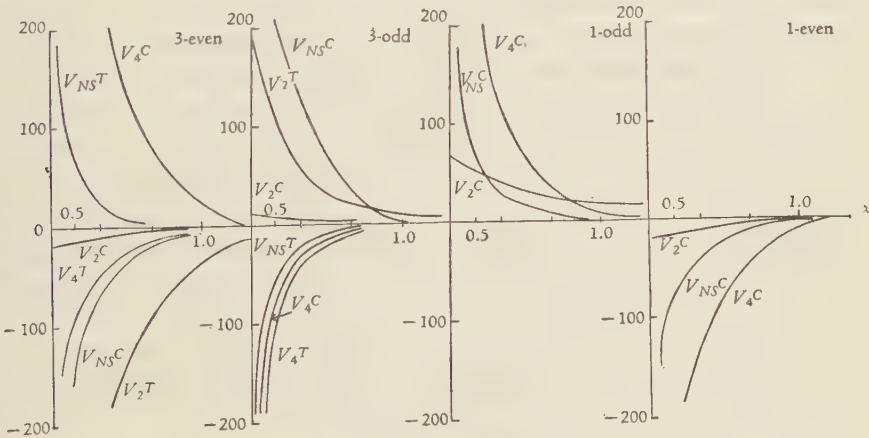


Fig. 1 Velocity independent no-pair potentials:

$$V = \mu/2 \cdot (g^2/4\pi)^2 (\mu/2M)^4 (V^C + V^T S_{12}) \text{ for } f^2=0, (g^2/4\pi) (\mu/2M)^2 = 0.08.$$

V_2 : 2nd order static, V_4 : 4th order static, V_{NS} : nonstatic contributions.

* The main contents of this paragraph have been reported in this journal.⁷⁾

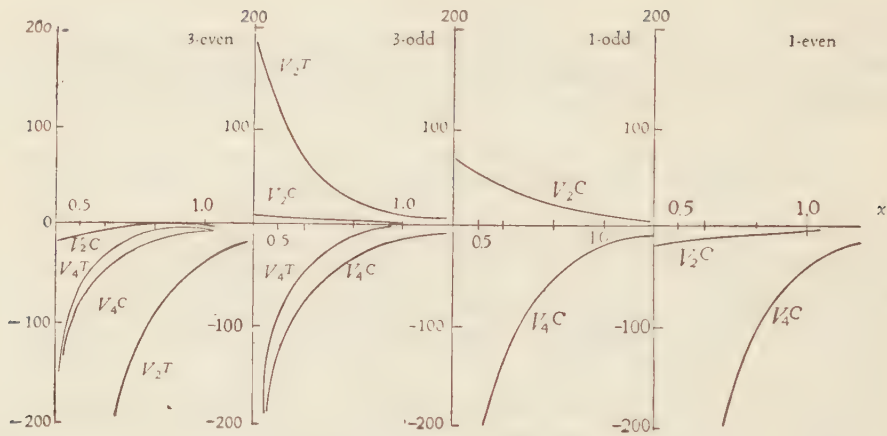


Fig. 2 Velocity independent potentials:
 $V = \mu/2 \cdot (g^2/4\pi)^2 (\mu/2M)^4 (V^C + V^T S_{12})$ for $f^2 = 1$, $(g^2/4\pi) (\mu/2M)^2 = 0.08$.
 V_2 : 2nd order, V_4 : 4th order contributions.

small because of the cancellation between the second order and the fourth order contributions. Further, the sixth order contributions are also large in this state, and more thorough investigations are needed. The potentials in the two extreme cases with $f^2 = 0$ and $f^2 = 1$ are shown in Figs. 1 and 2.

§ 4. Validity of the expansion in powers of $p/2M$

In the evaluation of the nonstatic corrections we use the expansion in powers of $p/2M$. We discuss the validity of this approximation taking up the no-pair contributions as an illustration. The approximation used consists of two steps. At first $(\omega_{p-p'} \pm (E_p - E_{p'}))^{-1}$

is approximated by $\{1 \mp (E_p - E_{p'}) / \omega_{p-p'}\} / \omega_{p-p'}$ and then E_p by $p^2/2M$. The first approximation is not so bad, but the second is formidably crude for large values of p . The final contribution is of the form

$$(E_p - E_{p-k} - E_{p-k'} + E_{p-k-k'}) / 2\omega_{kl}^2 \tag{15}$$

which is approximated by $(\mathbf{k} \cdot \mathbf{k}') / 2M\omega_{kl}^2$. This approximation is valid for $k_1, k_2, p \ll M$ and the expression (15) is vanishingly small if one of k_1, k_2 and p is very large.

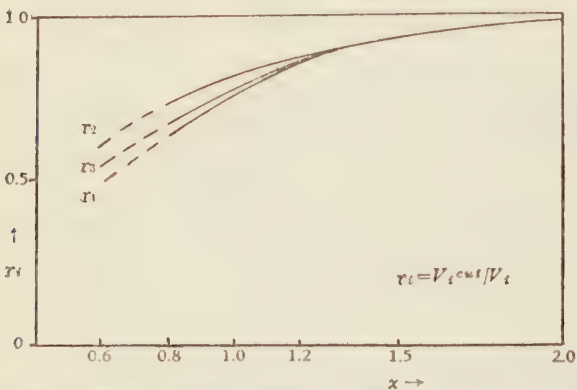


Fig. 3 The ratio of the nonstatic corrections calculated with and without "cutting off" factor:
 $V \sim (3/2 + (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}))$, $V_2 \sim (3/2 + (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})) (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})$,
 $V_3 \sim (3/2 + (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})) S_{12}$.

Hence, we multiply $(kk')/2M\omega_k^2$ by a factor $\alpha^4/(\alpha^2+k^2)(\alpha^2+k'^2)$ to improve the approximation. Then the contribution corresponding to large k_1 and/or k_2 is strongly damped. We put $\alpha=7\mu$ and perform the integration over k_1 and k_2 . Here it must be noted that this "cutting off" is not of physical nature, but merely a mathematical approximation. The ratio of magnitudes of the integrals with and without "cutting off factor" is shown in Fig. 3. The errors in the nonstatic contributions estimated by the use of the "cutting off" procedure are about 20% at $x \sim 1$ and 10% at $x \sim 1.5$. Although the errors in other contributions may not be the same with this value, it may give some indications concerning the magnitude of the errors introduced by the power series expansion.

§ 5. Concluding remarks

From the above consideration the following is concluded. The nonstatic corrections are small and negligible for the large separation $x > 1.0$. They are large for the small separation. But in this region the approximations used are not reliable quantitatively, and other effects such as the higher order corrections and the existence of heavy mesons are supposed to contribute to the nuclear forces. Concerning the nonstatic corrections evaluated, they would not alter the qualitative features of the static potentials except for the case of the no-pair central force in the triplet odd state and of the one-pair contributions. The spin-orbit coupling terms obtained by Klein⁸ are cancelled by the inclusion of the one-pair terms. As seen in § 2, the separation of the no-pair terms from the one- and the two-pair terms is not admissible and leads to absurdity in the evaluation of the velocity dependent potentials. The velocity independent no-pair contributions (11.b) correspond to the first order velocity independent corrections in the $Ps(pv)$ theory, but the velocity dependent terms (11.d) are not so. The effects of nucleon recoil to the potentials have been studied by I. Sato⁹⁾ and by G. Eder¹⁰⁾,* but the differences in the approximation methods adopted and in the types of processes considered do not allow the direct comparison with our results. Nishijima and Sindo¹¹⁾ have evaluated the effects of nucleon recoil for the $\tau_i \sigma \cdot \nabla \phi_i$ coupling, but their results do not seem to be in agreement with ours.

Similar attempts to obtain the nonstatic corrections in the $Ps(pv)$ theory by means of the normalized Tamm-Dancoff method¹²⁾ have been performed by S. Sato^{13,14)} and by K. Itabashi and I. Sato¹³⁾ independently. S. Sato has used the coupling $\tau_i(\sigma \cdot \nabla \phi_i + \nabla_i \tau_i)$ and has obtained the Galilei invariant spin-orbit coupling. K. Itabashi and I. Sato have used the $\tau_i \sigma \cdot \nabla \phi_i$ coupling only and have treated the so-called probability operator separately.

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* Eder's method seems to lead to the velocity dependent potentials which are nonhermitic

Appendix

Evaluation the of potentials in § 2

To evaluate the integral

$$\int f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}, \sigma) g(k_1^2, k_2^2) e^{i(k_1 r_1 + k_2 r_2)} d\mathbf{k}_1 d\mathbf{k}_2, \quad (\text{A} \cdot 1)$$

we interchange the order of integration and differentiation (although it is not permitted for this integral in the strict sense)

$$\begin{aligned} \int f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}, \sigma) g(k_1^2, k_2^2) e^{i(k_1 r_1 + k_2 r_2)} d\mathbf{k}_1 d\mathbf{k}_2 &\rightarrow f(\nabla_1/i, \nabla_2/i, \mathbf{p}, \sigma) \\ &\times \int g(k_1^2, k_2^2) e^{i(k_1 r_1 + k_2 r_2)} d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned} \quad (\text{A} \cdot 2)$$

The results in § 2 are obtained by combining the following formulas.

$$(\mathbf{k}_1 \cdot \mathbf{k}_2) \rightarrow -(\mathbf{r}_1 \cdot \mathbf{r}_2) D_1 D_2 \quad (\text{A} \cdot 3)$$

$$(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \rightarrow (\mathbf{r}_1 \cdot \mathbf{r}_2)^2 D_1^2 D_2^2 + (\mathbf{r}_1)^2 D_1^2 D_2 + (\mathbf{r}_2)^2 D_1 D_2^2 + 3 D_1 D_2 \quad (\text{A} \cdot 4)$$

$$\begin{aligned} (\mathbf{k}_1 \cdot \mathbf{k}_2)^3 \rightarrow &-(\mathbf{r}_1 \cdot \mathbf{r}_2)^3 D_1^3 D_2^3 - 3(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{r}_1)^2 D_1^3 D_2^2 - 3(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{r}_2)^2 D_1^2 D_2^3 \\ &- 15(\mathbf{r}_1 \cdot \mathbf{r}_2) D_1^2 D_2^2 \end{aligned} \quad (\text{A} \cdot 5)$$

$$(i\sigma^{(1)} \cdot \mathbf{k}_1)(i\sigma^{(2)} \cdot \mathbf{k}_1) \rightarrow (\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_1) D_1^2 + (\sigma^{(1)} \cdot \sigma^{(2)}) D_1 \quad (\text{A} \cdot 6)$$

$$\begin{aligned} (i\sigma^{(1)} \cdot \mathbf{k}_1)(i\sigma^{(2)} \cdot \mathbf{p})(\mathbf{k}_1 \cdot \mathbf{p}) \rightarrow &(\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{r}_1) D_1^2 \\ &+ (\sigma^{(1)} \cdot \mathbf{p})(\sigma^{(2)} \cdot \mathbf{p}) D_1 \end{aligned} \quad (\text{A} \cdot 7)$$

$$\begin{aligned} (i\sigma^{(1)} \cdot \mathbf{k}_1)(i\sigma^{(2)} \cdot \mathbf{k}_1)(\mathbf{k}_1 \cdot \mathbf{p})^2 \rightarrow &-(\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_1)(\mathbf{p} \cdot \mathbf{r}_1)^2 D_1^4 \\ &- \{2(\sigma^{(1)} \cdot \mathbf{p})(\sigma^{(2)} \cdot \mathbf{r}_1)(\mathbf{p} \cdot \mathbf{r}_1) + 2(\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{r}_1) + (\mathbf{p} \cdot \mathbf{r}_1)^2(\sigma^{(1)} \cdot \sigma^{(2)}) \\ &+ (\mathbf{p})^2(\sigma^{(2)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_1)\} D_1^3 - \{(\mathbf{p})^2(\sigma^{(1)} \cdot \sigma^{(2)}) + 2(\sigma^{(1)} \cdot \mathbf{p})(\sigma^{(2)} \cdot \mathbf{p})\} D_1^2 \end{aligned} \quad (\text{A} \cdot 8)$$

$$\begin{aligned} (i\sigma^{(1)} \cdot \mathbf{k}_1 \times \mathbf{k}_2)(i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2) \rightarrow &-(\sigma^{(1)} \cdot \mathbf{r}_1 \times \mathbf{r}_2)(\sigma^{(2)} \cdot \mathbf{r}_1 \times \mathbf{r}_2) D_1^2 D_2^2 \\ &+ \{(\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_1) - (\mathbf{r}_1)^2(\sigma^{(1)} \cdot \sigma^{(2)})\} D_1^2 D_2 + \{(\sigma^{(1)} \cdot \mathbf{r}_2)(\sigma^{(2)} \cdot \mathbf{r}_2) \\ &- (\mathbf{r}_2)^2(\sigma^{(1)} \cdot \sigma^{(2)})\} D_1 D_2^2 - 2(\sigma^{(1)} \cdot \sigma^{(2)}) D_1 D_2 \end{aligned} \quad (\text{A} \cdot 9)$$

$$\begin{aligned} (i\sigma^{(1)} \cdot \mathbf{k}_1 \times \mathbf{k}_2)(i\sigma^{(2)} \cdot \mathbf{k}_1 \times \mathbf{k}_2)(\mathbf{k}_1 \cdot \mathbf{k}_2) \rightarrow &(\mathbf{r}_1 \cdot \mathbf{r}_2)(\sigma^{(1)} \cdot \mathbf{r}_1 \times \mathbf{r}_2)(\sigma^{(2)} \cdot \mathbf{r}_1 \times \mathbf{r}_2) D_1^3 D_2^3 \\ &+ \{(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{r}_1)^2(\sigma^{(1)} \cdot \sigma^{(2)}) - (\mathbf{r}_1 \cdot \mathbf{r}_2)(\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_1)\} D_1^3 D_2^2 \\ &+ \{(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{r}_2)^2(\sigma^{(1)} \cdot \sigma^{(2)}) - (\mathbf{r}_1 \cdot \mathbf{r}_2)(\sigma^{(1)} \cdot \mathbf{r}_2)(\sigma^{(2)} \cdot \mathbf{r}_2)\} D_1^2 D_2^3 \\ &+ \{2(\mathbf{r}_1 \cdot \mathbf{r}_2)(\sigma^{(1)} \cdot \sigma^{(2)}) - (\sigma^{(1)} \cdot \mathbf{r}_1)(\sigma^{(2)} \cdot \mathbf{r}_2) - (\sigma^{(1)} \cdot \mathbf{r}_2)(\sigma^{(2)} \cdot \mathbf{r}_1)\} D_1^2 D_2^2 \end{aligned} \quad (\text{A} \cdot 10)$$

$$(i\sigma \cdot \mathbf{k}_1 \times \mathbf{k}_2)(\mathbf{p} \cdot \mathbf{k}_2) \rightarrow -(\mathbf{p} \cdot \mathbf{r}_2)(\sigma \cdot \mathbf{r}_1 \times \mathbf{r}_2) D_1 D_2^2 - (\sigma \cdot \mathbf{r}_1 \times \mathbf{p}) D_1 D_2 \quad (\text{A} \cdot 11)$$

$$\begin{aligned} (i\sigma \cdot \mathbf{k}_1 \times \mathbf{k}_2)(\mathbf{p} \cdot \mathbf{k}_1 + \mathbf{k}_2) \rightarrow &-(\mathbf{p} \cdot \mathbf{r}_1)(\sigma \cdot \mathbf{r}_1 \times \mathbf{r}_2) D_1^2 D_2 - (\mathbf{p} \cdot \mathbf{r}_2)(\sigma \cdot \mathbf{r}_1 \times \mathbf{r}_2) D_1 D_2^2 \\ &+ (\sigma \cdot \mathbf{p} \times \mathbf{r}_1 - \mathbf{p} \times \mathbf{r}_2) D_1 D_2 \end{aligned} \quad (\text{A} \cdot 12)$$

$$\begin{aligned}
& (i\boldsymbol{\sigma} \cdot \mathbf{k}_1 \times \mathbf{k}_2) (\mathbf{k}_1 \cdot \mathbf{k}_2) (\mathbf{p} \cdot \mathbf{k}_2) \rightarrow (\boldsymbol{\sigma} \cdot \mathbf{r}_1 \times \mathbf{r}_2) (\mathbf{r}_1 \cdot \mathbf{r}_2) (\mathbf{p} \cdot \mathbf{r}_2) D_1^2 D_2^3 \\
& + \{ (\boldsymbol{\sigma} \cdot \mathbf{r}_1 \times \mathbf{r}_2) (\mathbf{r}_1 \cdot \mathbf{p}) + (\boldsymbol{\sigma} \cdot \mathbf{r}_1 \times \mathbf{p}) (\mathbf{r}_1 \cdot \mathbf{r}_2) \} D_1^2 D_2^2 + \{ (\boldsymbol{\sigma} \cdot \mathbf{r}_2 \times \mathbf{p}) \\
& + (\boldsymbol{\sigma} \cdot \mathbf{p} \times \mathbf{r}_2) \} D_1 D_2^2, \quad (\text{A} \cdot 13)
\end{aligned}$$

where $D_1 = (1/r_1 \cdot \partial / \partial \mathbf{r}_1)$ and $D_2 = (1/r_2 \cdot \partial / \partial \mathbf{r}_2)$,

$$1/(2\pi)^3 \cdot \int d\mathbf{k}_1 e^{i\mathbf{k}_1 \mathbf{r}} / \omega_1^2 \rightarrow 1/4\pi \cdot e^{-\mu r} / r \quad (\text{A} \cdot 14)$$

$$1/(2\pi)^3 \cdot \int d\mathbf{k}_1 e^{i\mathbf{k}_1 \mathbf{r}} / \omega_1^4 \rightarrow 1/8\pi \mu \cdot e^{-\mu r} \quad (\text{A} \cdot 15)$$

$$\begin{aligned}
1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / (\omega_1 + \omega_2) & \sim -1/(2\pi)^6 \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} \omega_1 / \omega_2 (\omega_1 \perp \omega_2) \\
& \rightarrow 1/(2\pi)^3 \cdot (\mu^3 / r_1 r_2) K_2(\mu(r_1 + r_2)). \quad (\text{A} \cdot 16)
\end{aligned}$$

$$1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / \omega_1 \omega_2 (\omega_1 + \omega_2) \rightarrow 1/(2\pi)^3 \cdot (\mu / r_1 r_2) K_1(\mu(r_1 + r_2)) \quad (\text{A} \cdot 17)$$

$$\begin{aligned}
1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / \omega_1^2 (\omega_1 + \omega_2) \\
\rightarrow 1/(2\pi)^3 \cdot \{ (\mu / r_1 r_2) e^{-\mu r_1} K_1(\mu r_2) - (\mu / r_1 r_2) K_1(\mu(r_1 + r_2)) \} \quad (\text{A} \cdot 18)
\end{aligned}$$

$$\begin{aligned}
1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / \omega_1^2 \omega_2^2 (\omega_1 + \omega_2) \\
\rightarrow 1/(2\pi)^3 \cdot \mu^2 \{ r_1 e^{-\mu r_2} K_0(\mu r_1) + r_2 e^{-\mu r_1} K_0(\mu r_2) - (r_1 + r_2) K_0(\mu(r_1 + r_2)) \} \quad (\text{A} \cdot 19)
\end{aligned}$$

$$\begin{aligned}
1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / \omega_1^3 \omega_2 (\omega_1 + \omega_2) \\
\rightarrow 1/(2\pi)^3 \cdot \mu^2 \{ -r_2 e^{-\mu r_1} K_0(\mu r_2) + (r_1 + r_2) K_0(\mu(r_1 + r_2)) \} \quad (\text{A} \cdot 20)
\end{aligned}$$

$$\begin{aligned}
1/(2\pi)^6 \cdot \int d\mathbf{k}_1 d\mathbf{k}_2 e^{i\mathbf{k}_1 \mathbf{r}_1 + i\mathbf{k}_2 \mathbf{r}_2} / \omega_1 \omega_2 (\omega_1 + \omega_2)^3 \\
\rightarrow 1/(2\pi)^3 \cdot \{ 4\mu(r_1^2 + r_2^2) / r_1 r_2 \cdot \int_{\mu(r_1 + r_2)}^{\infty} dx K_1(x) / x \\
+ 4\mu|r_1^2 - r_2^2| / r_1 r_2 \cdot \int_{\mu|r_1 - r_2|}^{\infty} dx K_1(x) / x - 4(r_1 + r_2) / r_1 r_2 \cdot \int_0^{\mu|r_1 + r_2|} dx \int_x^{\infty} dy K_0(y) \}. \quad (\text{A} \cdot 21)
\end{aligned}$$

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Explicit Formulae of Beta-Gamma Directional Correlation

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The β - γ directional correlation function is expressed as a series in the Legendre polynomials:

$$\mathcal{C}(\theta) = \sum_{L_1 \leq L_2} \sum_n c_{L_1 L_2}^{(2n)} b_{L_1 L_2}^{(1n)} P_{2n}(\cos \theta),$$

where $c_{L_1 L_2}^{(2n)}$'s are the numerical coefficients which depend upon the spin assignments of the nuclei and the multiplicities of γ -ray. Their numerical values are tabulated for the most interesting transition schemes. The parameters for the β -ray, $b_{L_1 L_2}^{(1n)}$, are also given in the convenient formulae to analyse the experimental data.

§ 1. Introduction

Many convenient formulae and tables are prepared nowadays for the analysis of the γ - γ angular correlations,^{1,2,3)*} for experimental physicists to analyse their data easily with the help of these formulae and tables. There are, however, few literatures suitable for this purpose in the case of β - γ directional correlation. The theory of β - γ directional correlation was first constructed by Falkoff and Uhlenbeck⁽¹⁾ which suffer, however, from the following three restrictions: 1) Nuclear charge Z is zero, 2) pure interactions assumed, 3) only one reduced matrix element responsible for β -decay.

We know that the formulae of β -decay in the approximation $Z=0$ are usually not available for the analysis except in special cases. Moreover, from our experience, we are certain that it is necessary to take the mixed interactions and consequently to consider the interference terms between reduced matrix elements. In view of this unsatisfactory situation, the theories of β - γ directional correlation are developed both by Yamada, Kato and the present author,^{5,6,7)} and by Fuchs,⁽⁸⁾ Biedenharn and Rose⁽⁹⁾ with none of the above three restrictions.

Until now, there are seven elements $K^{42,9)}$ $As^{76,10)}$ $Rb^{86,11)}$ $Sb^{122,12)}$ $Sb^{124,11,13)}$ $I^{126,9)}$ $Tm^{170,14,15)}$ for which an anisotropy of β - γ angular distribution has been observed experimentally. They were not yet fully analysed theoretically, except $Sb^{124,16)}$ and $Tm^{170,7,17)}$ by the theoretical research group of β -decay at the University of Tokyo. This is due, as Frauenfelder⁽¹⁸⁾ writes in Siegbahn's book, to the complexity of the formulae for β -decay. The aim of the present paper is to construct the formulae more suitable for the analysis of the experimental data on the β - γ directional correlation.

In Section 2, we reformulate the theory which was developed in reference 5. In Section

* Both the directional and the polarization correlations.

3 we give the formulae and the associated tables, and in Section 4 we mention some remarks on their applications. The formulae for VTP, the selection rules for reduced matrix elements and the correction factors for β -ray spectra are also given in Appendices. I, II and III, respectively.

§ 2. From Clebsch-Gordan coefficients to Racah coefficients

In this section we try to rewrite the formalism of Yamada and Morita^{5)*} in terms of the Racah coefficients,¹⁹⁾ because in their formulae the summation over the magnetic quantum numbers in the Clebsch-Gordan coefficients²¹⁾ are somewhat tedious.** We use the same notations as theirs.

A nucleus transits from an initial state with J, m to an intermediate state with J', m' , emitting the first particle under the influence of the interaction $a[T(\mathbf{X}_i)] \mathcal{Y}_{L_1 M_1}(\mathbf{X}_i) \mathcal{Y}_{L_1 M_1}^*(\mathbf{A}_i)$. After the lapse of very short time, this nucleus transits again from the state with J', m' to a final state with J'', m'' , emitting the second particle under the influence of the interaction $a'[T(\mathbf{X}_k)] \mathcal{Y}_{L_1' M_1'}(\mathbf{X}_k) \mathcal{Y}_{L_1' M_1'}^*(\mathbf{A}_i)$. The relative probability amplitude in this successive transition, $J, m \rightarrow J', m' \rightarrow J'', m''$, is

$$a[T(\mathbf{X}_i)] \mathcal{M}(\mathbf{X}_i) (J L_1 m M_1 | J L_1 J' m') \mathcal{Y}_{L_1 M_1}^*(\mathbf{A}_i) \cdot a'[T(\mathbf{X}_k)] \mathcal{M}(\mathbf{X}_k) (J' L_1' m' M_1' | J' L_1' J'' m'') \mathcal{Y}_{L_1' M_1'}^*(\mathbf{A}_k). \quad (1)$$

There is also an alternative way of transition in which these two particles are emitted: $J, m \rightarrow J', m''' \rightarrow J'', m''$, under the influence of the interactions $a[T(\mathbf{X}_j)] \mathcal{Y}_{L_2 M_2}(\mathbf{X}_j) \mathcal{Y}_{L_2 M_2}^*(\mathbf{A}_j)$ and $a'[T(\mathbf{X}_i)] \mathcal{Y}_{L_2' M_2'}(\mathbf{X}_i) \mathcal{Y}_{L_2' M_2'}^*(\mathbf{A}_i)$. The relative probability amplitude is however the same as in the case of eq. (1) with suitable change of the suffices. Consequently, the directional correlation function $\hat{C}(\theta)$, the relative probability that two particles are emitted in the directions making an angle θ , is

$$\begin{aligned} \hat{C}(\theta) = & S_1 S_2 \sum_{L_1 L_2 L_1' L_2'} \sum_{m m' m'' m'''} a^*[T(\mathbf{X}_i)] a[T(\mathbf{X}_j)] a'^*[T(\mathbf{X}_i)] a'[T(\mathbf{X}_j)] \\ & \cdot \mathcal{M}^*(\mathbf{X}_i) \mathcal{M}(\mathbf{X}_j) \mathcal{M}^*(\mathbf{X}_k) \mathcal{M}(\mathbf{X}_i) \mathcal{Y}_{L_1 M_1}(\mathbf{A}_i) \mathcal{Y}_{L_2 M_2}^*(\mathbf{A}_j) \\ & \cdot \mathcal{Y}_{L_1' M_1'}(\mathbf{A}_k) \mathcal{Y}_{L_2' M_2'}^*(\mathbf{A}_i) (J L_1 m M_1 | J L_1 J' m') (J L_2 m M_2 | J L_2 J' m''') \\ & \cdot (J' L_1' m' M_1' | J' L_1' J'' m'') (J' L_2' m''' M_2' | J' L_2' J'' m''), \end{aligned} \quad (2)$$

which corresponds to eq. (1) of YM.

If we choose the direction of motion of the first particle as the fixed axis, the interferences between m'' and m''' ($\vdash m''$) vanish.²¹⁾ Then, $M_1 = M_2$. We shall hence omit the suffices of M and M' hereafter.

We define the angular distribution function as follows:

$$\begin{aligned} F_{L_1 L_2}''(\theta) = & S \sum_{T L_1} \sum_{T L_2} \varepsilon \{ a^*[T(\mathbf{X}_i)] a[T(\mathbf{X}_j)] \mathcal{M}^*(\mathbf{X}_i) \mathcal{M}(\mathbf{X}_j) \\ & \cdot \mathcal{Y}_{L_1 M}(\mathbf{A}_i) \mathcal{Y}_{L_2 M}^*(\mathbf{A}_j) + \text{c.c.} \}, \end{aligned} \quad (3)$$

* We refer it as YM hereafter.

** Though, for each given state of total angular momentum, the summation is not actually so troublesome as it is apparent in the general expression.

with $\varepsilon=1/2$ for square terms,
and $\varepsilon=1$ for cross terms.

Here we can take $L_1 \leq L_2$ without losing generality. This equation becomes by the reduction of two solid harmonics and parity condition²⁾,

$$F_{L_1 L_2}^M(\theta) = \sum_n (-)^M (L_2 L_1 - MM | L_2 L_1 2n 0) b_{L_1 L_2}^{(2n)} P_{2n}(\cos \theta), \quad (4)$$

with $L_2 - L_1 \leq 2n \leq L_1 + L_2$, $n = \text{integer}$.

Here $b_{L_1 L_2}^{(2n)}$ is parameter which depends upon the properties of the emitted particle.

Substituting eq. (4) in eq. (2), $\mathcal{W}(\theta)$ becomes,

$$\begin{aligned} \mathcal{W}(\theta) = & \sum_{L_1 \leq L_2} \sum_{L_1' \leq L_2'} \sum_{n n'} \sum_{m m' m'' M M'} (-)^{M+M'} (J L_1 m M | J L_1 J' m') (J L_2 m M | J L_2 J' m') \\ & \cdot (L_2 L_1 - MM | L_2 L_1 2n 0) (J' L_1' m' M' | J' L_1 J'' m'') (J' L_2' m' M' | J' L_2 J'' m'') \\ & \cdot (L_2' L_1' - M' M' | L_2' L_1' 2n' 0) b_{L_1 L_2}^{(2n)} b_{L_1' L_2'}^{(2n')} P_{2n'}(\cos \theta), \end{aligned} \quad (5)$$

with $L_2 - L_1 \leq 2n \leq L_1 + L_2$ and $L_2' - L_1' \leq 2n' \leq L_1' + L_2'$.

After summing over the magnetic quantum numbers and omitting the irrelevant common factor we obtain the following expression of $\mathcal{W}(\theta)$:

$$\begin{aligned} \mathcal{W}(\theta) = & \sum_{L_1 \leq L_2} \sum_{L_1' \leq L_2'} \sum_n (-)^{L_1' + L_2'} W(L_1 J' L_2 J' J; 2n) W(L_1' J' L_2' J'; J'' 2n) \\ & \cdot b_{L_1 L_2}^{(2n)} b_{L_1' L_2'}^{(2n')} P_{2n}(\cos \theta), \end{aligned} \quad (6)$$

which corresponds to eq. (29) or (31) of YM.

n' is equal to n from the orthogonality of the Clebsch-Gordan coefficients. The subsidiary conditions for n are just the triad restrictions for the arguments of Racah coefficients. Moreover, well known condition,²²⁾ $2n_{\text{Max}} \leq \text{Min}(L_1 + L_2, L_1' + L_2', 2J')$, is obtained from these triad restrictions. Especially, the angular correlation is isotropic when the intermediate state has spin $J'=0$ or $1/2$. Strictly speaking, n must be distinguished by the groups of L_1, L_2, L_1', L_2' , so that it should be written as $n(L_1, L_2, L_1', L_2')$. We first sum over n for a group of (L_1, L_2, L_1', L_2') , and then for another group of (L_1, L_2, L_1', L_2') , and so on for every possible group.

From eq. (24) of YM and eq. (4), $b_{L_1 L_2}^{(2n)}$'s of the γ -ray are :

$$\begin{aligned} \text{Dipole : } & b_{11}^{(0)} = -2\sqrt{3} |\alpha|^2, \quad b_{11}^{(2)} = -\sqrt{6} |\alpha|^2. \\ \text{Quadrupole : } & b_{22}^{(0)} = 2\sqrt{5} |\beta|^2, \quad b_{22}^{(2)} = -5\sqrt{2/7} |\beta|^2, \quad b_{22}^{(4)} = -4\sqrt{10/7} |\beta|^2. \end{aligned} \quad (7)$$

Interference between dipole and quadrupole : $b_{12}^{(2)} = 2\sqrt{30} \text{Re}(\alpha\beta^*)$.

These parameters, $b_{L_1 L_2}^{(2n)}$, do not depend on "electric or magnetic" radiation. α and β are the reduced matrix elements of the dipole and the quadrupole radiations, respectively. $b_{L_1 L_2}^{(2n)}$'s of β -ray are given in the subsequent section. The relations between $b_{L_1 L_2}^{(2n)}$'s and $a_{L_1 L_2}^{(2n)}$'s which were introduced by eqs. (19)–(23) of YM are obvious, and will not be repeated here.

§ 3. Explicit formulae and associated tables of β - γ directional correlation

In this section we derive the explicit formulae and their associated tables to analyse

the experimental data of the β - γ angular correlation.

To simplify eq. (6), we make a restriction based on the experimental evidences,⁹⁻¹⁵⁾ i.e., the second emitted particle, γ -ray, has a unique multipolarity, $L_1' = L_2' = L'$. The β - γ angular correlation function is defined by

$$\mathcal{C}(\theta) = \sum_{L_1 \leq L_2} \sum_n c_{L_1 L_2}^{(2n)} b_{L_1 L_2}^{(2n)} P_{2n}(\cos \theta), \quad \text{Formula I}$$

where the summation over n is performed for every possible pair of L_1 and L_2 .

$$c_{L_1 L_2}^{(2n)} = W(L_1 J' L_2 J'; J 2n) W(L' J' L' J'; J'' 2n) b_{L' L'}^{(2n)}$$

is numerical factor given in Tables and depends upon the spin assignments of the nuclei and the multiplicities of γ -ray. $b_{L' L'}^{(2n)}$ is parameter for γ -ray given by eq. (7). $b_{L_1 L_2}^{(2n)}$ is parameter for β -ray given by the following formulae.

In the experiments⁹⁻¹⁵⁾ up to the present, the nuclei in which an anisotropy of β - γ directional correlation is observed have the final state with spin $J'' = 0$. Hence, we tabulate the values of $c_{L_1 L_2}^{(2n)}$ only for these transition schemes. In addition, if the final nuclei are even-even ones, their first excited states have spin $J' = 2^{(2n)}$ in most cases. Accordingly, the $c_{L_1 L_2}^{(2n)}$'s which correspond with the dipole radiation in Tables I and II are perhaps unnecessary.

If the β -decay is an allowed transition, the theoretical $\mathcal{C}(\theta)$ is isotropic. Hence, the parameters for β -ray are given for the first and the second forbidden transitions. Moreover, the interaction of β -decay is perhaps ST or STP based on the experimental evidences of e - ν angular correlations on He^{6, 24)} P^{32, 25)} and Ne^{19, 26)}. We express here $b_{L_1 L_2}^{(2n)}$'s only for STP.

Parameters for β -ray in the case of STP

$F_{L_1 L_2}^M(\theta)$'s^{5, 6, 7)} of β -ray were given in terms of $L_i, M_i, N_i^{(27)}$; $L_i^-, M_i^-, N_i^{-(28)}$; $L_{-i}, M_{-i}, N_{-i}^{(29)}$ and $L_{-i}^-, M_{-i}^-, N_{-i}^{-(30)}$. From those formulae of $F_{L_1 L_2}^M(\theta)$'s and eq. (4) we calculate $b_{L_1 L_2}^{(2n)}$'s in a small $(\alpha Z)^2$ approximation²⁷⁾ which is usually used in the theory of β -decay.

1) First forbidden transition

The following parameters $b_{L_1 L_2}^{(2n)}$ must be multiplied by the factor $|\mathfrak{M}(\beta\sigma \times \mathbf{r})|^2$, which will not be written explicitly. The ratios between the reduced matrix elements are introduced as follows:

$$\begin{aligned} G_P \mathfrak{M}(\beta i_3) / G_P \mathfrak{M}(\beta\sigma \times \mathbf{r}) &= (\alpha Z / 2\rho) v, \quad \mathfrak{M}(\beta\sigma \cdot \mathbf{r}) / \mathfrak{M}(\beta\sigma \times \mathbf{r}) = i w, \\ G_S \mathfrak{M}(\beta \mathbf{r}) / G_T \mathfrak{M}(\beta\sigma \times \mathbf{r}) &= -ix, \quad \mathfrak{M}(\beta \mathbf{a}) / \mathfrak{M}(\beta\sigma \times \mathbf{r}) = (\alpha Z / 2\rho) \gamma, \\ \mathfrak{M}(B_{i_3}^1) / \mathfrak{M}(\beta\sigma \times \mathbf{r}) &= iz, \end{aligned}$$

where v, w, x, y and z are all real numbers⁵⁾.

Formulae II

$$\begin{aligned} b_{00}^{(0)} &= (\alpha Z / 2\rho)^2 (v + w)^2 + (\alpha Z / 2\rho) \{ -K + (p^2 / W) \} (2w / 3) (v + w) \\ &\quad + \{ (1 / 9) (K^2 + p^2) - (2Kp^2 / 9W) \} w^2, \\ b_{11}^{(0)} &= -\sqrt{3} [(\alpha Z / 2\rho)^2 (x - y + 1)^2 \\ &\quad + (\alpha Z / 2\rho) \{ (2K / 3) (x - 1) (y - x - 1) + (2p^2 / 3W) (x + 1) (x + 1 - y) \} \\ &\quad + \{ (1 / 6) (K^2 + p^2) (2x^2 + 1) + (2Kp^2 / 9W) (1 - x^2) \}], \end{aligned}$$

$$\begin{aligned}
b_{11}^{(2)} &= \sqrt{3/2} [(\alpha Z/2\rho) (2p^2/3W) (2x-1) (x+1-\gamma) \\
&\quad + \{ (2Kp^2/9W) (1-2x) (x-1) + (p^2/6) (4x^2-1) \}], \\
b_{02}^{(2)} &= (1/\sqrt{6}) [(\alpha Z/2\rho) (2p^2/W) (v+w) + \{ (2p^2/3) - (2Kp^2/3W) \} w] z, \\
b_{12}^{(2)} &= [(\alpha Z/2\rho) (p^2/W) (x-\gamma+1) + \{ (Kp^2/3W) (1-x) + (p^2/2) \}] z, \\
b_{22}^{(0)} &= (\sqrt{5}/12) (K^2+p^2) z^2, \quad b_{22}^{(2)} = -\sqrt{7/2} (p^2/12) z^2, \quad b_{22}^{(4)} = 0.
\end{aligned}$$

In the first forbidden transition, there are nine transition schemes, if the γ -ray is the dipole or the quadrupole radiation and the final state of the nucleus has spin $J''=0$. $c_{J_1 J_2}^{(2n)}$'s are given in Table I.

Table I

$a\sqrt{b}/c\sqrt{d}$ must be read as $a\sqrt{b}/(c\sqrt{d})$. $c_{J_1 J_2}^{(2n)}$'s are normalized arbitrarily. Vanishing values of $c_{J_1 J_2}^{(2n)}$'s are not written.

β : First	$c_{00}^{(0)}$	$c_{11}^{(0)}$	$c_{11}^{(2)}$	$c_{02}^{(2)}$	$c_{12}^{(2)}$	$c_{22}^{(0)}$	$c_{22}^{(2)}$
γ : Dipole							
3-1-0						1	$1/\sqrt{70}$
2-1-0		1	$1/10\sqrt{2}$		$3/10\sqrt{2}$	$-\sqrt{3/5}$	$\sqrt{21/10}\sqrt{2}$
1-1-0	1	$-1/\sqrt{3}$	$1/2\sqrt{6}$	$1/\sqrt{10}$	$\sqrt{3}/2\sqrt{10}$	$1/\sqrt{5}$	$\sqrt{7}/10\sqrt{2}$
0-1-0		1	$1/\sqrt{2}$				
γ : Quadrupole							
4-2-0						1	$-\sqrt{10/7}\sqrt{7}$
3-2-0		1	$-1/7\sqrt{2}$		$-\sqrt{3/7}$	$-\sqrt{3/5}$	$-2\sqrt{6/7}\sqrt{7}$
2-2-0	1	$-1/\sqrt{3}$	$-1/2\sqrt{6}$	$-1/\sqrt{14}$	$-1/2\sqrt{14}$	$1/\sqrt{5}$	$3/14\sqrt{14}$
1-2-0		1	$-1/2\sqrt{2}$		$1/2\sqrt{2}$	$-\sqrt{3/5}$	$\sqrt{3}/2\sqrt{14}$
0-2-0						1	$-\sqrt{5/14}$

2) Second forbidden transition

The following parameters $b_{J_1 J_2}^{(2n)}$ must be multiplied by the factor $|\Re(T_{ij}^3)|^2$, which will not be written explicitly. The ratios between the reduced matrix elements are introduced as follows:

$$G_S \Re(R_{ij}^3)/G_T \Re(T_{ij}^3) = -i(x/2), \quad \Re(A_{ij}^3)/\Re(T_{ij}^3) = (\alpha Z/2\rho)(\gamma/2),$$

$$\Re(S_{ijk}^3)/\Re(T_{ij}^3) = iz,$$

where x , y and z are real.

Formulae III

$$\begin{aligned}
b_{22}^{(0)} &= \sqrt{5} [(\alpha Z/2\rho)^2 \{ (K^2/48) (2x-\gamma+2)^2 + (p^2/48) (x-\gamma+1)^2 \} \\
&\quad + (\alpha Z/2\rho) \{ (K^3/60) (x-1) (y-2x-2) + (K^2 p^2/36W) (x+1) (2x+2-\gamma) \\
&\quad + (K p^2/36) (x-1) (y-x-1) + (p^4/60W) (x+1) (x-\gamma+1) \} \\
&\quad + \{ (1/360) (K^4+p^4) + (K^2 p^2/108) \} (3x^2+2) \\
&\quad + (1/90W) (K^3 p^2 + K p^4) (1-x^2) \}],
\end{aligned}$$

$$\begin{aligned}
b_{22}^{(2)} &= -\sqrt{7/2}[(\alpha Z/2\rho)^2(p^2/48)(x-y+1)^2 + (\alpha Z/2\rho)\{(K^2p^2/36W)(2x+2-y)x \\
&\quad + (Kp^2/36)(1-x)(1+x-y) + (p^4/84W)(2x^2+3x-2xy-y+1)\} \\
&\quad + \{(K^3p^2/90W)(1-x)x + ((K^2p^2/108) + (p^4/252))(3x^2+1) \\
&\quad + (Kp^4/126W)(-2x^2+x+1)\}], \\
b_{22}^{(4)} &= (\sqrt{35/3}\sqrt{2})[(\alpha Z/2\rho)(p^4/70W)(3x^2+x-3xy+2y-2) \\
&\quad + \{(Kp^4/105W)(-3x^2+5x-2) + (p^4/420)(9x^2-4)\}], \\
b_{23}^{(2)} &= -\sqrt{7/30}[(\alpha Z/2\rho)\{(K^2p^2/18W)(2x-y+2) + (p^4/42W)(x-y+1)\} \\
&\quad + \{(K^3p^2/45W) + (Kp^4/63W)\}(-x+1) + (K^2p^2/27) + (p^4/63)\}z], \\
b_{23}^{(4)} &= \sqrt{7/3}[(\alpha Z/2\rho)(p^4/84W)(x-y+1) + \{(Kp^4/126W)(-x+1) + (p^4/126)\}z], \\
b_{33}^{(0)} &= -(\sqrt{7}/72)\{(1/15)(K^4+p^4) + (2K^2p^2/9)\}z^2, \\
b_{33}^{(2)} &= (\sqrt{7}/6\sqrt{3})\{(K^2p^2/45) + (p^4/105)\}z^2, \\
b_{33}^{(4)} &= (\sqrt{11}p^4/540\sqrt{14})z^2, \quad b_{33}^{(6)} = 0.
\end{aligned}$$

In the second forbidden transition, there are four transition schemes, if the γ -ray is the dipole or the quadrupole radiation and the final state of the nucleus has spin $J''=0$. $c_{L_1L_2}^{(2n)}$'s are given in Table II.

Table II
The remarks are the same as those of Table I.

β : Second	$c_{22}^{(0)}$	$c_{22}^{(2)}$	$c_{22}^{(4)}$	$c_{23}^{(2)}$	$c_{23}^{(4)}$	$c_{33}^{(0)}$	$c_{33}^{(2)}$	$c_{33}^{(4)}$
γ : Dipole								
4-1-0						1	$1/4\sqrt{3}$	
3-1-0	1	$1/\sqrt{70}$		$1/\sqrt{14}$		$-\sqrt{5/7}$	$\sqrt{15/4}\sqrt{7}$	
γ : Quadrupole								
5-2-0						1	$-5/14\sqrt{3}$	$-V_2^2/21V_{11}$
4-2-0	1	$-V_{10}/7V_7^-$	$-V_2^2/63V_7^-$	$-5V_5^2/14V_7^-$	$-5V_2^2/63V_7^-$	$-\sqrt{5/7}$	$-5V_{15}/28V_7^-$	$-2V_{110}/63V_7^-$

Example of application

We shall consider the transition scheme, $4_- \rightarrow 2_+ \rightarrow 0_+$. In this case the β -decay is first forbidden and γ -ray is quadrupole radiation.

From Table I, $c_{22}^{(0)}=1$ and $c_{22}^{(2)}=-\sqrt{10/7}\sqrt{7}$.

From Formulae II, $b_{22}^{(0)}=(\sqrt{5}/12)(K^2+p^2)z^2$ and $b_{22}^{(2)}=-\sqrt{7/2}(p^2/12)z^2$.

We substitute these values as $c_{L_1L_2}^{(2n)}$'s and $b_{L_1L_2}^{(2n)}$'s in Formula I. Then, the β - γ directional correlation function is

$$\mathcal{W}(\theta) = (\sqrt{5}/12)(K^2+p^2)z^2 + (\sqrt{5}/7)(p^2/12)z^2 P_2(\cos\theta).$$

Omitting the irrelevant common factor, $\mathcal{W}(\theta)$ becomes,

$$\mathcal{W}(\theta) = (K^2+p^2) + (p^2/7)P_2(\cos\theta),$$

where p and K are the momentum of the electron and the energy of the neutrino, respec-

tively. This is the final result to be compared with the experimental data.

Similar procedures are applicable for the other cases of the transition schemes.

§ 4. Some remarks for application

As we have shown in the example in the last section, the procedure of the application of Formulae and Tables to the analysis of β - γ directional correlation is obvious. Firstly we assume that the transition scheme, i.e., the spin assignment, is already known. We substitute in $\mathcal{W}(\theta)$, Formula I, the values of $c_{L_1 L_2}^{(2n)}$'s, Table I or II, and $b_{L_1 L_2}^{(2n)}$'s, Formulae II or III. The parameters for β -ray, $b_{L_1 L_2}^{(2n)}$, can be calculated, when Z , W_0 and W are known. To our regret, this theoretical $\mathcal{W}(\theta)$ has ambiguities based on the ratios of the reduced matrix elements, v, \dots, z , which can not be calculated exactly for the lack of our knowledge of the wave functions of the nuclei. However, some of the ratios are estimated⁽²⁰⁾ as nearly equal to 1 or 2, and they can be calculated approximately^(30,31) by the single particle shell model⁽²²⁾ or the Bohr model.⁽³²⁾ Naturally the selection rule (see, Appendix II) of the angular momenta reduces the number of these ratios. Comparing the theoretical $\mathcal{W}(\theta)$ with the experimental one we obtain the values of ratios, v, \dots, z , which give the good β - γ directional correlation. In this case, we must also treat the correction factor of the β -ray spectrum with the same values of u, \dots, z . This procedure reduces also the number of the ratios. These values may give us some informations for the model of the nuclei and for the interaction of β -decay.

To the contrary, we assume that the values of v, \dots, z , are known. In this case, the theoretical $\mathcal{W}(\theta)$ changes variously by the spin assignments of the nuclei. Consequently, we can determine these spins from the comparison with the experimental $\mathcal{W}(\theta)$.

Generally speaking, the $(\alpha Z/2\rho)^2$ terms have an important rôle in $\mathcal{W}(\theta)$ as in the correction factor. $(\alpha Z/2\rho)$ is nearly equal to 10 for the medium nuclei. If we evaluate the theoretical $\mathcal{W}(\theta)$ only with $(\alpha Z/2\rho)^2$ terms, it becomes isotropic or reduces an anisotropy, because there are no $(\alpha Z/2\rho)^2$ terms in $b_{L_1 L_2}^{(2)}$ for the first forbidden transition and in $b_{L_1 L_2}^{(4)}$ for the second forbidden transition. As the almost first forbidden spectra ($\Delta J=0, \pm 1$) have an allowed shape except in a few cases, one usually takes these $(\alpha Z/2\rho)^2$ terms⁽³⁴⁾ for the correction factor. However, the so-called cancellation effect⁽³⁵⁾ may occur in some nuclei. In such cases, we must take all terms of $(\alpha Z/2\rho)^{0,1 \text{ and } 2}$ for both the correction factor and the $\mathcal{W}(\theta)$. In addition, we must consider the finite de Broglie wave length effect,⁽³⁶⁾ the finite nuclear size correction,^(37,38) etc., for $b_{L_1 L_2}^{(2n)}$'s in the case of the drastic cancellation. These corrections make the calculation of $\mathcal{W}(\theta)$ very tedious.

If the γ -ray is the mixed multipole radiations, we take eq. (6) as $\mathcal{W}(\theta)$.

The complete tables of $c_{L_1 L_2}^{(2n)}$'s will be published for the transition schemes with non-zero J'' on the issue of the Bulletin of Kobayasi Institute of Physical Research. (5, No. 2, (1955)). The theoretical analysis of the β - γ directional correlation of K^{42} , As^{76} , Rb^{86} , Sb^{122} , I^{126} is under way by us in connection with the branching ratios of β -decays on the shell structure and Bohr Model of these nuclei.

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Appendix I. $b_{L_1 L_2}^{(2n)}$'s for VTP

We express here the Formulae for VTP, because it seems that some questions remain for this interaction. One of these questions is concerned with the meson theory of ρ^0 -decay.¹⁴⁾

1) First forbidden transition

The ratios between the reduced matrix elements are introduced as follows:

$$G_V \mathfrak{M}(\mathbf{r})/G_T \mathfrak{M}(\beta \boldsymbol{\sigma} \times \mathbf{r}) = ix, \quad G_V \mathfrak{M}(\alpha)/G_T \mathfrak{M}(\beta \boldsymbol{\sigma} \times \mathbf{r}) = -(\alpha Z/2\rho) \gamma'.$$

v, w, γ and z are the same as those of STP. x and γ' are also real numbers.

Formulae IV

$$b_{00}^{(0)} = b_{00}^{(0)} \text{ in STP,}$$

$$b_{11}^{(0)} = -\sqrt{3} [(\alpha Z/2\rho)^2 \{ (x-\gamma')^2 + (1-\gamma)^2 + (2/W)(\gamma-1)(x-\gamma') \} \\ + (\alpha Z/2\rho)(2/3) \{ (K + (p^2/W))(x(x-\gamma') + 1-\gamma) + (K/W)(x(\gamma-2) + \gamma') \} \\ + \{ (1/6)(K^2 + p^2)(2x^2 + 1) + (2Kp^2/9W)(x^2 + 1) \}],$$

$$b_{11}^{(2)} = \sqrt{3/2} [(\alpha Z/2\rho)(2p^2/3W) \{ 2x(x-\gamma') - 1 + \gamma \} \\ + \{ (2Kp^2/9W)(2x^2 - 1) + (p^2/6)(4x^2 - 1) \}],$$

$$b_{02}^{(2)} = b_{02}^{(2)} \text{ in STP,}$$

$$b_{12}^{(2)} = [(\alpha Z/2\rho)(p^2/W)(1-\gamma) + \{ (Kp^2/3W) + (p^2/2) \}] z,$$

$$b_{32}^{(2n)} = b_{32}^{(2n)} \text{ in STP, } n=0, 1, 2.$$

2) Second forbidden transition

The ratios between the reduced matrix elements are introduced as follows:

$$G_V \mathfrak{M}(R_{ij})/G_T \mathfrak{M}(T_{ij}\beta) = ix/2, \quad G_V \mathfrak{M}(A_{ij})/G_T \mathfrak{M}(T_{ij}^B) = -(\alpha Z/2\rho)(\gamma'/2),$$

where x and γ' are real. z is the same as that of ST.

Formulae V

$$b_{33}^{(0)} = \sqrt{5} [(\alpha Z/2\rho)^2 \{ (K^2/48) \{ (2x-\gamma')^2 + (2-\gamma)^2 \} + (K^2/24W)(2x-\gamma')(\gamma-2) \\ + (p^2/48) \{ (x-\gamma')^2 + (1-\gamma)^2 \} + (p^2/24W)(x-\gamma')(\gamma-1) \} \\ + (\alpha Z/2\rho) \{ (K^3/60) + (K^2 p^2/36W) \} \{ x(2x-\gamma') + 2-\gamma \} \\ + (K^3/60W) \{ x(\gamma-4) + \gamma' \} \\ + \{ (Kp^2/36) + (p^4/60W) \} \{ x(x-\gamma') + 1-\gamma \} + (Kp^2/36W) \{ x(\gamma-2) + \gamma' \} \} \\ + \{ (1/360)(K^4 + p^4) + (K^2 p^2/108) \} (3x^2 + 2) \\ + (1/90W)(K^3 p^2 + Kp^4)(x^2 + 1) \},$$

$$b_{32}^{(2)} = -\sqrt{7/2} [(\alpha Z/2\rho)^2 \{ (p^2/48) \{ (x-\gamma')^2 + (1-\gamma)^2 \} + (p^2/24W)(1-\gamma)(\gamma'-x) \} \\ + (\alpha Z/2\rho) \{ (K^2 p^2/36W)(2x-\gamma')x + (Kp^2/36) \{ (x-\gamma')x + 1-\gamma \} \}$$

$$\begin{aligned}
 & + (p^4/84W) (2x(x-y') + 1-y) + (Kp^2/36W) (x(y-2) + y') \} \\
 & + \{ (K^3p^2/90W) x^2 + (K^2p^2/108) + (p^4/252) \} (3x^2+1) \\
 & + (Kp^4/126W) (2x^2+1) \}], \\
 b_{22}^{(4)} & = (\sqrt{35/3} \sqrt{2}) [(\alpha Z/2\rho) (p^4/70W) (3x(x-y') + 2(y-1)) \\
 & + \{ (Kp^4/105W) (3x^2-2) + (p^4/420) (9x^2-4) \}], \\
 b_{23}^{(3)} & = -\sqrt{7/30} [(\alpha Z/2\rho) \{ (K^2p^2/18W) (2-y) + (p^4/42W) (1-y) \} \\
 & + (1/9) \{ (K^3p^2/5W) + (K^2p^2/3) + (Kp^4/7W) + (p^4/7) \}] z, \\
 b_{23}^{(4)} & = \sqrt{7/3} [(\alpha Z/2\rho) (p^4/84W) (1-y) + (1/126) \{ (p^4K/W) + p^4 \}] z, \\
 b_{33}^{(2n)} & = b_{33}^{(cn)} \text{ in ST, } n=0, 1, 2, 3.
 \end{aligned}$$

Appendix II. Selection rules for reduced matrix elements

1) First forbidden transition

v, \dots, z have their non-zero values only in the following cases with parity change

Yes. 1 signifies the pure $\mathfrak{M}(\beta\sigma \times \mathbf{r})$ term.

$$v, w: J \longrightarrow J.$$

$$1, x, y, y': J \longrightarrow J \text{ (except } 0 \longrightarrow 0), J \longleftrightarrow J+1.$$

$$z: J \longrightarrow J \text{ (except } 0 \longrightarrow 0, 1/2 \longrightarrow 1/2), J \longleftrightarrow J+1 \text{ (except } 0 \longleftrightarrow 1), \\ J \longleftrightarrow J+2.$$

2) Second forbidden transition

x, y, y' and z have their non-zero values only in the following cases with parity change

No. 1 signifies the pure $\mathfrak{M}(T_{ij}\beta)$ term.

$$1, x, y, y': J \longleftrightarrow J+2.$$

$$z: J \longleftrightarrow J+2 \text{ (except } 0 \longleftrightarrow 2), J \longleftrightarrow J+3.$$

Appendix III. Correction factors of β -ray spectra

1) First forbidden transition

$$C_1 = (b_{00}^{(0)} - \sqrt{1/3} b_{11}^{(0)} + \sqrt{1/5} b_{22}^{(0)}) |\mathfrak{M}(\beta\sigma \times \mathbf{r})|^2. \quad \text{Formula VI}$$

2) Second forbidden transition

$$C_2 = (\sqrt{1/5} b_{22}^{(0)} - \sqrt{1/7} b_{33}^{(0)}) |\mathfrak{M}(T_{ij}^3)|^2. \quad \text{Formula VII}$$

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Theory of Sound Waves and Collective Description

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The linearized equation of the density matrix is applied to generalize the theory of sound waves due to Bloch and Tomonaga. It has been shown that the true scattering matrix is given by the oscillator hamiltonian obtained by this method. The generalized theory of sound waves, which may be considered as a generalized Hartree approximation, is not only capable of yielding the correct plasma frequency and the screening interaction of metallic electrons but facilitates the analysis of the higher-order perturbation. The relationship to Bohm and Pines' collective description is pointed out and a polarization field is introduced in place of the longitudinal electric-field. As an example, we consider the electron-lattice interaction, and it has been shown that the stability of the atomic lattice increases under the influence of the Coulomb interaction. The limit of applicability of this method is examined by the theory of transformation function. In the Appendix, the correlation energies of some alkali metals are obtained.

§ 1. Introduction

Since 1934 it has been stressed by Bloch,¹⁾ Bohr,²⁾ Landau³⁾ and others that the hydrodynamical description is effective in dealing with such systems in which the interaction is so strong and the density is so large that the particle number is no more regarded as a good quantum number. These systems such as metallic electrons, nuclear matter and liquid helium are sometimes called quantum fluids. As has already been shown in our preceding papers,^{5),7)} we can obtain the hydrodynamical equations from the total hamiltonian of the system in question. Since these equations are essentially non-linear, we meet with great mathematical difficulties in solving them. The most tractable solution will be given by the sound approximation. Recently it has been shown by Tomonaga⁴⁾ that the hamiltonian of sound waves, once presented by Bloch,¹⁾ proves to be equal to the true hamiltonian of fermions so far as the one-dimensional problem is concerned. In his proof, it is assumed that excited particles and holes are restricted in the neighborhood of the Fermi surface. Though this assumption prevails also in the three-dimensional case, the hydrodynamical picture collapses, because the fermion hamiltonian cannot be represented by that of sound waves, therefore the linearization of the hydrodynamical equation does not serve as a good approximation any more. From our point of view, it is not the density operator but the density matrix that is to be linearized. In Section 2 we shall summarize the results obtained in foregoing papers referred to as I⁵⁾ and it will be shown that the plasma frequency and its normal coordinate result from the linearized equations of density matrices.

By making use of the linearized solution obtained in Section 2, we shall propose in

Section 3 a method of canonical transformation in order to separate out the collective motion from the total system and to know what is the other motion of the system than the collective one. The results obtained there will be compared with those of Pines and Bohm.⁽⁵⁾

In a preceding paper⁽⁷⁾ we considered the coordinate of the collective oscillation as representing an external variable describing a polarization of the medium. This polarization field was described by the creation operator b^* and the destruction operator b , and the interaction energy of electrons with these oscillators was eliminated by the canonical transformation which will be again applied in Section 3. There, in relation to the subsidiary condition, we shall point out the difficulty involved in Bohm-Pines' theory.

As mentioned above, the energy of oscillators proves to be equal to the correct fermion energy so far as the amount of the excitation energy is smaller than a critical value for the one-dimensional case. Is it true for three dimensional problems? Recently this question has been answered by Schafroth⁽⁸⁾ for the free fermion assembly and his result leads to ours obtained in I. In our three dimensional theory there appears a set of oscillators whose momenta are parallel to a single direction, say z -axis. It will be shown that the true transformation function of the S -matrix can be obtained from the hamiltonian of oscillators given in Section 2. This fact enables us to find the higher-order correlation energy, and we can consider that this method is a sort of the Hartree approximation to take account of the higher-order correlation effects.

As an example, we consider the electron-lattice interaction. By performing the summation of the infinite series of the perturbation, it will be shown that the singularity involved in the expression of the second-order perturbation energy of Fröhlich does not appear and the Coulomb interaction increases the stability of the atomic lattice so that the upper limit of the coupling constant F appearing in Fröhlich's theory is shifted to a larger value compared with that in the absence of the Coulomb interaction.

In Section 5, we examine the validity of the above-mentioned linearization approximation in which excited particles and holes are assumed to be restricted in the neighborhood of the Fermi surface and the oscillators of different directions are treated as independent of one another. It will be shown that such approximation is justified for the cases, in which the excitation energy is not very large and the interaction is a long-range one, and further the wave functions in the momentum space are nearly uniform so that the wave functions are almost unchanged against small translations in the momentum space. Though these conditions seem to restrict the applicability of the theory so much, we think that the linearized solution of the density matrix is more useful as the first approximation than starting with the free particle hamiltonian in dealing with the many fermion problems.

In the appendix the correlation energy of metallic electrons will be computed and compared with those given by Wigner and more recently by Pines.⁽⁹⁾

§ 2. Linearization

Our object is the hamiltonian of the form

$$H = K + (1/2) \sum_k G(k) \rho(k) \rho(-k) - NG_0/2, \quad (2.1)$$

$$K = (\hbar^2/2m) \sum_k k^2 \psi_k^* \psi_k, \quad \rho(k) = \sum_l \psi_{l-(k/2)}^* \psi_{l+(k/2)}, \quad N = \rho(0), \quad G_0 = \sum_k G(k),$$

where ψ_k and ψ_k^* are the fourier coefficients of the quantized wave functions and $G(k)$ is that of the interaction potential.

With aids of the commutation relations

$$[\psi_k, \psi_l^*]_+ = \delta_{kl}, \quad [\psi_k, \psi_l]_+ = [\psi_k^*, \psi_l^*]_+ = 0, \quad (2.2)$$

we obtain the equation of the density matrix $\psi_{K-(k/2)}^* \psi_{K+(k/2)}$ which becomes

$$\begin{aligned} \partial g(K, k) / \partial t + (i\hbar/m) (K \cdot k) g(K, k) \\ = (i/2\hbar) \sum_q G(q) [\rho(-q) \{g(K - (q/2), k+q) - g(K + (q/2), k+q)\} \\ + \{g(K - (q/2), k+q) - g(K + (q/2), k+q)\} \rho(-q)], \end{aligned} \quad (2.3)$$

where

$$g(K, k) = \psi_{K-(k/2)}^* \psi_{K+(k/2)}. \quad (2.4)$$

On finding the second order derivative of $g(K, q)$, we shall ignore the derivatives of operators $\rho(q)$ and $g(K, q)$ involved in the bracket of (2.3), then we are led to a linearized equation which assumes the form

$$\begin{aligned} \partial^2 g(K, k) / \partial t^2 + (\hbar/m)^2 (K \cdot k)^2 g(K, k) \\ = (1/m) (K \cdot k) \{g(K + (k/2), 0) - g(K - (k/2), 0)\} G(k) \rho(k), \end{aligned} \quad (2.5)$$

in which the last term has been derived by setting $g(K, q) = g(K, 0) \delta_{q,0}$. Here, in the meaning of the first approximation, we put

$$\begin{aligned} g(K, 0) = 1, \quad \text{if } |K| \leq k_0 \quad (\text{the maximum wave number}), \\ g(K, 0) = 0, \quad \text{if } |K| > k_0. \end{aligned} \quad (2.6)$$

Then the last term becomes zero for small values of K and is appreciable only in the neighborhood of the Fermi surface.

Now we introduce a number of operators $\rho_r(k)$ defined by

$$\rho_r(k) = \sum_K^* g(K, k), \quad \sum_K^* = \sum_K \{K^2 - (K \cdot k)^2 / k^2 = r^2\}, \quad (2.7)$$

in which the summation is carried out over phase points lying on a cylindrical surface with its axis parallel to k and of radius r . From the equation (2.5) we get the following simultaneous equations (see Appendix I):

$$\left. \begin{aligned} \partial^2 \rho_r(k) / \partial t^2 + k^2 c_r^2 \rho_r(k) + (S_r/m) k^2 G(k) \sum_{s=0}^{k_0} \rho_s(k) &= 0 \\ c_r &= \hbar(k_0^2 - r^2)^{1/2} / m, \end{aligned} \right\} \quad (2.8)$$

in which $\sum_r^* (K \cdot k)^2 g(K, k)$ has been replaced by $k^2(k_0^2 - r^2) \rho_r(k)$ and S_r is given by

$$\begin{aligned} S_r &= |k|^{-1} (k_0^2 - r^2)^{1/2} \left[\sum_{(K \cdot k) > 0}^* \{g(K - (k/2), 0) - g(K + (k/2), 0)\} \right. \\ &\quad \left. - \sum_{(K \cdot k) < 0}^* \{g(K - (k/2), 0) - g(K + (k/2), 0)\} \right] \\ &= (L^2/\pi) r (k_0^2 - r^2)^{1/2} = L r N_r, \end{aligned} \quad (2.9)$$

which means the number of phase points on the cylindrical surface. As mentioned in I

the solution of these simultaneous equations is given by a set of normal coordinates

$$\tilde{\zeta}_{\omega,k} = \sum_r \alpha(\omega) \rho_r(k) / (\omega^2 - k^2 c_r^2), \quad (2.10)$$

in which the normalization factor is determined by

$$\alpha(\omega)^2 \sum_r S_r / m (\omega^2 - k^2 c_r^2)^2 = 1, \quad (2.11)$$

or
$$\alpha(\omega_{\max}) \simeq (m/N)^{1/2} \omega_p^2 = k^2 N^{1/2} G(k) / m^{1/2}, \quad (2.12)$$

and the eigenfrequencies are obtained from the dispersion equation

$$1 = \sum_r (S_r k^2 G(k) / m) / (\omega^2 - k^2 c_r^2), \quad (2.13)$$

from which the maximum frequency, the plasma frequency becomes

$$\omega_{\max}(k) = \omega_p \{ 1 + (3/10) k^2 (\hbar k_0 / m \omega_p)^2 + 0 (\hbar |k| k_0 / m \omega_p)^4 \}, \quad (2.14)$$

$$\simeq \omega_p \{ 1 + (\hbar c / \omega_p)^2 / 2 \}, \quad \bar{c}^2 = (3/5) (\hbar k_0 / m)^2. \quad (2.14')$$

Other frequencies correspond to the excitation energies of the individual particles as will be shown in the next section. In the following we shall be concerned merely with the normal coordinate of the maximum frequency and its canonical conjugate variable σ_k which is given by

$$\sigma_k = (\alpha/m) \sum_r S_r \pi_r(k) / (\omega^2 - k^2 c_r^2), \quad (2.15)$$

where $\pi_r(k)$ is defined by

$$\pi_r(k) = (i\pi \hbar / k r L^2) \{ \sum_{(K,l)>0}^* g(K, -k) - \sum_{(K,k)<0}^* g(K, -k) \}, \quad (2.16)$$

$$[\rho_r(k), \pi_s(l)] = i\hbar \delta_{rs} \delta_{kl}, \quad k \parallel l. \quad (2.17)$$

Using (2.11) which means the unitarity condition of the transformation to the set of normal coordinates, we can easily prove that σ_k is canonically conjugate to $\tilde{\zeta}_k$ (see Appendix II).

An approximation, in which excited particles and holes are assumed to be restricted in the neighborhood of the Fermi surface, has been used in the above derivation. This assumption will require some weakness of the interaction energy. This point will be discussed again in Section 5.

§ 3. Collective description

We try to express the total hamiltonian in terms of the collective variables $\tilde{\zeta}_k$ and σ_k defined in the preceding section. For boson assemblies⁷⁾ we considered the interaction of individual particles transmitted by a collective oscillation field which was considered as an external field. There the canonical transformation of the form

$$U = 1 - \sum_K \sum_j |k_j G(k) [N / 2mb\omega]^{1/2} [\omega - (\hbar/m) (K \cdot k)]^{-1} [g(K, -k) b_k^* - g(K, k) b_k] \quad (3.1)$$

was applied to cancel the interaction energy. Recently Bohm and Pines⁶⁾ have presented the so-called collective description introducing the longitudinal electric field in place of the collective oscillation field. Now we can go over to Bohm-Pines' description starting with our classical view point and making use of the results given in Section 2. We believe that

the calculation has thus been much simplified.

Firstly, the following transformation function is introduced.

$$U_1 = \exp(i/\hbar \cdot \sum_k \hat{\xi}_k p_k), \quad [q_k, p_l] = i\hbar \delta_{kl}, \quad (3.2)$$

which is equivalent to (3.1) and p_k is the fourier coefficient of the canonical conjugate of the external variable subject to the subsidiary condition

$$q_k = 0. \quad (3.3)$$

With the aid of (3.2) the total hamiltonian turns out into

$$H_1 = U_1^{-1} H U_1 = K + \sum_k \dot{\xi}_k p_k + (1/2) \sum_k k^2 p_k p_{-k} + (1/2) \sum_k G(k) \rho(k) \rho(-k) - N G_0/2, \quad (3.4)$$

and the subsidiary condition becomes

$$q_k - \hat{\xi}_k = 0. \quad (3.5)$$

In the above the following identities are applied:

$$\dot{\xi}_k = (i/\hbar) [K, \hat{\xi}_k] = k^2 \sigma_{-k}, \quad (3.6)$$

$$(i/\hbar) [\hat{\xi}_k, \hat{\xi}_l] = k^2 \delta_{k-l}, \quad (3.7)$$

and

$$\begin{aligned} \dot{\rho}_r(k) &= (i/\hbar) [K, \rho_r(k)] = S_r k^2 \pi_r(-k)/m, \\ \dot{\pi}_r(k) &= (i/\hbar) [K, \pi_r(k)] = -m c_r^2 \rho_r(-k)/S_r. \end{aligned}$$

where the dotted quantities represent the change with time in the absence of the interaction energy.

Next we go over to the collective representation by the transformation of the form

$$U_2 = \exp(-i/\hbar \cdot \sum_k \sigma_k q_k), \quad (3.8)$$

due to which the hamiltonian splits into the collective part H_c and the individual part $H_{i.d.}$ as follows:

$$H = H_c + H_{i.d.}, \quad (3.9)$$

$$H_c = (1/2) \sum_k (k^2 p_k p_{-k} + (\omega(k)/k)^2 q_k q_{-k}), \quad (3.10)$$

$$H_{i.d.} = K - (1/2) \sum_k k^2 \sigma_k \sigma_{-k} + V_s, \quad (3.11)$$

$$V_s = (1/2) \sum_k G(k) \eta(k) \eta(-k), \quad \eta(k) = \rho(k) - \hat{\xi}_k, \quad (3.12)$$

where V_s represents the screening interaction and the subsidiary condition becomes

$$\hat{\xi}_k = 0. \quad (3.13)$$

We have used in (3.10) the following relation

$$(i/\hbar) [\dot{\sigma}_k, \sigma_{-k}] = (\alpha^2/m) \sum_r S_r c_r^2 / (\omega^2 - k^2 c_r^2)^2 = (\omega^2 - \omega_p^2)/k^2, \quad (3.14)$$

from which the dispersion equation (2.13) is obtained noticing (2.11). The canonical relation appeared in the above facilitates the calculations and the series of commutators appearing in the canonical transformation breaks up to the second order term. This is because of our assumption that the oscillators of different directions are considered as

independent of one another, and it resembles the random phase approximation by Bohm and Pines⁶⁾ (see Section 5).

In the one-dimensional problem we get

$$c_r = 0 \quad (r \neq 0), \quad \omega^2 - \omega_p^2 = k^2 c_0^2, \quad (3.15)$$

$$\xi_k = (m/N)^{1/2} \rho(k), \quad \sigma_k = (N/m)^{1/2} \pi(k), \quad (3.16)$$

$$H_{i.d.} = -(1/2) \sum_k \hbar |k| c_0 + NE_0 \quad (\text{the mean Fermi energy}). \quad (3.17)$$

Here we have applied Tomonaga's theory to write K in terms of ξ_k and σ_k , and $H_{i.d.}$ becomes constant. Therefore the total hamiltonian is given solely in terms of the collective variables as already shown by Tomonaga. But the result becomes different if we follow the method of Bohm and Pines. They obtained the expectation value of $H_{i.d.}$ in the representation that K is diagonal and they ignored the subsidiary condition. However, the operator ξ_k is not diagonal in this representation and $[K, \xi_k] \neq 0$. In fact, some indefinite terms can be involved in their calculation since the diagonal element of $\xi_k \xi_{-k}$ is not zero. In the following it will be shown that the diagonal element of (3.11) does not give the correct eigenvalue even for the one-dimensional case, if we ignore the subsidiary condition (3.13).

The expression of the individual particle energy $H_{i.d.}$ can be interpreted in an intuitive way. Before the above canonical transformation has been applied, the collective energy

$$H_{c.o.} = (1/2) \sum_k (k^2 \sigma_k \sigma_{-k} + (\omega(k)/k)^2 \xi_k \xi_{-k}), \quad (3.19)$$

was given in terms of the wave functions (2.2) in the old representation. The expression of the remainder of the hamiltonian obtained by subtracting (3.19) from the total hamiltonian becomes just equal to (3.11). In the old representation the collective part and individual part are not completely commutative with each other, while in the representation (3.9) they are commutative. In this new representation the subsidiary condition becomes necessary so as to hold the commutativity of the two parts of the hamiltonian. Expanding the denominator of (2.15) in the power series of $(c_r^2 - c^2)k^2/\omega_p^2$ and taking the first order term, we get the diagonal element of the second term of (3.11) in the form

$$-(1/2) \left\langle \sum_k k^2 \sigma_k \sigma_{-k} \right\rangle_{d.i.} = -(b^2/2mN) \sum_{K,k} (K \cdot k/k)^2 N_{K-(k/2)} (1 - N_{K+(k/2)}), \quad (3.20)$$

$$N_K = \psi_K^* \psi_K. \quad (3.21)$$

Since the collective coordinate almost disappears for large wave numbers such as $|k| \sim k_c = (\omega_p/c)$, the major contribution comes from the summation over k within the limit of $|k| = k_c$. In the one-dimensional case this term turns out to be

$$-\delta_0 = -(1/4) \hbar c_0 \sum |k|, \quad (3.22)$$

for the completely degenerate state, therefore the expectation value of (3.11) becomes larger than the value of (3.17) by the factor δ_0 , twice of which is the zero-point energy of the collective oscillation in the absence of the interaction energy. As mentioned in the above, K involves the term $(c_0^2/2) \sum_k \xi_k \xi_{-k}$, the diagonal element of which is just equal to δ_0 , hence we overestimate the expectation value of $H_{i.d.}$ by the amount δ_0 if we ignore the subsidiary condition (3.13). For the three dimensional case the corresponding quantity

to δ_0 is

$$\delta_0 = (L\hbar^2/4\pi Nm) \sum_k \sum_{K, |K|=k_0} |k|^{-1} (K \cdot k)^2 = (3\hbar^2/16m) k_0 \sum |k|, \quad (3.23)$$

twice of which becomes the average of the sum of the zero-point energy of the total set of oscillators whose momenta are proportional to k 's:

$$(K\hbar/4\pi N) \{c_r S_r dr \sum |k| = 2\delta_0. \quad (3.24)$$

We shall see in the next section that the energy of a set of oscillators whose momenta are parallel to a single direction, say z -axis, becomes one component (z -component) of the kinetic energy of electrons. However, it is difficult in the three dimensional case to get any more close relationship between the hamiltonian of the total set of oscillators and the kinetic energy of electrons. Here we give up to compare the hamiltonian of oscillators with that of electrons, and instead try to find the transformation function. The point of our method consists in dealing with the theory of transformation function to evade the difficulties with regard to the subsidiary condition and to estimate the errors caused by this sound approximation.

A criterion for the applicability of this method is given by estimating the amount of the excitation kinetic energy defined by

$$K_{\text{ex.}} = \langle K \rangle_{\text{AV.}} / N - E_F \quad (\text{Fermi energy}), \quad (3.25)$$

as mentioned in Tomonaga's theory.⁴⁾ This criterion reveals that, as will be shown in Appendix III, the problem of the metallic cohesion is within the limit of the applicability.

Another limit of the applicability appears in relation to the non-commutativity of operators of different directions in the three dimensional problem. The detailed account of these limits of the applicability will be given in the following sections.

§ 4. An application of sound approximation

In the above, the normal coordinate of the plasma oscillation and the collective representation have been obtained by dealing with a set of oscillators represented by ρ_r and π_r . In the following it will be shown that the energy of these oscillators is useful in obtaining the energy eigenvalue of metallic electrons.

We introduced in I a set of canonical variables defined by

$$q_{xy}(k) = (m/N_{xy})^{1/2} \{\rho_{xy}^+(k) + \rho_{xy}^-(k)\}, \quad (4.1)$$

$$p_{xy}(k) = (i\hbar\pi/kL) (N_{xy}/m)^{1/2} \{\rho_{xy}^+(-k) - \rho_{xy}^-(-k)\}, \quad (4.2)$$

$$N_{xy} = L(k_0^2 - x^2 - y^2)^{1/2} / \pi,$$

$$\rho_{xy}^+(k) = \sum_{z>0} \psi_{x,y,z-k/2}^* \psi_{x,y,z+k/2}, \quad \rho_{xy}^-(k) = \sum_{z<0} \psi_{x,y,z-k/2}^* \psi_{x,y,z+k/2}, \quad (4.3)$$

$$[q_{xy}(k), p_{x'y'}(k')] = i\hbar \delta_{xx'} \delta_{yy'} \delta_{kk'}. \quad (4.4)$$

The momenta of these oscillators are parallel to the z -axis and the equations of them are derived from the z -component of the kinetic energy;

$$K_z = (\hbar^2/2m) \sum_z \sum_{x,y} \psi_{x,y,z}^* \psi_{x,y,z}, \quad (4.5)$$

because the x - and the y -component of the kinetic energy are commutative with them.

By a similar treatment to Tomonaga's theory, it can be shown that (4.5) proves to be identical with the energy of oscillators

$$K_{xy} = (1/2) \sum_k \sum_{x,y} \{ k^2 p_{xy}(k) p_{xy}(-k) + c_{xy}^2 q_{xy}(k) q_{xy}(-k) - \hbar^2 k |c_{xy}| \} + E_0/3, \quad (4.6)$$

$$c_{xy} = \hbar(k_0^2 - x^2 - y^2)^{1/2}/m, \quad E_0 = (3N/5)\zeta, \quad N = (L^3/6\pi^2)k_0^3, \quad \zeta = \hbar^2 k_0^2/2m, \quad (4.7)$$

where N is the total number of electrons and ζ is the maximum Fermi energy. The operators of different directions are not commutative with one another, hence they are not independent. In our approximation, however, they are assumed to be commutative and there is no correlation between any two of different directions. Henceforth, all the quantum states are restricted to be isotropic in the momentum space. To get the equations of another direction, we must rotate the z -axis to the preferred direction.

After these preliminaries, we consider the transformation functions of the many electrons interacting with the lattice-phonons. In the interaction representation the transformation operator becomes

$$U(t, -\infty) = \sum_{n=0}^{\infty} (1/n!) (-i/\hbar)^n \int \cdots \int_{-\infty}^t T[H_i(t^{(1)}) H_i(t^{(2)}) \cdots H_i(t^{(n)})] dt^{(1)} \cdots dt^{(n)}, \quad (4.8)$$

where T is the chronological operator defined by

$$\begin{aligned} T[A(t), B(t')] &= A(t) B(t') \quad \text{for } t > t', \\ &= B(t') A(t) \quad \text{for } t < t', \end{aligned} \quad (4.9)$$

and

$$H_i(t) = \exp\{ (i/\hbar) K_{os} \} H_i(0) \exp\{ (-i/\hbar) K_{os} \}. \quad (4.10)$$

For example, setting $H_i(t) = \sum_k c_k(t) q_{xy}(k)$, we get

$$\begin{aligned} H_i(t) &= \sum_k c_k(t) \left\{ \sum_{z>0} \psi_{x,y,z-(k/2)}^\dagger \psi_{x,y,z+(k/2)} \exp(-ic_{xy}kt) \right. \\ &\quad \left. + \sum_{z<0} \psi_{x,y,z-(k/2)}^* \psi_{x,y,z+(k/2)} \exp(ic_{xy}kt) \right\}, \end{aligned}$$

from which we can find the true expression by replacing c_{xy} by $\hbar z/m$ for $z > 0$ or $-\hbar z/m$ for $z < 0$. In this way the true transformation function is obtained, if $H_i(t)$ is expressed by a linear combination of $q_{xy}(k)$.

As is well-known the energy change is given by

$$JE = U(0, -\infty)^{-1} H U(0, -\infty) - H + H_i. \quad (4.11)$$

The point of our approximation consists in assuming that the interaction energy is separable with respect to the wave number;

$$H_i = \sum_k H_i(k)/2,$$

and any two quantities, $\int H_i(k)dt$ and $\int H_i(k')dt$ are commutative with each other, if $k \neq k'$. Accordingly, the energy change JE becomes separable with respect to k and its each member JE_k is calculated from

$$H(k) = \sum_{x,y} \{ k^2 p_{xy}(k) p_{xy}(-k) + c_{xy}^2 q_{xy}(k) q_{xy}(-k) - \hbar^2 |k| c_{xy} \} + H_i(k). \quad (4.12)$$

In the following, instead of dealing with the transformation function directly, we try

to find ΔE_k by the perturbation method. Owing to this separation, the calculation is so much simplified that we can compute the higher order perturbation effects. This approximation resembles Macke's method⁽¹⁾ of the many electron problem, in which it is assumed that any state with a single excited particle of a given excitation momentum combines only with those in which a number of particles are excited in the same direction with the prescribed momentum. To illustrate this method, we deal with the electron-phonon interaction which plays an important role in Fröhlich's theory⁽²⁾ of superconductivity.

If a conduction electron of wave number K absorbs or emits a lattice-phonon of wave number k , it gets the wave number $K+k$ or $K-k$, so the interaction energy assumes the form

$$H_i = \sum_k \sum_K (2\omega/\hbar)^{1/2} M(k) g(K, k) q(-k), \quad (4.13)$$

$$M(k)^2 = (2F/3N_0) \hbar \omega_c^2, \quad F \simeq 1, \quad N_0: \text{number of atoms,} \\ \omega: \text{phonon frequency,}$$

where $q(k)$ is the phonon coordinate and we write (4.13) in the form

$$H_i = (1/2) \sum_k H_i(k), \quad (4.14)$$

where

$$H_i(k) = \sum_{x,y} c'_{xy}(k) \{ q_{xy}(k) q(-k) + q_{xy}(-k) q(k) \}, \quad (4.15)$$

$$c'_{xy}(k) = (2\omega N_{xy}/m\hbar)^{1/2} M(k).$$

Now our problem is to find eigenvalues of the hamiltonian of the form

$$H(k) = H_0(k) + H_i(k), \quad (4.16)$$

$$H_0(k) = \sum_{x,y} \{ k^2 p_{xy}(k) p_{xy}(-k) + c_{xy}^2 q_{xy}(k) q_{xy}(-k) - \hbar |k| c_{xy} \} \\ + p(k) p(-k) + \omega(k)^2 q(k) q(-k), \quad (4.17)$$

$$[q(k), p(l)] = i\hbar \delta_{kl}, \quad (4.18)$$

and we get the eigenfrequencies from the dispersion equation

$$\sum_s k^2 c_s'^2(k) / (\Omega^2 - \Omega_s^2) = \Omega^2 - \omega^2, \quad \Omega_s = |k| c_s, \quad s = (x, y), \quad (4.19)$$

and they are given by the perturbation method in the form

$$\Omega_s(k)^2 = k^2 c_s^2 - \sum_{p=0}^{\infty} \sum_{t_1 \dots t_p} k^2 c_s'(k)^2 c_{t_1}'(k)^2 \dots c_{t_p}'(k)^2 / (\omega^2 - \Omega_s^2)^{p+1} (c_{t_1}^2 - c_s^2) \dots (c_{t_p}^2 - c_s^2), \quad (4.20)$$

except for terms depending on the surface of the box confining the system. The total energy change is given by the following sum over s and k , and taking the principal value we get

$$\Delta E = (1/2) \sum_k \sum_s \Delta \Omega_s(k)^2 k^{-2} Q_s(k) Q_s(-k) \\ = (1/2) \sum_k \sum_s c_s'(k)^2 [1 + D_s(k)]^{-1} [\Omega_s^2 - \omega(k)^2]^{-1} Q_s(k) Q_s(-k), \quad (4.21)$$

$$D_s(k) = 2\nu F \omega(k)^2 [\Omega_s^2 - \omega(k)^2]^{-1} [1 - (\Omega_s/2\Omega_0) \ln(\Omega_0 + \Omega_s/\Omega_0 - \Omega_s)], \quad (4.22)$$

$$\nu = N/N_0,$$

in which Q_s 's are the transformed coordinates of oscillators written in terms of the new

wave functions in the form

$$Q_{xy}(k) = (m/N_{xy})^{1/2} \sum_z \mathcal{V}_{x,y,z-k/2}^* \mathcal{V}_{x,y,z+k/2}. \quad (4.23)$$

The diagonal element of (4.21) becomes

$$\langle \Delta E \rangle = \sum_k \sum_K F(K, k) [1 + D_K(k)]^{-1} N_{K-x/2} (1 - N_{K+k/2}), \quad (4.24)$$

$$= \sum_k \sum_K F(K, k) [1 - D_K(k) + D_K(k)^2 \dots] N_{K+k/2} (1 - N_{K+k/2}), \quad (4.25)$$

$$F(K, k) = (\omega(k)/\hbar) M(k)^2 / [(\hbar/m)^2 (K \cdot k)^2 - \omega(k)^2]. \quad (4.26)$$

The first term of (4.25) is of the same form as that given by Fröhlich's theory. For values of K which make $F(K, k)$ infinitely large, we get

$$F(K, k) [1 + D_K(k)]^{-1} \simeq M(k)^2 / 2\nu F \hbar \omega(k) = \zeta / 3\nu N_0, \quad (4.27)$$

which shows that the correction term $D_K(k)$ acts to cancel the singularity in the perturbation energy. The above formulation may be considered as a Hartree approximation to take account of the higher-order perturbation effect.

Next we consider the frequency shift of lattice-phonon and we get the observed frequency⁵⁾ by

$$\omega'(k)^2 = \omega(k)^2 [1 - 2\nu F(1 - \delta_\omega)], \quad (4.28)$$

$$\delta_\omega = (1 - 2\nu F)^{1/2} (\sigma/k_0) \tanh^{-1} [k_0 / (1 - 2\nu F)^{1/2} \sigma], \quad \sigma^2 = (m\omega/\hbar k)^2, \quad (4.29)$$

in which δ_ω is negligible since $(\sigma/k_0)^2 \simeq 10^{-5}$, and $2\nu F$ becomes smaller than one lest the observed frequency ω' should be imaginary. On the other hand, Fröhlich's condition of superconductivity reads

$$1/3 < \nu F,$$

so that the interaction constant is restricted as follows:

$$1/3 < \nu F < 1/2. \quad (4.30)$$

As will be shown in the next section, the upper limit is somewhat extended due to the Coulomb interaction.

§ 5. Concluding remarks

The sound approximation presented in Section 2 and Section 4 is based on the assumption that 1) excited particles and holes are restricted in the neighborhood of the Fermi surface and 2) the oscillators of the different directions are regarded as independent of one another.

The first assumption is reflected in the following points. As was obtained in a preceding paper⁷⁾ the dispersion equation and the normal coordinate are expressed by

$$1 = \{k^2 G(k)/m\} \sum_K \psi_K^* \psi_K / [\{\omega(k) - \hbar(K \cdot k)/m\}^2 - (\hbar k^2/2m)^2], \quad (5.1)$$

$$\hat{\xi}_k = (N/m)^{1/2} k^2 G(k) \sum_K \psi_{K+(k/2)}^* \psi_{K-(k/2)} / \{\omega(k)^2 - (\hbar/m)^2 (K \cdot k)^2\}. \quad (5.2)$$

On finding the corresponding expressions (2.13) and (2.10), we have expanded the electron momenta at the Fermi top and set the number operator N_K to be unity within the Fermi sphere. Though the expression of the denominator of (5.1) is different from

that of (2.13), the maximum frequency obtained from (5.1) becomes identical with (2.14) up to the second order term. From (5.2) we get (2.10) by setting K involved in the denominator to be the Fermi top value. We can say the same thing with the transformation function and the true transformation function is obtained in our scheme by this rule of replacing as already mentioned in Section 4.

The second assumption is effective in dealing with the transformation which is explicitly written in the form

$$U(0, -\infty) = 1 - \hbar^{-1} \sum_k \sum_K M(k) \bar{\Psi}_{K+(k/2)}^* \bar{\Psi}_{K-(k/2)} [B_k / \{ (\hbar/m) (K \cdot k) - \omega(k) \} + B_{-k}^* / \{ (\hbar/m) (K \cdot k) + \omega(k) \}], \quad (5.3)$$

where the creation and the destruction operators are defined by

$$q_i = (\hbar/2\omega(k))^{1/2} (B_{+k}^* + B_i), \quad p_i = i(\hbar\omega(k)/2)^{1/2} (B_k^* - B_{-i}), \quad (5.4)$$

As was already shown by Kitano and Nakano,¹⁰⁾ in the second order perturbation energy there appear such terms as

$$\sum_{k, k'} \sum_{\tau, \tau'} \sum_K (M(k) M(k') / 2\hbar) [\{ (\hbar/m) (K + (k/2)) \cdot k + \omega(k) \}^{-1} - \{ (\hbar/m) (K - k' + (k/2)) \cdot k + \omega(k) \}^{-1}] \bar{\Psi}_{K+k}^* \bar{\Psi}_{K+k'} B_k^* B_{k'}$$

which take account of the inelastic scattering of phonons. These terms are ignored in our approximation, which would be justified if the wave function do not change abruptly from point to point in the momentum space. In fact, for such wave functions Ψ 's, we may consider that

$$\sum_K^* (\bar{\Psi}_{K-(k/2)}^* \bar{\Psi}_{K+(k/2)} - \bar{\Psi}_k^* \bar{\Psi}_{K+k}) \Psi = 0. \quad (5.6)$$

If the summation were carried out over the whole space it would vanish completely, but, in general, it is not zero and there survive terms with their momenta in the neighborhood of the Fermi surface. Even in this neighborhood, we assume that the wave functions are unchanged against small translations so that (5.6) is satisfied with a sufficient accuracy.

These assumptions require some weakness of the interaction energy and the interaction to be long-range. In other words, the fourier coefficient of the interaction potential becomes small as the wave number increases, therefore the wave functions with distant wave numbers are not combined with one another. In this respect some questions are left about the electron-lattice interaction, in which excitations with large wave numbers might be important.

The Coulomb interaction might make some contribution to Fröhlich's theory. Introducing the Coulomb interaction,⁵⁾ it has been shown that the upper limit of Fröhlich's condition is pushed up to the value of $1/2 + \lambda(k)$, $\lambda(k) = (e/k)^2 (3\pi N_0 / V \epsilon^2)$, after making some elementary manipulations. For the cut-off wave number, the allowable range of νF is shifted from $1/2$ to a certain value larger than one. For Na, $\lambda(k_c) = 0.8$, and for Zn, $\lambda(k_c) = 0.6$. As is shown by this calculation the Coulomb interaction acts to increase the stability of the atomic lattice.

To the problems where the excitations associated with large wave numbers become important, the theory of sound waves is inapplicable. In fact, as pointed out by Schafroth,⁸⁾ it fails to give the diamagnetism. But it is always possible in our theory to come back

to the usual treatment through the transformation function and to obtain the diamagnetism. The screening interaction of electrons and the mesonic interaction of nucleons are out of the limit of the applicability. But, even in these problems, the idea of the sound approximation⁽²⁾ will be useful in the evaluation of the transformation function (see Appendix III).

The author thanks Professor Husimi for his interest in this problem and he is indebted much to Dr. Nakano and Mr. Kitano for the discussions. He sincerely appreciates the valuable discussions and the kind informations of Professor Tomonaga and of Professor Wentzel during the early stages of the work.

Appendix I

The second order derivative of the density matrix is expressed by

$$\begin{aligned}
 & \partial^2 g(K, k) / \partial t^2 + (\hbar/m)^2 (K \cdot k)^2 g(K, k) \\
 &= (1/m) (K \cdot k) \sum_q G(q) [\rho(-q) \{g(K-q/2, k+q) - g(K+q/2, k+q)\}] [\\
 & - (1/\hbar) \sum_q G(q)] [q \cdot j(-q) \{g(K-q/2, k+q) - g(K+q/2, k+q)\}] [\\
 & + (1/m) \sum_q G(q)] [(k+q) \cdot \{ (K-q/2) g(K-q/2, k+q) \\
 & - (K+q/2) g(K+q/2, k+q) \} \rho(-q)] [\\
 & - (1/\hbar)^2 \sum_q G(q)] [\rho(-q)] [\sum_p G(p) \rho(-p) \{g(K-q/2+p/2, k+q+p) \\
 & + g(K+q/2-p/2, k+q+p) \\
 & - g(K+q/2+p/2, k+q+p) - g(K-q/2-p/2, k+q+p)\}] [], \quad (\text{A} \cdot 1 \cdot 1) \\
 & j(q) = (\hbar/m) \sum_q Q_{q-q/2}^{(*)} \psi_{q+q/2}^2
 \end{aligned}$$

where the notation $[[AB]]$ means $(AB+BA)/2$. If we restrict ourselves to such states as subject to the condition (5.6), we have only to do with the following terms

$$\begin{aligned}
 & (1/m) (K \cdot k) G(k) [\rho(k) \{g(K+k/2, 0) - g(K-k/2, 0)\}] [\\
 & - (1/\hbar) G(k)] [k \cdot j(k) \{g(K+k/2, 0) \\
 & - g(K-k/2, 0)\}] [- (1/2m) \sum_q G(q) (k+q) \cdot q] [g(K-q/2, k+q) \\
 & + g(K+q/2, k+q) \rho(-q)] []. \quad (\text{A} \cdot 1 \cdot 2)
 \end{aligned}$$

The last term is negligible since we are considering that the magnitude of k and q are small compared with k_0 . The second term can also be ignored, since both k and $j(k)$ are small quantities and $k \cdot j(k)$ is set to be zero, and finally summing up (A.1.1) with respect to k over the cylindrical surface mentioned in Section 2, we are led to the equation (2.7).

Appendix II. Unitary transformation

We have obtained in Section 2 a set of normal coordinates. The transformation of the set of normal coordinates is equivalent to an eigenvalue problem of the following quadratic form

$$Y(k) = \sum_s C_s^2 X_s^2 + (1/2) \sum_{s,t} (st N_s N_t / m^2)^{1/2} LG(k) X_s X_t. \quad (\text{A} \cdot 2 \cdot 1)$$

Therefore the transformation matrix is given by

$$U_{ws}(k) = \alpha_w (L N_s)^{1/2} / m^{1/2} (\omega^2 - k^2 c_s^2), \quad (\text{A} \cdot 2 \cdot 2)$$

and the normalization factor is determined by the unitarity condition such as

$$1 = \sum_s \alpha_w^2 L N_s / m (\omega^2 - k^2 c_s^2)^2, \quad (\text{A} \cdot 2 \cdot 3)$$

$$\alpha_w \simeq (m/N)^{1/2} \omega_p^2 \quad \text{for} \quad \omega = \omega_{\max}. \quad (\text{A} \cdot 2 \cdot 4)$$

Appendix III. Correlation energy

By the sound approximation we can estimate the correlation energy of metallic electrons which is defined by

$$e_{\text{corr.}} = E_0 - E_F - E_{\text{exch.}}, \quad (\text{A} \cdot 3 \cdot 1)$$

where E_0 is the ground-state energy and E_F is the well-known Fermi energy. The exchange energy of the Hartree-Fock approximation $E_{\text{exch.}}$ is given by

$$E_{\text{exch.}} = -0.458 e^2 / r_0, \quad (\text{A} \cdot 3 \cdot 2)$$

in which r_0 is the atomic radius.

The expression of the perturbation energy corresponding to (4.24) reads (see (6.22), Section 6 of I⁽⁵⁾)

$$\langle \Delta E \rangle = (1/2) \sum_k \sum_K [k^2 G(k) \{k^2 + D_K(k)\}^{-1} N_K (1 - N_{K+K}) - (2\pi N e^2 / V k^2) N_K], \quad (\text{A} \cdot 3 \cdot 3)$$

$$D_K(k) = D_K(t) = \lambda^2 (1 - t \tanh^{-1} t), \quad (\text{A} \cdot 3 \cdot 4)$$

$$\lambda^2 = 3\omega_p^2 / 2c_0^2, \quad t = (K \cdot k)_{|k|=k_0} / k_0 |k|, \quad (\text{A} \cdot 3 \cdot 5)$$

which is divided into two parts; the self-energy e_s and the exchange energy $e_{\text{exch.}}$, hence the correlation energy in question is given by

$$e_{\text{corr.}} = e_s + e_{\text{exch.}} - E_{\text{exch.}}. \quad (\text{A} \cdot 3 \cdot 6)$$

Concerning the self-energy the calculation is straightforward and we get

$$\begin{aligned} e_s &= (2\pi e^2 / VN) \sum_k \sum_K [\{k^2 + D_K(k)\}^{-1} - (1/k^2)] N_K \\ &= (-\lambda^2 e^2 / 2\pi) \int_{-1}^1 dt \int_0^\infty X(t)^2 \{k^2 + \lambda^2 X(t)^2\}^{-1} dk \\ &= (-\lambda e^2 / 2\pi) \int_{-1}^1 \tan^{-1} [k / \lambda X(t)] \Big|_{k=0}^\infty X(t) dt, \end{aligned} \quad (\text{A} \cdot 3 \cdot 7)$$

$$X(t) = (1 - t \tanh^{-1} t)^{1/2}, \quad X(t_0) = 0, \quad t_0 \simeq 0.83. \quad (\text{A} \cdot 3 \cdot 8)$$

The integration over t results in

$$e_s \simeq -(t_0/2) e^2 \lambda \int_0^1 (1 - y^2)^{1/2} dy = -1.13 r_0^{-1/2} \text{ erg.}, \quad (\text{A} \cdot 3 \cdot 9)$$

within errors of as much as 3 percent.

The expression of the exchange energy resembles that of the screening interaction studied by Macke⁽⁹⁾ and Wohlfarth.⁽¹⁴⁾ Here we examine their results by performing the following integration

$$e_{\text{exch.}} = - (1/2N) \int_0^{k_0} 4\pi K^2 dK \int_0^{k_0} (2\pi/K) K' dK' \int_{|K-K'|}^{K+K'} k G(k) dk, \quad G(k) = 4\pi e^2/V(k^2 + k'^2).$$

(A·3·10)

For an electron of wave number *K*, the energy is given by

$$J(K) = - \frac{e^2 k_0}{2\pi} \left\{ 2 + \frac{1}{2} \left(\frac{1-x^2+l^2}{x} \right) \ln \frac{(1+x)^2+l^2}{(1-x)^2+l^2} - 2l \tan^{-1} \left(\frac{1+x}{l} \right) \right. \\ \left. - 2l \tan^{-1} \left(\frac{1-x}{l} \right) \right\},$$

$$x = K/k_0, \quad l = \lambda/k_0,$$

(A·3·11)

which is of the same form as that obtained by Wohlfarth. The integration over *K* yields

$$e_{\text{exch.}} = - \frac{3e^2 k_0}{4\pi} \left\{ 1 - \frac{1}{6} l^2 + \left(\frac{1}{2} l^2 + \frac{l^4}{24} \right) \ln(1+4l^{-2}) - \frac{4}{3} l \tan^{-1} \left(\frac{2}{l} \right) \right\}.$$

(A·3·12)

However, in our case, *l* is not a constant value and it depends on *t*. The calculation is simplified by taking the average concerning the magnitude of the momentum and the relative direction *t*. The exchange energy is obtained by setting $l^2 = \langle l^2 \rangle = k^2 \langle X(t)^2 \rangle$, $k_0^2 = k^2/2k_0^2$ in (A·3·12). In the sound approximation the self-energy has been over estimated since the magnitudes of electron wave numbers are assumed to be the Fermi-top value. Therefore the true energy value will be smaller than (A·3·9) by the amount of order 1/4. The correlation energies of Li, Na and Cs are listed in the following table.

Table A·1

	<i>r</i> ₀	$\langle 1^2 \rangle^{1/2}$	$e_s' = \frac{3}{4} e_s$	<i>e</i> _{exch.}	<i>E</i> _{exch.}	<i>e</i> _{corr.}	<i>e</i> _{corr.} (Pines)	<i>e</i> _{corr.} (Wigner) ⁽¹⁵⁾
Li	1.72	0.82	−4.05	−1.10	−3.80	−1.4	−1.01	−0.94
Na	2.10	0.91	−3.72	−0.80	−3.11	−1.4	−0.92	−0.86
Cs	3.04	1.09	−3.05	−0.08	−2.16	−1.0	−0.81	−0.73
unit	Å	Å ^{−1}	(e. v.)					

We have calculated the higher-order correlation energy which *does involve* the correlation of antiparallel spins, which was obtained in I by Tomenaga's theory. The correlation of antiparallel spins is not only involved in the self-energy *e*_s, but responsible for some part of the change in the Hartree-Fock exchange energy *E*_{exch.}. The other part of this change comes from the correlation of parallel spins due to the Coulomb interaction. These correlation effects reduce the exchange energy cooperatively.

The first assumption of the sound approximation will be justified if the excitation

kinetic energy, the upper limit of which is given by $e_{\text{corr.}}$ for the lowest state, is smaller than a critical value.⁴⁾ In the three dimensional case, there appears the variable t and the major contribution to $e_{\text{corr.}}$ comes from the region of t between 0.5 and 0.83, therefore we can find that the critical value lies between $0.2E_F$ and $0.8E_F$. The correlation energies obtained are about $0.5E_F$ so that we can say our problem is within the limit of the applicability.

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Note added in proof Recently, after this paper was submitted, Prof. Tomonaga (*Prog. Theor. Phys.* **13** (1955), 467, 482) has presented a general theory of the collective description. Our method of auxiliary variables given in Section 3 has some parallelism with this theory. Instead of applying the random phase approximation of Bohm or our second assumption mentioned in Section 5, he has introduced the statistical concept of the mean deviation of an operator. This concept was useful in the hydrodynamical description of quantum mechanics in estimating errors arising from the anharmonic terms⁷⁾ (see also *Proc. International Conference Theoret. Phys. Kyoto, 1953*, p. 901). In Tomonaga's theory, the orthogonality of the collective variable to the internal variable is of importance, while in our method there appears the problem of the subsidiary condition and all the wave functions are required to satisfy the subsidiary condition. However, in practice, it will sometimes be more convenient to make a device to solve the eigenvalue problem regardless with the subsidiary condition. In this respect, we have proposed to adopt another second transformation $U_2' = U_2 U_1$ instead of U_2 . Then the transformed hamiltonian, say $H_{i.d.}$, becomes commutative with both ξ_k and σ_k . This fact represents that $H_{i.d.}$ is written solely in terms of the internal variables which are orthogonal to the collective variables as has been achieved in Tomonaga's theory. The proof for this is given by the following equations:

$$U = U_1 U_2 U_1, \quad [U, \xi_k] = -(\xi_k + q_k)U, \quad [U^{-1}, \xi_k] = U^{-1}(\xi_k + q_k), \quad [U, \sigma_k] = -(\sigma_k + p_k)U, \\ [U^{-1}, \sigma_k] = U^{-1}(\sigma_k + p_k),$$

therefore

$$[U^{-1} H_1 U, \xi_k] = [U^{-1} H_1 U, \sigma_k] = 0. \quad \text{Q.E.D.}$$

We have been able to show how the orthogonality of variables emphasized by Tomonaga is reflected in the method of auxiliary variables.

On the Meson-Theoretical Potentials

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Properties of the second order nuclear potentials derived from the symmetrical pseudoscalar meson theory are examined with respect to such phenomena that the fourth and higher order nuclear forces do not take a serious role. Informations originating from p - p scattering at low and intermediate energies furnish a strong confirmation of the second order potentials. Corrections which are required to the second order potentials by the experimental data are estimated and are found to be not inconsistent with the fourth order potentials.

§ 1. Introduction

Since the type of the π -meson was found to be pseudoscalar, nuclear forces between two nucleons have been given in various ways, adopting the symmetrical pseudoscalar meson theory. It has been made clear that the fourth order nuclear potentials in the static approximation are stronger than the second order ones inside the meson Compton wave length i.e., 1.40×10^{-13} cm¹⁾. It has also been estimated that the sixth order nuclear potentials take a serious role when the inter-nucleon distance x in the unit of the meson Compton wave length is smaller than 0.6.²⁾ In this region, involved effect due to multiple scattering³⁾, effect due to simultaneous presence of three or more mesons none of which are emitted or absorbed by the same nucleon⁴⁾, non-static corrections⁵⁾, and so on, are also large and the neglect of them is not justified.

On the other hand, properties of the static second and fourth order potentials in the region $x > 0.6$ have been analyzed according to the so-called "Taketani principle", which is quite useful in such circumstances.⁶⁾

The nuclear potentials constructed from the symmetrical pseudoscalar meson theory are almost the same whether the coupling type is pseudoscalar or pseudovector, as far as the region $x > 0.6$ is concerned, if the effects of the nucleon pair formation appearing in the former case are reduced by some causes, e.g., by the radiative effects. However, different methods to derive the nuclear forces give different potential shapes for either of the coupling types.

TMO potentials⁷⁾ are the results of one extreme method of derivation. In this method, the dissociation probability, P_d , representing the probability of the system to be in the states where one or more mesons are present is approximated by the perturbation method up to the second order and becomes much larger than unity when the two nucleons come very close together.⁸⁾ The potentials derived by the other extreme method are BW potentials⁹⁾ where P_d is irrespective of the inter-nucleon distance and vanishes identically.⁹⁾ It

is to be noted that the difference between TMO and BW potentials is of the fourth or higher order. However, both TMO potentials and BW potentials can reproduce experimental data qualitatively. Therefore FST potentials,⁹⁾ which restrict the dissociation probability to $0 \leq P_D \leq 1$, are also expected to be the case.

Thus, it has been well known that the meson-theoretical nuclear forces have very suitable characteristics to fit the experimental data of the two-nucleon system qualitatively. However, if one wants to go a step further into a quantitative treatment of the nuclear force problems, one encounters many obstacles to be surmounted. For instance, how should one treat the difference in the fourth order potentials originating from every different method to construct the potentials? Or is it possible to draw any quantitative conclusions on the meson-theoretical nuclear forces in the region $x > 0.6$ without being obscured by their uncertainties in the region $x < 0.6$?

It is highly desirable to exploit the meson-theoretical nuclear forces as quantitatively as possible, avoiding the obstacles described above. In this paper we investigate from experimental data to what extent the second order potentials are reliable in the region $x > 1$. Furthermore we compare qualitatively the experimentally required corrections to the second order potentials with the fourth and higher order potentials. As will be shown in Sec. 2, the most suitable experiment for this purpose is the p - p scattering at low energy. The properties of the central forces are discussed in Sec. 3, comparing them with the p - p scattering experiment at 3.9 Mev, while those of the tensor forces are investigated in Sec. 4 using the 18.3 Mev data. The questions above can now be answered in the affirmative on the basis of these results. Conclusions summarized in Sec. 5 furnish a strong confirmation of the symmetrical pseudoscalar meson theory.

§ 2. General remarks on phenomenological analysis

Many conclusions have been drawn concerning the properties of the meson-theoretical nuclear forces. First of all, let us see to what extent they are reliable qualitatively and to what extent they are confirmed quantitatively discriminating the contributions of the second order potentials from those of the higher order ones.

At low energy, the second and fourth order nuclear forces derived from the meson theory reproduce the low energy parameters of both the triplet and the singlet states, with the coupling constant $g^2/4\pi$ of about 0.08*, if suitable phenomenological potentials inside $x=0.6$ are assumed^{7,9,10,11)}. However, emphasis should be laid on the fact that all these parameters are defined mainly by the wave functions of the S -states, and are dependent, in consequence, on the shape and magnitude of the fourth order and the phenomenological inside potentials, as will be seen.

According to a calculation of Blatt and Kalos¹²⁾, it is impossible to explain simultaneously all the low energy data assuming the second order potentials alone. This is also the case

*) $g^2/4\pi$ is the coupling constant with the pseudovector coupling. It is related to the equivalent pseudoscalar coupling constant $G^2/4\pi$ as $g^2/4\pi = (\mu/2M)^2 G^2/4\pi$.

no matter how a phenomenological attractive potential is added to the second order potentials, as far as the attractive potential is independent on the spin and τ -spin operators. Actually, for $g^2/4\pi \sim 0.08$, the second order potential of the singlet even state is too weak to reproduce the experimental data of the scattering length and the effective range at one time. A strong attraction of the fourth order in this state is expected to play an important role.

For BW potentials, it is known from a calculation that various deuteron parameters are dependent on the strength of the inside phenomenological tensor potential⁽⁹⁾. For TMO potentials a large cut off of the attractive nature for the central potential is necessary for the deuteron to be bound⁽⁹⁾.

Thus, low energy parameters are connected with the second order potentials as well as with the fourth order and the phenomenological inside potentials.

There exists, however, one exception that is attributable mainly to the second order potentials alone. Namely, a slight difference between the experimental values of the n - p and p - p singlet effective range can be explained by the mass difference between π^\pm - and π^0 -mesons⁽¹¹⁾. If one attributes this difference to some interactions effective when two nucleons come very close together, it seems that one can not help assuming they are very singular, which can hardly be justified.⁽¹¹⁾ However, as the problem of the mass difference itself is very involved, it seems to be premature to conclude the validity of the meson-theoretical nuclear forces from this fact alone.

At intermediate energy the meson-theoretical potentials can also reproduce both n - p and p - p scattering data qualitatively⁽⁹⁾⁽¹⁰⁾⁽¹¹⁾⁽¹⁵⁾, which is mainly due to their following properties.

The singlet odd potential is repulsive, while the averaged effect of the triplet odd potentials is attractive. The effects of these two states cancel each other, resulting in the n p angular distribution that is nearly symmetric about 90° . However, if only the second order potentials are assumed, large backward n - p scattering arises actually owing to the operator $(\tau_1 \cdot \tau_2)(\sigma_1 \cdot \sigma_2)$ which makes both the singlet and triplet odd central forces repulsive⁽¹⁶⁾. So, the fourth order central potential in the triplet odd state is expected to play an important role. The total cross section of the n - p scattering is reduced to the experimental value when BW potentials are adopted if suitable phenomenological potentials inside $x=0.6$ are assumed⁽¹⁰⁾⁽¹⁷⁾, while all of the phenomenological potentials proposed so far reproduce too large ones. However, the total cross section is also very sensitive to the parameters of the inside potentials⁽¹⁰⁾.

On the other hand, the p - p scattering experimental data restrict the triplet odd state potentials such that the central force is very weak and the tensor force is not so strong as the other state ones. Actually the meson-theoretical potentials averaged over x have the above required properties⁽¹¹⁾. As the second order potentials are in general very weak compared with the fourth order ones, the singlet even state potential has a quite short range. The meson-theoretical singlet D -wave phase shift is much smaller than those due to the usual phenomenological potentials. This is mainly due to the properties of the second order potential and is fitting to the isotropic p - p angular distribution⁽⁹⁾⁽¹¹⁾⁽¹⁵⁾. The above two facts are favourable to the meson theory, though the quantitative comparison is

of course impossible.

Recently, the pseudoscalar meson-theoretical potentials have been applied to some problems of nuclei other than deuteron¹⁷⁾. These results are indeed interesting, but to discuss about them is outside the scope of this paper, as no conclusive knowledge about two-body nuclear forces can be got from them.

It follows from the remarks above that there are many facts indicating the qualitative validity of the meson theory of nuclear forces. At the same time one finds that the quantitative knowledge about it is rather poor.

In order to push the theory forward so that it can be treated quantitatively, it is highly desirable to investigate its predictions with respect to such phenomena that the fourth order and the phenomenological inside potentials are not definitely effective.

As *S*-waves are distorted by the second order potential as well as by the higher order ones, their properties are not suitable for our purpose. So it is interesting to inquire into the properties of the *P*-waves at energy as low as one can. However, the *n-p* scattering is not useful because *P*-waves interfere with *S*-waves which are of phenomenological nature. On the contrary, the triplet *P*-waves in the *p-p* scattering distorted by the nuclear forces interfere with the well known Coulomb force scattering, which is quite favourable to our analysis.

The presence of non-central forces separates the triplet *P*-wave phase shift into three. This makes the analysis very complicated. First we study the properties of the central force from the averaged *P*-wave phase shift at 3.9 Mev where the effect of the tensor force can be replaced safely by an equivalent central force. Then we investigate the properties of the tensor force from the *p-p* scattering at 18.3 Mev. In this energy region, the angular distribution is determined mainly by each of the three *P*-wave phase shifts which can no longer be replaced by an equivalent central phase shift without contradiction if the tensor force is as strong as the meson-theoretical one. The impact parameters of the *P*-wave, the inverse of the wave number in the unit of the meson Compton wave length, are about 3.3 and 1.5 at 3.9 and 18.3 Mev respectively. These parameters are very useful,²¹⁾ so the *P*-waves are not so much affected by the potentials in the region $x < 1$ where the fourth order potentials are important.

§ 3. The central force*

The differential scattering cross section for *p-p* scattering in the center of mass system can be expressed as follows;

$$d\sigma_{p-p}(\theta)/d\Omega = d\sigma_{Mott}/d\Omega + {}^1d\sigma_N/d\Omega + {}^1d\sigma_{NC}/d\Omega + {}^3d\sigma_N/d\Omega + {}^3d\sigma_{NC}/d\Omega, \quad (1)$$

where the first term $d\sigma_{Mott}/d\Omega$ gives the pure Coulomb scattering, the second term is due to the nuclear force of the singlet even state and the third term the interference between that and the Coulomb force. The fourth and the fifth terms for the triplet odd state correspond just to the second and the third terms for the singlet even state.

*) Some part of the results in this section has been published already in this journal, reference 18.

When non-central forces are present the triplet P -wave phase shift is separated into three, i.e., δ_1' , $J=0, 1, 2$ denoting the total angular momentum. ${}^3d\sigma_N/d\Omega$ and ${}^3d\sigma_{NC}/d\Omega$ are expressed by them as follows¹⁰⁾;

$${}^3d\sigma_N/d\Omega = (3/k^2) (C_0 + C_2 \cos^2 \theta), \quad (2)$$

where k is the wave number in the center of mass system and

$$C_0 = (1/3) (\sin {}^3\delta_1^0)^2 + (3/4) (\sin {}^3\delta_1^1)^2 + (13/12) (\sin {}^3\delta_1^2)^2 \\ - (2/3) \sin {}^3\delta_1^0 \sin {}^3\delta_1^2 \cos({}^3\delta_1^0 - {}^3\delta_1^2) - (3/2) \sin {}^3\delta_1^1 \sin {}^3\delta_1^2 \cos({}^3\delta_1^1 - {}^3\delta_1^2), \quad (2')$$

$$C_2 = (3/4) (\sin {}^3\delta_1^1)^2 + (7/4) (\sin {}^3\delta_1^2)^2 + 2 \sin {}^3\delta_1^0 \sin {}^3\delta_1^2 \cos({}^3\delta_1^0 - {}^3\delta_1^2) \\ + (9/2) \sin {}^3\delta_1^1 \sin {}^3\delta_1^2 \cos({}^3\delta_1^1 - {}^3\delta_1^2), \quad (2'')$$

and

$${}^3d\sigma_{NC}/d\Omega = (1/4) (\eta/k^2) \{ (-\operatorname{cosec}^2(\theta/2) \cos \varphi_1^s + \sec^2(\theta/2) \cos \varphi_1^c) \\ \times (\sin 2{}^3\delta_1^0 + 3 \sin 2{}^3\delta_1^1 + 5 \sin 2{}^3\delta_1^2) \\ + (\operatorname{cosec}^2(\theta/2) \sin \varphi_1^s - \sec^2(\theta/2) \sin \varphi_1^c) (9 - \cos 2{}^3\delta_1^0 - 3 \cos 2{}^3\delta_1^1 - 5 \cos 2{}^3\delta_1^2) \} \cos \theta, \\ \text{where} \quad (3)$$

$\eta = e^2/\hbar v$, v being the relative velocity,

$$\sigma_1 - \sigma_0 = \tan^{-1} \eta,$$

$$\varphi_1^s = 2\eta \ln \sin(\theta/2) + 2(\sigma_1 - \sigma_0), \quad \varphi_1^c = 2\eta \ln \cos(\theta/2) + 2(\sigma_1 - \sigma_0).$$

When a central force alone is present, they are much simplified and expressed in terms of one P -wave phase shift Δ alone as follows;

$${}^3d\sigma_N/d\Omega = (27/k^2) (\sin \Delta)^2 \cos^2 \theta, \\ {}^3d\sigma_{NC}/d\Omega = (9/2) (\eta/k^2) \{ \sec^2(\theta/2) \sin \Delta \cos(\Delta + \varphi_1^c) \\ - \operatorname{cosec}^2(\theta/2) \sin \Delta \cos(\Delta + \varphi_1^s) \} \cos \theta. \quad (4)$$

Expanding eqs. (2), (2'), (2''), (3) and (4) under the assumption of small ${}^3\delta_1'$ and Δ , and comparing their first order terms with each other, one finds that, if the relation

$$\Delta = (1/9) ({}^3\delta_1^0 + 3{}^3\delta_1^1 + 5{}^3\delta_1^2) \quad (5)$$

is satisfied, a central and a non-central forces, combined to reproduce ${}^3\delta_1'$, are equivalent to another fictitious central force reproducing Δ given by eq. (5).

The p - p scattering at low energy determines at best this Δ . According to the accurate results of Wisconsin University, Δ is definitely negative¹¹⁾, of which the experimental value only at 4.203 Mev is not entirely consistent with the other six ones in its magnitude because of the experimental difficulties. The dependence of Δ on energy at the other six data from 1.855 to 3.899 Mev is consistent in themselves, so it would be sufficient to compare the theoretical results with $\Delta = -0.109 \pm 0.020$ at 3.899 Mev.

The definite information, that J is negative at low energy, originating from the $p-p$ scattering data, is particularly significant. In the zero energy limit, the algebraic relations,

$${}^3\delta_1 J(\text{exact}) > {}^3\delta_1 J(\text{Born approximation}), \quad (6)$$

$$\text{so that} \quad \Delta(\text{exact}) > \Delta(\text{Born approximation}), \quad (7)$$

are proved if each potential effective to each ${}^3\delta_1 J$ is positive or negative definite⁽²¹⁾. So at such a low energy, eq. (7) may well hold too. On the other hand, $J(\text{Born approximation})$ is determined by the central force alone, even though non-central forces are present, whether they are tensor type or velocity-dependent type such as $\mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$, \mathbf{L} denoting the relative orbital angular momentum. Thus J provides clues concerning the central force V_c of the triplet odd state. The integral $-\int_0^\infty x^2 V_c(x) j_1''(xkb/\mu c) dx$ must be negative in order to make $J(\text{Born approximation})$ negative, which is a necessary, but not a sufficient condition for $J(\text{exact})$ to be negative by eq. (7). Namely V_c must be repulsive as a whole.

The symmetrical pseudoscalar meson theory predicts the same second order nuclear potential irrespective of the methods to construct them, i.e.,

$$V^{(2)} = V_c^{(2)} + S_{12} V_t^{(2)} \quad (8)$$

$$\text{where} \quad V_c^{(2)} = (1/3) (g^2/4\pi) \mu (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) e^{-x}/x, \quad (9)$$

$$V_t^{(2)} = (1/3) (g^2/4\pi) \mu (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) (1 + 3/x + 3/x^2) e^{-x}/x. \quad (10)$$

$V_c^{(2)}$ is repulsive in the triplet odd state, though it is not so strong owing to the eigenvalue $(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) = 1$ in this state.

In this way, our attention is focused on two problems. Firstly, does this second order slight repulsive central force reproduce negative J , overwhelming the contributions from the fourth and the higher terms and from the tensor potential which is effective to make J positive? Second, if so, how will be the numerical comparison with the experimental data?

To see the contribution from the second order potentials clearly, we take into account only $V_c^{(2)} + S_{12} V_t^{(2)}$ at the first step cutting them all to be zero in the region $x < 1$. For $g^2/4\pi = 0.10$, we have ${}^3\delta_1^0 = 1.502^\circ$, ${}^3\delta_1^1 = -0.694^\circ$ and ${}^3\delta_1^2 = -0.280^\circ$. They result in a negative $J = -0.207^\circ$. The practical error coming from the approximation (5) is estimated in the Appendix, which is far less than 30%.

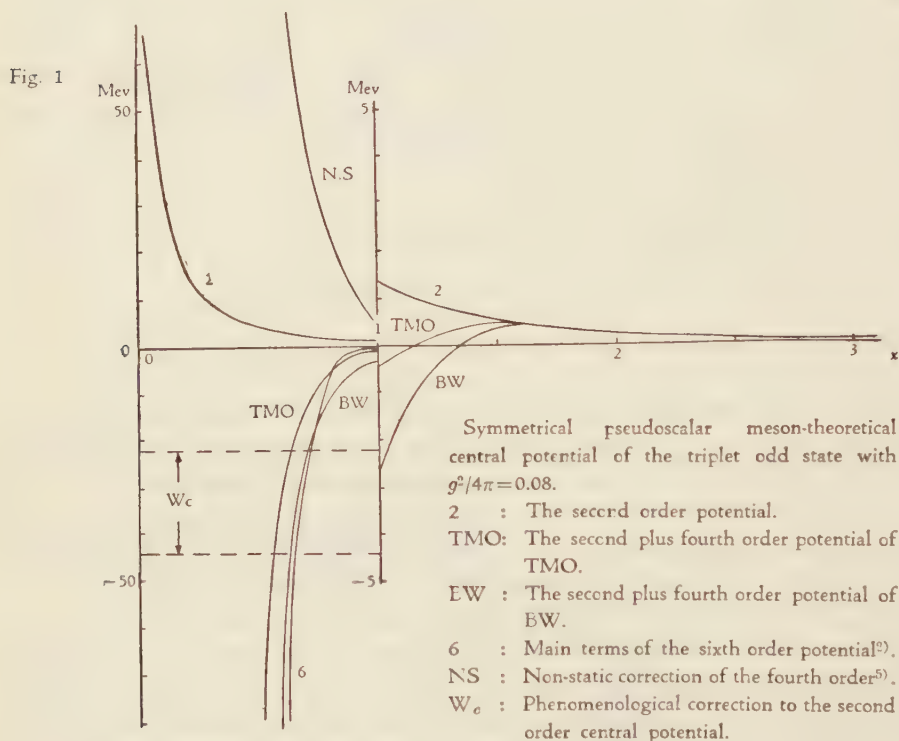
The result described above shows that even the weak central repulsion of the second order is repulsive enough to reproduce a negative J and that an attraction must be added to it. Then, how much is the order of magnitude of this attraction?

When this attraction is approximated by a square well with its range X and its depth W_c , which is superposed on $V_c^{(2)}$, the results for $g^2/4\pi = 0.08$ are listed in Table 1 using the Born approximation for W_c . The magnitude of $V_c^{(2)}$ corresponding to each X is also shown. We see from this Table and Fig. 1 that the additional attraction must not cancel the repulsion of $V_c^{(2)}$ in the region $x \gtrsim 3$ and that it may be rather strong if confined in the region of small x . This is quite natural as the impact parameter of the P -wave at 3.9 Mev is about 3.3. The importance of the repulsion in the region $x \gtrsim 3$ is confirmed also from the fact that ${}^3\delta_1^2 = -0.280^\circ$ is negative. The potential effective for this state is $V_c^{(2)}$

$-0.4V_c^{(2)}$, and is attractive all the while in the region $x \leq 2.732$ and repulsive only outside it. It is this latter repulsion that makes ${}^3\partial_1^{(2)}$ negative.

Table 1. The added attraction to $V_c^{(2)}$ of the range X and the depth W_c . $V_c^{(2)}$ at each X is also shown.

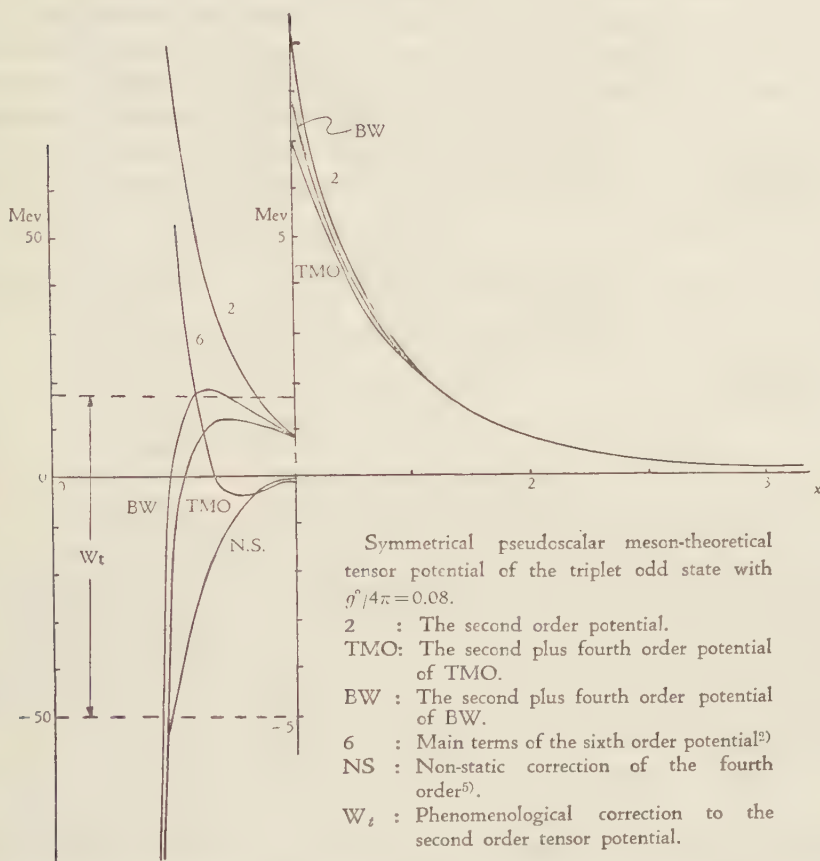
X	1.0	2.0	3.0
W_c (Mev)	$-22.5 \sim -46.6$	$-0.73 \sim -1.52$	$-1.03 \sim -2.14 \times 10^{-2}$
$V_c^{(2)}$ (Mev)	1.38	0.254	5.0×10^{-2}



The values of J obtained when the additional attraction is assumed to be the fourth order TMO or BW central potential are listed in the Table of reference 18 and are in good agreement with the experiment, e.g., $J = -0.14^\circ$ or -0.10° according as the TMO or BW second plus fourth order potentials are adopted in the region $x \geq 0.3$. It is a little unfortunate for inquiring into its detailed properties further that this fourth order potential is not so large as those of the other states, owing to an unexpected cancellation of various terms composing it and, in consequence, it is rather smaller than the non-static fourth order corrections⁵⁾ or the sixth order ones²⁾ even in the region $x=1 \sim 0.6$ in this state alone as shown in Fig. 1 2.

Thus it is summarized that the effect of the second order central potentials may well

Fig. 2



be present and may well be a main contributing factor to negative Δ , and that the necessary additional attraction has a short range and is of the same order of magnitude as the TMO or BW fourth order central potential. The properties of the tensor force will be examined in the next section. It is of particular interest to note that the negative Δ is unfavourable to usual phenomenological potentials proposed so far, for though it requires the exchange character of about $(0.38 + 0.62P_M)$, P_M denoting the space exchange operator²²⁾, the usual phenomenological potentials have too weak interactions in the odd states. This again confirms the superiority of the meson theory of nuclear forces.

§ 4. The tensor force

When the tensor force in the triplet odd state is much stronger than the central force, the former almost determines the triplet p - p scattering amplitude in the intermediate energy region. This is just the case at 18.3 Mev for the meson-theoretical nuclear forces. Namely the neglect of the terms proportional to $(\partial_1^2 V)^2$ is not justified and use must be made of

eqs. (2)~(3) instead of eqs. (4) and (5).

We have already known from the analysis of OF¹¹⁾ how the triplet scattering ${}^3d\sigma_N/d\Omega + {}^3d\sigma_{N'}/d\Omega$ is at 18.3 Mev provided the singlet even potential is assumed to be TMO or BW one, which is quite similar to each other in this state. It is worth noting that the results of OF remain valid, even though the singlet even potential is not the meson-theoretical one. It is the purpose of this section to see to what extent the meson-theoretical tensor force explains the p p scattering experiment in detail, making use of the results of OF.

For the central potential, adopt $V_c^{(2)}$ and the additional attraction of the range 1. Namely it has the shape

$$V_c = \begin{cases} V_c^{(2)} \text{ (meson-theoretical)} & x > 1, \\ W_c \text{ (a phenomenological constant)} & x < 1. \end{cases} \quad (11)$$

And correspondingly, adopt for the tensor potential.

$$V_t = \begin{cases} V_t^{(2)} \text{ (meson-theoretical)} & x > 1, \\ W_t \text{ (a phenomenological constant)} & x < 1. \end{cases} \quad (12)$$

We have now much information about the central potential $V_c^{(2)}$ and W_c . On the other hand, it seems to be quite natural from the preceding results to assume the presence of the meson-theoretical tensor potential of the second order $V_t^{(2)}$. In the region $x > 1$, $V_t^{(2)}$ is expected to be a good approximation for the meson-theoretical tensor potential as can be seen from Fig. 2. Then we will estimate the magnitude of W_t , that is permitted by the experimental data following the analysis of OF. The magnitude of W_t thus obtained will give some information about the higher order tensor potential.

The results in the case of $g^2/4\pi=0.08$ are tabulated in Table 2. The permitted

Table 2. The sets of W_c and W_t , and the phase shifts ${}^3\delta_1^J$ due to them. The values of the parameters permitted by the experiment are; $0 < C_0 < 0.027$, $0 < C_2 < 0.01$, $-0.018 < D < 0.006$. (See the reference OF¹¹⁾, Table IV. D is identical with Δ there.)

W_c (Mev)	W_t (Mev)	${}^3\delta_1^0$	${}^3\delta_1^1$	${}^3\delta_1^2$	C_0	C_2	D
-22.5	16.9	15.53	-3.96	0.92	0.026	0.008	-0.012
	0.0	8.94	-3.61	0.80	0.011	0.004	0.000
	-42.4	5.10	-1.26	0.46	0.003	0.001	-0.011
	-53.0	4.70	0.00	0.43	0.002	0.001	-0.023
-34.6	14.8	16.33	-3.78	1.20	0.028	0.010	-0.018
	0.0	9.57	-3.44	1.03	0.012	0.004	-0.007
	-31.8	5.05	-1.72	0.75	0.004	0.002	-0.013
	-42.4	5.27	-0.57	0.68	0.002	0.002	-0.022
-46.6	10.6	10.16	-3.55	1.49	0.027	0.011	-0.023
	0.0	10.32	-3.26	1.38	0.013	0.006	-0.014
	-11.6	8.19	-2.75	1.26	0.008	0.004	-0.015
	-21.2	7.03	-2.07	1.15	0.006	0.003	-0.017

region of W_t is from 17 to -50 Mev. As can be seen from Fig. 2, this shows that $V_t^{(2)}$ must be reduced in the region $x < 1$, whose deviation is in the direction as one would expect it to be, comparing with $V_t^{(1)}$ (TMO) or $V_t^{(1)}$ (BW). Its magnitude is not inconsistent with them as can be seen from the numerical comparison summarized in Table 3, though nothing quantitative can be said owing to the large uncertainty for the permitted W_t , which is entailed by the experimental errors of p - p scattering, the uncertainty of $^1d\sigma_N/d\Omega + ^1d\sigma_{Nc}/d\Omega$ and that of W_c .

Table 3. A crude comparison of the phenomenological correction $W_t - V_t^{(2)}$ with $V_t^{(4)}$ (TMO) or $V_t^{(4)}$ (BW) is made. Namely the ratio of the integral $\int_0^\infty V_t^{(4)} x^2 j^2 dx$ to $I \equiv \int_0^1 (W_t - V_t^{(2)}) x^2 j^2 dx$ is estimated. $j \equiv j_1 \left(x \frac{\hbar}{\mu c} k \right)$, where k is the wave number at 18.3 Mev.

$W_t = 17 \text{ Mev}$	$\int_{0.59}^\infty V_t^{(4)} \text{ (TMO)} x^2 j^2 dx / I = 1.0$ $\int_{0.5}^\infty V_t^{(4)} \text{ (BW)} x^2 j^2 dx / I = 1.0$
$W_t = -50 \text{ Mev}$	$\int_{0.3}^\infty V_t^{(4)} \text{ (TMO)} x^2 j^2 dx / I = 0.27$ $\int_{0.3}^\infty V_t^{(4)} \text{ (BW)} x^2 j^2 dx / I = 0.20$

It would be safe to say from the results above that $V_t^{(2)}$ itself is confirmed to be quite reliable in the region $x > 1$. When the condition $17 \text{ Mev} > W_t > -50 \text{ Mev}$ is satisfied, which is not inconsistent with $V_t^{(4)}$ (TMO) or $V_t^{(4)}$ (BW), $V_t^{(2)}$ can reproduce the experimental data. If the assumption of $V_t^{(2)}$ there is false, W_t is expected to differ much from the meson-theoretical tensor potential reduced to their average over x in the region $x < 1$, as the effect due to a small deviation of $V_t^{(2)}$ from the actual potential shape in the region $x > 1$ can be cancelled only by a quite large change of W_t in the region $x < 1$ at such energy.

Small variation of $g^2/4\pi$ would not alter these results. If a very large $\mathbf{L} \cdot \mathbf{S}$ forces of the fourth or higher order are present, about which we do not know precisely now, the resultant W_t would more or less be different.

§ 5. Conclusions

The properties of the second order meson-theoretical potentials $V_c^{(2)} + S_{12} V_t^{(2)}$ in the triplet odd state have been examined, by comparing them with such phenomena that the fourth or higher order contributions do not give serious effects.

The analysis at low energy gives the principal support to $V_c^{(2)}$. It is the main contributing factor to the negative A . We can also draw inference therefrom that an attraction of rather short range must be added, which has the same order of magnitude with the fourth order central potential in TMO or BW treatment.

The analysis at 18.3 Mev shows that $V_t^{(2)}$ is quite reliable qualitatively and that it is reduced in the region $x < 1$ by a correction term. This tendency agrees well with the

meson-theoretical tensor force, though nothing quantitative can be said owing to the large uncertainty in W_t .

These facts furnish a strong confirmation of the validity of the second order potentials in the region $x \approx 1$, whose coupling constant $g^2/4\pi$ is about 0.08 and constitute a spectacular success of the meson theory of nuclear force. Particularly, they suggest that a quantitative treatment of this theory will become much more significant.

As was stated in Sec. 2, the singlet D -wave phase shift due to the meson-theoretical potential is expected to be very small compared with the usual phenomenological ones. If one can determine it from scattering experiments at as low energy as he can, it might afford another good criterion to the meson theory of nuclear force.

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Appendix

An estimation of the terms proportional to $({}^3\partial_1 J)^2$.

In Sec. 3, the effects of the central and tensor force giving rise to ${}^3d\sigma_N/d\Omega$ and ${}^3d\sigma_{NC}/d\Omega$ have been approximated as if a central force yielding one phase shift J given by eq. (5) is present alone. We examine whether the errors caused by this approximation take serious effect on determining the J from the experimental angular distribution. If so we have to compare the angular distribution due to ${}^3\partial_1 J$ with raw materials of the experiments, without resorting to Δ of eq. (5).

The angular distribution of p - p scattering experiment at low energy deviates from that assuming only 1S -wave, if 3P -waves take part in it. This P -wave anomaly is appreciable in the region $\theta < 30^\circ$. Can the terms proportional to $({}^3\partial_1 J)^2$ be safely neglected there compared with the terms proportional to ${}^3\partial_1 J$?

From eqs. (2) ~ (5), we find:

The term $(1/k^2)a_1(\theta)$, proportional to ${}^3\partial_1 J$ or Δ is

$$a_1(\theta) = f(\theta) \times \begin{cases} (\sin 2 {}^3\partial_1^0 + 3 \sin 2 {}^3\partial_1^1 + 5 \sin 2 {}^3\partial_1^2) : \text{non-central,} \\ 9 \sin 2\Delta : \text{central alone,} \end{cases} \quad (\text{A} \cdot 1)$$

where
$$f(\theta) = (\eta/4) \left(-\frac{\cos \varphi_1^s}{\sin^2 (\theta/2)} + \frac{\cos \varphi_1^c}{\cos^2 (\theta/2)} \right) \cos \theta.$$

The terms $(1/k^2)a_2(\theta)$ and $(1/k^2)a'_2(\theta)$, proportional to $({}^3\partial_1 J)^2$ and Δ^2 are

$$a_2(\theta) = g(\theta) \times \begin{cases} (9 - \cos 2 {}^3\partial_1^0 - 3 \cos 2 {}^3\partial_1^1 - 5 \cos 2 {}^3\partial_1^2) : \text{non-central,} \\ 18 \sin^2 \Delta : \text{central alone,} \end{cases} \quad (\text{A} \cdot 2)$$

where
$$g(\theta) = (\eta/4) \left(\frac{\sin \varphi_1^s}{\sin^2 (\theta/2)} - \frac{\sin \varphi_1^c}{\cos^2 (\theta/2)} \right) \cos \theta,$$

$$\text{and } a_2'(\theta) = \cos^2 \theta \times \begin{cases} 3 C_2 & : \text{non-central,} \\ 27 \sin^2 \theta & : \text{central alone,} \end{cases} \quad (\text{A} \cdot 3)$$

where C_2 is given by eq. (2'').

The set of ${}^3\hat{o}_1^J$, whose magnitudes are the largest that we have calculated, is given in the case when $V^{(2)} = V_e^{(2)} + S_{12} V_t^{(2)}$ with $g^2/4\pi = 0.10$ are assumed in the region $x > 1$ and are cut to zero in the region $x < 1$, and is given in Sec. 3. $a_1(\theta)$, $a_2(\theta)$ and $a_2'(\theta)$ obtained by these ${}^3\hat{o}_1^J$'s on the one hand (suffix NC) and by assuming J alone on the other (suffix C) are tabulated in the Table.

As the actual tensor force would not be so strong as that given above, the actual ${}^3\hat{o}_1^J$'s are estimated much smaller. Thus we can see from the Table that the effect of the neglected terms is far less than 30% and does not affect the main results. The same conclusion, that the improvement of the fit can not be obtained by inclusion of quadratic terms in ${}^3\hat{o}_1^J$ is stated also in reference 22.

Table

θ	$a_1(\theta)_{NC}$	$a_1(\theta)_C$	$a_2(\theta)_{NC}$	$a_2(\theta)_C$	$a_2'(\theta)_{NC}$	$a_2'(\theta)_C$	ratio
10°	0.7662	0.7673	-0.0030	-0.0003	0.0340	0.0004	0.039
20°	0.1544	0.1545	-0.0003	-0.0000 ₃	0.0310	0.0004	0.195
30°	0.0774	0.0775	-0.0001	-0.0000 ₁	0.0263	0.0003	0.334

The ratio in the Table is $\{(a_1 + a_2 + a_2')_{NC} - (a_1 + a_2 + a_2')_C\} / (a_1 + a_2 + a_2')_C$.

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Double Meson Production in the Intermediate-Coupling Theory, I

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The intermediate-coupling theory is applied to the problem of the meson production in meson-nucleon collisions. For simplicity, the model used is a charged scalar meson field interacting with a fixed nucleon. The amplitudes of the ordinary scattering and the meson production are obtained in the "one-level approximation".

§ 1. Introduction and summary

The intermediate-coupling theory has been formulated to exhibit its validity in a region of moderate coupling strength. The essential point of this theory consists in the disconnection of meson configurations between bound and unbound ones. Mesons in bound configuration are bound to the nucleon and make the so-called meson cloud around the nucleon. The various states of the meson cloud correspond to the various isobar levels of the nucleon. Mesons in unbound configurations, which are called "*s*-mesons", move in the outer space. When a meson collides with the nucleon, there arise the resonance-like interactions between the incident *s*-meson and the meson cloud.

Actually in this case the "one-level approximation" is allowed; that is, only one nucleon level is excited in the course of the collision process. In the scattering problems hitherto treated the presence of more than one *s*-meson can be neglected. It seems that the one level approximation is closely connected with the neglect of two or more *s*-meson configurations and therefore the process of the double meson production cannot be treated in this approximation.¹⁾

However, it is found that the one-level approximation has more contents than in the case of "*Scattering Problem in the Intermediate-Coupling Theory, I*" by Z. Maki, M. Sato and S. Tomonaga.²⁾ That is: though only one nucleon level is excited in the course of the collision process, this excited nucleon can fall down to any lower level. For instance, in π^-p scattering problem using a charged scalar meson field the initial nucleon in the ground level absorbs a negative π meson and is excited to the third level from the ground. In case this excited nucleon falls down again to the ground level, the so-called ordinary scattering occurs like M.S.T. When the energy of incident meson is enough large, the double meson production will occur as follows: Namely the excited nucleon falls down to the

second level and emits a positive s -meson and if the second level is not stable, the nucleon in the second level falls still more down to the ground emitting a negative s -meson. In this way there exist in the final state a nucleon in the ground level and two s -mesons. In the case of π^+p scattering problem by M. S. T., however, there is no need of the above considerations, the excited level being the next higher from the ground.

Especially, for some values of the effective coupling constant V of the meson-nucleon interaction and the cut off momentum k_m , an isobar level is stable and hence a process of charge-exchange scattering occurs. Experimentally there exists no stable isobar level. From this fact we must impose a certain physical condition on the values of V and k_m in order to prohibit the existence of the stable isobar level. The model we used is but a toy. Even if we use a more realistic model, a similar feature will remain *mutatis mutandis*.

Recently the cross-section of double meson production is experimentally ascertained to be comparatively large at 0.6 and 1.5 Gev². It seems that the relativistic effects are important. But in the final state there are two mesons with comparatively low energy, for which the reaction effects are predominant. Thus the intermediate-coupling theory will be useful in this energy region. The calculations for the more realistic model compared with the experimental results will soon appear in this journal.

§ 2. π^-p collisions

The formulation is almost the same as M.S.T. The only different point is that the model used is the charged scalar meson field instead of the charged longitudinal vector meson field in M.S.T. Accordingly the Hamiltonian of our system is of the form

$$H = K_{00} \mathcal{Q} + \int K_{0s} A^* a_s ds + \int K_{0s'} B^* b_{s'} ds' + \int K_{s0} a_s^* A ds + \int K_{s'0} b_{s'}^* B ds' + \int S (a_s^* a_s + b_s^* b_s) ds. \quad (1)$$

We have used the following abbreviations:

$$\begin{aligned} \mathcal{Q} &= A^* A + B^* B - V[(A + B^*)\tau_+ + (A^* + B)\tau_-], \\ A - V\tau_- &= A, \quad A^* - V\tau_+ = A^*, \\ B - V\tau_+ &= B, \quad B^* - V\tau_- = B^* \end{aligned} \quad (2)$$

and

$$\begin{aligned} K_{00} &= \int \varphi_0^*(k) K \varphi_0(k) dk, \\ K_{0s} &= \int \varphi_0^*(k) K \varphi_s(k) dk, \\ K_{s0} &= \int \varphi_s^*(k) K \varphi_0(k) dk, \end{aligned} \quad (3)$$

with

$$\varphi_0(k) = \frac{1}{K} - \frac{1}{K\sqrt{K}}, \quad K^2 = \int \frac{d^3k}{K^3}, \quad K = \sqrt{k^2 + \mu^2} \text{ and } S = \sqrt{s^2 + \mu^2}, \quad (4)$$

and $V=g\mathbf{K}$ is the effective coupling constant and μ the rest mass of π -meson, $\varphi_s(\mathbf{k})$ are determined by

$$(K-S)\varphi_s(\mathbf{k})=\varphi_0(\mathbf{k})\int\varphi_0^*(\mathbf{k})K\varphi_s(\mathbf{k})d\mathbf{k} \quad (5)$$

with outgoing wave boundary condition and form a complete orthonormal set of functions together with $\varphi_0(\mathbf{k})$. A (or B) and a_s (or b_s) are the annihilation operators of positive (or negative) charged bound and unbound meson with wave vector s and satisfy the commutation relations

$$[A, A^*]=[B, B^*]=1,$$

$$[a_s, a_{s'}^*]=\delta(s-s') \text{ etc.}$$

The first term of (1) describes the "clothed" nucleon and its possible isobars and must be treated numerically to determine the levels in Fig. 1. The last term of (1) is the Hamiltonian of the s -meson and the others are the interaction Hamiltonian of the clothed nucleon and s -meson. They play an essential rôle in our process, the term $K_{0s'} B^* b_{s'}$, for example, describes the absorption of negative s -meson with wave vector s' and simultaneous transition of clothed nucleon.

Here we restrict ourselves only to the problem of π^-p collisions. As discussed in § 1, we must take account of three levels indicated in Fig. 1. The process of the double meson production occurs in the following way:

- 1) The incident π^- with wave vector \mathbf{p} is absorbed by the proton thereby exciting it to the excited neutron level N' . This transition matrix element is denoted by $\beta^* = (N'|B^*|P)$.
- 2) N' falls down to the N^- isobar level emitting a positive s -meson. This matrix

element is $\gamma = (N^-|A|N')$.

- 3) Subsequently N^- falls down to the ground level, with the matrix element $\alpha = (N|B|N^-)$. The matrix element γ is not much smaller than the other matrix elements; for example $\gamma=0.558$ for $V^2=1$. Therefore we cannot neglect γ in comparison with α and β^* . We assign s or s' for a positive or negative s -mesons and 0, 1, 2 for nucleon levels in the order of their energy values. The wave function of our system is given by

$$\Psi = \phi_2^0 |0\rangle_2 + \int \phi_1^s |s\rangle_1 d\mathbf{s} + \int \phi_0'^{s'} |s'\rangle_0 d\mathbf{s}' + \int \phi_0^{ss'} |ss'\rangle_0 d\mathbf{s} d\mathbf{s}', \quad (6)$$

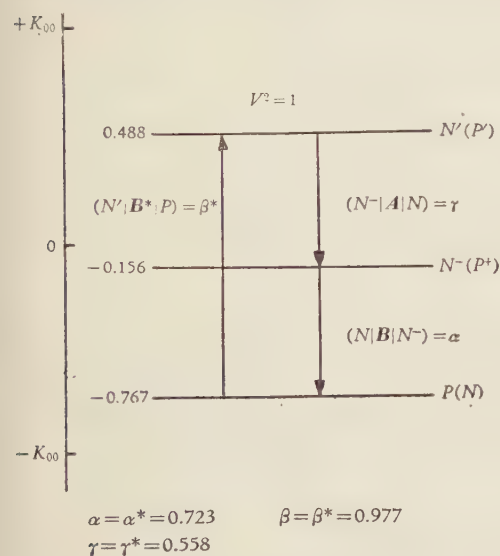


Fig. 1

where ϕ 's are the probability amplitudes and $|>$'s are the state vectors. For example, $\phi_0^{ss'}$ means the probability amplitude of the state, represented by $|ss'>_0$, where the nucleon is in the ground level and there are two s -mesons. We put (6) into the Schrödinger equation

$$H\mathcal{F}=E\mathcal{F}, \quad (7)$$

and we get four integral equations as follows :

$$(K_{00}\Omega_{20}-E_0)\phi_2^0+\gamma^*\int K_{0s}\phi_1^s ds+\beta^*\int K_{0s'}\phi_0^{ss'} ds'=0, \quad (8)$$

$$(S+K_{00}\Omega_{10}-E_0)\phi_1^s+\gamma K_{s0}\phi_2^0+\alpha^*\int K_{0s'}\phi_0^{ss'} ds'=0, \quad (9)$$

$$(S-E_0)\phi_0^{ss'}+\beta K_{s'0}\phi_2^0=0, \quad (10)$$

$$(S+S'-E_0)\phi_0^{ss'}+\alpha K_{s'0}\phi_1^s=0, \quad (11)$$

with $E-K_{00}\Omega_0=E_0$ and $\Omega_{n0}=\Omega_n-\Omega_0$, where Ω_n is the n -th eigenvalue of Ω .

Next we solve these equations under such boundary conditions that the initial state consists of a plane wave and a scattered wave and the final state only of outgoing waves. From (10), (11) we get

$$\phi_0^{s'}=\delta(s'-p_0)-\beta K_{s'0}\delta_+(S'-E_0)\phi_2^0, \quad (10')$$

$$\phi_0^{ss'}=-\alpha K_{s'0}\delta_+(S+S'-E_0)\phi_1^s \quad (11')$$

and substitute (10') into (8). We obtain

$$(K_{00}\Omega_{20}-E_0-|\beta|^2\int K_{0s'}K_{s'0}\delta_+(S'-E_0)ds')\phi_2^0+\gamma^*\int K_{0s}\phi_1^s ds+\beta^*K_{0p_0}=0. \quad (8')$$

Using the next relation (5.5') in M.S.T.,

$$\int K_{0s'}K_{s'0}\delta_+(S'-E_0)ds'=K_{00}-E_0-1/\int\delta_+(K-E_0)\varphi_0^2(k)dk, \quad (12)$$

we can rewrite the coefficient of ϕ_2^0 as follows ;

$$\begin{aligned} & K_{00}\Omega_{20}-E_0-|\beta|^2(K_{00}-E_0-1/\int\delta_+(K-E_0)\varphi_0^2(k)dk) \\ & =|\beta|^2(1+I(E_0)F_2(E_0))/I(E_0) \end{aligned} \quad (13)$$

with

$$F_2(E_0)=[K_{00}(\Omega_{20}-|\beta|^2)+E_0(|\beta|^2-1)]/|\beta|^2 \quad (14)$$

and

$$I(E_0)=\int\delta_+(K-E_0)\varphi_0^2(k)dk. \quad (15)$$

The bracket term in the left side of (13) contains both real and imaginary parts. The former shows the energy shift of the level 2 through the s -meson-nucleon interaction and the latter shows that the level 2 becomes unstable by emitting one s -meson. Then (8') becomes

$$|\beta|^2(F_2(E_0)+1/I(E_0))\phi_2^0+\gamma^*\int K_{0s}\phi_1^s ds+\beta^*K_{0p_0}=0. \quad (8'')$$

Substituting (11') into (9), we get

$$|\alpha^2| (F_1(Q) + 1/I(Q)) \phi_1^s + \gamma K_{s0} \phi_2^0 = 0 \quad (16)$$

with

$$F_1(Q) = [K_{00}(\Omega_{10} - |\alpha^2|) + Q(|\alpha^2| - 1)]/|\alpha^2| \quad (17)$$

and

$$Q = E_0 - S. \quad (18)$$

It should be noted that the term $1 + I(Q)F_1(Q)$ may have a zero point because $I(Q)$ is real for $Q < \mu$. If $1 + I(Q)F_1(Q)$ has a zero point, the level 1 is stable. As regards this question, we shall discuss in § 4. Considering this fact we obtain from (16)

$$\phi_1^s = -(\gamma/|\alpha^2|) K_{s0} I(Q) \partial_+ (1 + I(Q)F_1(Q)) \phi_2^0, \quad (19)$$

and the substitution of (19) into (8'') yields

$$\phi_2^0 = -(\beta^*/|\beta^2|) K_{0p0} (1/X(E_0)) \quad (20)$$

with

$$X(E_0) = F_2(E_0) + 1/I(E_0) - (\gamma/|\beta^2| |\alpha^2|) \int K_{s0} K_{s0} I(Q) \partial_+ (1 + I(Q)F_1(Q)) ds. \quad (21)$$

Finally we get from (19), (20), (10') and (11')

$$\phi_1^s = (\gamma\beta^*/|\alpha^2||\beta^2|) (K_{0p0}/X(E_0)) K_{s0} I(Q) \partial_+ (1 + I(Q)F_1(Q)), \quad (22)$$

$$\phi_0^{s'} = \delta(s' - p_0) + (K_{0p0}/X(E_0)) K_{s'0} \partial_+ (S' - E_0), \quad (23)$$

$$\phi_0^{ss'} = (\gamma/\alpha\beta) (K_{0p0}/X(E_0)) K_{s0} K_{s'0} I(Q) \partial_+ (1 + I(Q)F_1(Q)) \partial_+ (S + S' - E_0). \quad (24)$$

§ 3. Derivation of scattering amplitudes

In § 2 we obtained the amplitudes $\phi_0^{s'}$, ϕ_1^s and $\phi_0^{ss'}$ in s -representation. As is shown in the Appendix there are general relations between the amplitudes in s -representation and in k -representation.

a) Ordinary scattering $\pi^- + p \rightarrow \pi^- + p$.

We get at once from (23)

$$R^{ord}(p_0) = K_{0p0}(\varphi_0(p_0) + K_{0p0}/X(E_0)), \quad (25)$$

using the relations (5.2), (5.3) and (5.4) in M.S.T. When we put $\gamma=0$ and use α and Ω_{10} instead of β and Ω_{20} , the expression (25) becomes just equal to (9.1) in M.S.T.

b) Double meson production $\pi^- + p \rightarrow \pi^- + \pi^+ + n$.

Using (A.6), we get

$$R^d(K, K') = (\gamma/\alpha\beta) (K_{0p0}/X(E_0)) \{K_{0k}/(1 + I(K)F_1(K))\} I(K') K_{0k'}, \quad (26)$$

where $K + K' = E_0$.

From (25), (26) we obtain the total cross-sections $\sigma^{ord}(E_0)$ and $\sigma^d(E_0)$ as follows:

$$\sigma^{ord}(E_0) = (4\pi^2)^2 4\pi E_0^2 |K_{0p0}(\varphi_0(p_0) + K_{0p0}/X(E_0))|^2, \quad (27)$$

$$\sigma^d(E_0) = (4\pi)^2 (4\pi)^2 (E_0/p_0) \left| \langle \gamma/\alpha_1 \hat{\gamma} \rangle (K_{0\mu_0}/X(E_0)) \right|_1^2 \int_{\mu}^{E_0-\mu} dK' kK k'K' |I(K') K_{0k} K_{0k'}| \times 1/(1+I(K)F(K))|^2. \quad (28)$$

The total cross-sections for two values of the cut off momentum: $k_m = 4\mu$ and $k_m = 6.65\mu$, are shown in Fig. 2.

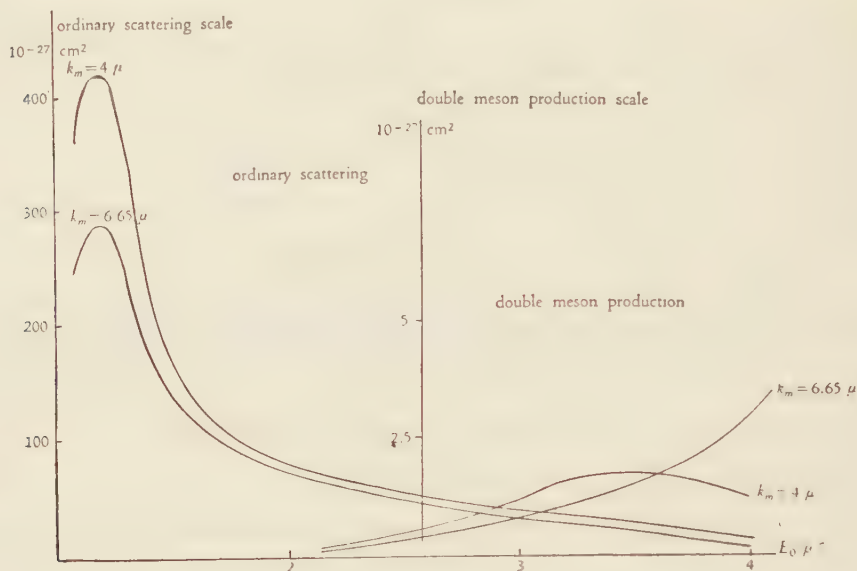


Fig. 2

The results depend on the cut off momentum k_m . For $k_m = 4\mu$ the level 1 becomes stable and therefore the charge-exchange scattering occurs.

In our model the total cross section of double meson production σ^d is not too small in comparison with that of ordinary scattering σ^{ord} . Moreover, for $k_m = 4\mu$ there is a maximum at $E_0 \approx 3.5\mu$. The level distance between the level 0 and 2 is almost equal to 3.5μ . At first sight the double meson production seems to be due to the resonance with the isobar level. But checking up the calculations, we find that the rise of the cross section is accidental.

The curve of σ^{ord} has a sharp peak at $E_0 \approx 1.2\mu$. The difference between our paper and M. S. T. is ascribed to the model: the charged scalar in our case and the charged longitudinal vector meson field in M.S.T.

§ 4. Is the level N^- stable?

If $1 + I(Q)F_1(Q)$ has a zero point in the region $Q < \mu$, it happens that a positive meson is scattered out and then the level N^- is stable and cannot emit one more meson. Therefore the charge-exchange scattering $\pi^- + p \rightarrow \pi^+ + N^-$ can arise. The amplitude $R^{ech}(p_1)$ and the total cross-section $\sigma^{ech}(E_0)$ are given from (22) by

$$R^{ech}(\mathbf{p}_1) = (\gamma/|\alpha|^2\beta) (K_{0j0}/X(E_0)) \{K_{0j1}/(1+I(Q)F_1(Q))'\}, \quad (29)$$

$$\sigma^{ech}(\mathbf{p}_1) = (4\pi^2)^2 4\pi (E_0 E_1 p_1/p_0) |R^{ech}(\mathbf{p}_1)|^2, \quad (30)$$

where $Q=E_0-E_1$ and $(1+I(Q)F_1(Q))'=\partial(1+I(Q)F_1(Q))/\partial Q|_{Q=E_0-E_1}$, and E_1 is the energy and \mathbf{p}_1 is the momentum satisfying $1+I(E_0-E_1)F_1(E_0-E_1)=0$. The value of σ^{ech} is thousand times smaller than that of ordinary scattering σ^{ord} .

Whether the level 1 is stable or not does not depend on the energy and charge of the incident meson, nor, on the special process considered. In order to understand this fact, we shall examine the following eigenvalue problem;

$$H\mathcal{P}=E\mathcal{P}, \quad (31)$$

assuming the wave function

$$\mathcal{P}=\phi_1^0|0\rangle_1+\int\phi_0^{s'}|s'\rangle_0ds'. \quad (32)$$

If an eigenvalue E satisfying the relation $E<\mu$ exists, the level N^- will be stable. From (31) we get

$$(E-K_{00}\mathcal{Q}_{10})\phi_1^0+\alpha^*\int K_{0s'}\phi_0^{s'}ds'=0, \quad (33)$$

$$(E-S')\phi_0^{s'}+\alpha K_{s'0}\phi_1^0=0. \quad (34)$$

Under the boundary condition that there is no incident meson, we solve (34) and substitute $\phi_0^{s'}$ into (33). Then we get

$$[E-K_{00}\mathcal{Q}_{10}+|\alpha|^2\int K_{s'0}K_{0s'}\delta_+(S'-E)ds']\phi_1^0=0, \quad (35)$$

or using (8''), (15) and (17)

$$\{(1+I(E)F_1(E))/I(E)\}\phi_1^0=0. \quad (35')$$

Therefore when $1+I(E)F_1(E)$ has a zero point, a non-vanishing solution ϕ_1^0 exists. If it is so, the eigenvalue E must be smaller than μ because the function $I(E)$ is real for $E\leq\mu$ and complex for $E>\mu$. Hence the wave function does not extend so far. Thus the level N^- is originally stable. The coefficient of ϕ_1^0 in (35') is the same as that of ϕ_1^s in (16). When N^- level is stable in π^-p collisions, the fact is that N^- level is originally stable.

Whether $1+I(Q)F_1(Q)$ has a zero point in the region $E<\mu$ or not depends on the effective coupling constant V and the cut off momentum k_m . From the numerical calculations it is found that there is no pole for $k_m=6.65\mu$ and $V^2=1$, but one pole at $E<\mu$ for $k_m=4\mu$ and $V^2=1$. So long as $k_m<4\mu$, there is always one pole. Experimentally we cannot find such a stable level. To say inversely, there is some physical condition between the coupling constant of meson nucleon interaction and the cut off momentum. The allowable region of the cut off momentum k_m that we must take to prohibit the existence of the stable isobar levels varies corresponding to the value of V^2 , as shown in Fig. 3(a). The relation between g and V is also shown in Fig. 3(b). From these figures we find that the value of g cannot be too large for moderate coupling strength of V .

Such a result is due to the physical restriction: "The existence of the stable isobar level

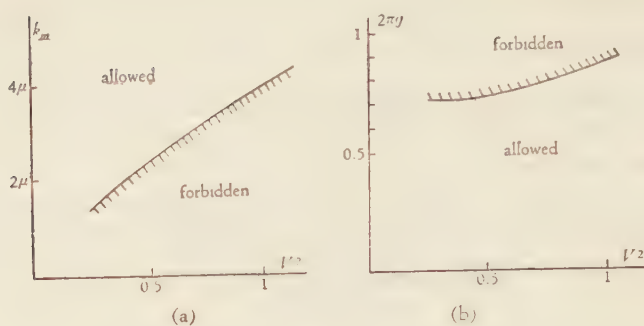


Fig. 3

is forbidden". Recently Sawada⁶⁾ treated the problem of nuclear force in the cut off theory and showed that the value of the coupling constant g must be restricted by the physical condition that the nucleon dissociation probability lies between 0 and 1. From these facts we can infer that in the cut off theory there must be always some physical condition between the coupling constant and the cut off momentum.

Appendix

The scattering amplitudes in k -representation can be derived as follows: In order to go back to k -representation from s -representation, we use the next relations;

$$\phi(k') = \int \phi^{s'} \varphi_{s'}(k') ds', \quad (\text{A} \cdot 1)$$

$$\phi(k) = \int \phi^s \varphi_s(k) ds, \quad (\text{A} \cdot 2)$$

$$\phi(k, k') = \int \phi(s, k') \varphi_s(k) ds = \int \int \phi^{s's'} \varphi_{s'}(k') \varphi_s(k) ds ds', \quad (\text{A} \cdot 3)$$

with

$$\varphi_s(k) = \delta(k-s) + K_{0s} \delta_+(K-S) \varphi_0(k), \quad (\text{A} \cdot 4)$$

which is the solution of (5) given by (4.5) in M.S.T. As is well known the following relations exist:

$$\begin{aligned} \int e^{ikr} \delta_+(K-E_0) F(k) dk &\approx \frac{4\pi}{2ir} \int e^{ikr} \delta_+(K-E_0) F(k) K dk \\ &\approx \frac{4\pi^2}{r} E_0 F(p_0) e^{ip_0 r}, \end{aligned} \quad (\text{A} \cdot 5)$$

where $F(k)$ is regular and \approx means the neglect of the terms of order $1/r^2$ and higher⁷⁾. Therefore the wave amplitude $\phi(r)$ in configuration space is given by

$$\begin{aligned} \phi(r) &= \int \phi(k) e^{ikr} dk = \int \int \phi^s \varphi_s(k) e^{ikr} dk ds \\ &= \int \phi^s ds \int e^{ikr} [\delta(k-s) + \delta_+(K-S) K_{0s} \varphi_0(k)] dk \\ &= \int \phi^s ds [e^{isr} + (4\pi^2/r) e^{i p_0 r} S K_{0s} \varphi_0(s)]. \end{aligned}$$

Now, when $\phi^s = \partial_+(S - E_0)F(S)$ is given in s -representation we get using (A.5)

$$\phi(\mathbf{r}) = (4\pi^2/r) e^{ip_0 r} E_0 F(E_0) [1 + 8i\pi^2 p_0 E_0 K_{0p_0} \varphi_0(\mathbf{p}_0)],$$

which corresponds to $\phi(\mathbf{k}) = \partial_+(K - E_0)R(K)$ with

$$R(E_0) = F(E_0) [1 + 8i\pi^2 p_0 E_0 K_{0p_0} \varphi_0(\mathbf{p}_0)]. \quad (\text{A.6})$$

This is the required relation.

In the general case of $\phi^s = \partial_+(f(S))F(S)$, where $f(S)$ has only one zero point at $S = E_1$, we get easily

$$R(E_1) = \frac{F(E_1)}{f'(E_1)} [1 + 8i\pi^2 p_1 E_1 K_{0p_1} \varphi_0(\mathbf{p}_1)], \quad (\text{A.7})$$

with

$$f'(E_1) = \partial f(S) / \partial S|_{S=E_1}.$$

In the case of charge-exchange scattering we can apply (A.7) and obtain at once:

$$\begin{aligned} R^{sch}(\mathbf{p}_1) &= (\gamma/|\alpha^2|\beta) (K_{0p_1}/X(E_0)) \{K_{p_10} [1 + 8i\pi^2 p_1 E_1 \varphi_0(\mathbf{p}_1) K_{0p_1}] \\ &\quad / [1 + I(E_0 - E_1) F_1(E_0 - E_1)]\} \\ &= (\gamma/|\alpha^2|\beta) (K_{0p_1}/X(E_0)) (K_{0p_1} / [1 + I(E_0 - E_1) F_1(E_0 - E_1)]), \end{aligned} \quad (\text{A.8})$$

using

$$K_{0s} = K_{s0} [1 + 8i\pi^2 s S K_{0s} \varphi_0(\mathbf{s})]. \quad (\text{A.9})$$

In the case of double meson production we first change s' into k using (A.11) in M.S.T.,

$$\begin{aligned} \phi(s, \mathbf{k}') &= \int \phi_0^{ss'} \varphi_{s'}(\mathbf{k}') ds' \\ &= -(\gamma/\alpha\beta) (K_{0p_0}/X(E_0)) K_{s0} \partial_+(1 + I(Q) F_1(Q)) \varphi_0(\mathbf{k}') \\ &\quad + (\gamma/\alpha\beta) (K_{0p_1}/X(E_0)) K_{s0} \partial_+(1 + I(Q) F_1(Q)) \partial_+(K' - E_0 + S) \\ &\quad \times [\varphi_0(\mathbf{k}') / \{\partial_+(K' - E_0 + S) \varphi_0^2(\mathbf{k}') dk'\}]. \end{aligned} \quad (\text{A.10})$$

Since the energy K' of the scattered negative s -meson must satisfy the condition

$$\mu < K' < S_{max} = E_0 - \mu,$$

only the second term contributes to double meson production. When $1 + I(Q) F_1(Q)$ is complex, $\partial_+(1 + I(Q) F_1(Q))$ means $[1 + I(Q) F_1(Q)]^{-1}$. Then the coefficient of $\partial_+(K' - E_0 + S)$ as a function of S is regular and using (A.6) we get

$$\begin{aligned} R^l(K, K') &= (\gamma/\alpha\beta) (K_{0p_0}/X(E_0)) \varphi_0(\mathbf{k}') \\ &\quad \times \frac{K_{K0} [1 + 8i\pi^2 k K \varphi_0(\mathbf{k}) K_{0p_0}]}{[1 + I(E_0 - E_1) F_1(E_0 - E_1)] \{\partial_+(K'' - K') \varphi_0^2(\mathbf{k}'') dk''\}} \\ &= (\gamma/\alpha\beta) (K_{0p_0}/X(E_0)) \{K_{0k'} K_{0k} I(K') / [1 + I(K) F_1(K)]\}, \end{aligned} \quad (\text{A.11})$$

with

$$K + K' = E_0.$$

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Letters to the Editor

Formal Theory of Nuclear Direct Interaction in Nuclear Scattering

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Formal theories of the elastic scattering of nucleons with complex nuclei, developed by Watson¹⁾ and by Francis and Watson²⁾, are extended so as to include the quasi-elastic scattering by nuclei, that is, the direct interaction in nuclear reactions.³⁾

Such process has been treated by Austern, Butler and McManus⁴⁾, by Hayakawa, Kawai and Kikuchi⁵⁾, by Hayakawa and Sasakawa⁶⁾ and by Hayakawa and Yoshida.⁷⁾ A-B-M's theory, in which the impulse approximation was made use of, was based upon the physical assumption, similar to the case of the deuteron stripping reaction, that the process takes place only near the boundary of the nucleus. On the other hand, H-K-K treated this process by using the semi-classical Monte-Carlo method, the quantum mechanical description of which was developed in H-S's, where the incident and emitted nucleon waves were described by the ones distorted by a real potential and the nuclear force was taken into account to the first order. One of the purposes of the present note is to point out the necessity of complex potential well, which is given by the elastic scattering experiments, rather than real one in describing the interaction of the incident and emitted waves with nucleus.

The total Hamiltonian of a system is separated into three terms,

$$H = H_0 + \sum_{k=1}^A V_k = H_N + T + \sum_{k=1}^A V_k, \quad (1)$$

where H_N , T and V_k are the nuclear Hamiltonian, the kinetic energy operator of the incident particle and its interaction potential with the k -th nucleon in the nucleus, respectively.

Following Watson's^{1), 2)} argument, we can replace V_k by t_k , t -matrix of Lippmann and Schwinger,³⁾ which is given by

$$t_k = V_k + V_k (E + i\epsilon - H_0)^{-1} t_k, \quad (2)$$

provided the number of nucleons in the nucleus is large.

The initial and final states are

$$\begin{aligned} H_0 \Phi_i &= E \Phi_i, & \Phi_i &= g_N \exp\{i\mathbf{k} \cdot \mathbf{r}\}, \\ H_0 \Phi_f &= E \Phi_f, & \Phi_f &= g_N' \exp\{i\mathbf{k}' \cdot \mathbf{r}\}, \end{aligned} \quad (3)$$

where g_N , \mathbf{k} , g_N' and \mathbf{k}' are the nuclear ground state wave function, the wave vector for the incident nucleon, the nuclear excited state wave function and the wave vector for the emitted nucleon, respectively.

We divide the interaction $t = \sum_{k=1}^A t_k$ into two parts,

$$t = t_c^a + I^a, \quad (4)$$

t_c^a is the diagonal part of t for nuclear state a and I^a the corresponding non-diagonal part. Therefore, for the initial state $t = t_c^i + I^i$, and for the final state $t = t_c^f + I^f$.

The wave equation for this system is described as follows,

$$\begin{aligned} \Psi^{(+)} &= \Phi_i + (E + i\epsilon - H_0)^{-1} t \Psi^{(+)} \\ &= [1 + (E + i\epsilon - H)^{-1} t] \Phi_i \\ &= [1 + (E + i\epsilon - H)^{-1} I^i] \\ &\quad \times [1 + (E + i\epsilon - H_0 - t_c^i)^{-1} t_c^i] \Phi_i \\ &= [1 + (E + i\epsilon - H)^{-1} I^i] \varphi_i^{(+)}, \end{aligned} \quad (5)$$

where $\varphi_i^{(+)}$ satisfies the following equation,

$$\varphi_i^{(+)} = \Phi_i + (E + i\epsilon - H_0)^{-1} t_c^i \varphi_i^{(+)}. \quad (6)$$

Using the formal relation,

$$\begin{aligned} (E + i\epsilon - H_0 - B)^{-1} A \\ = (E + i\epsilon - H_0)^{-1} [A + B(E + i\epsilon - H_0 - B)^{-1} A], \end{aligned} \quad (7)$$

(5) can be written as follows,

$$\begin{aligned} \Psi^{(+)} &= \varphi_i^{(+)} + (E + i\epsilon - H_0 - t_c^f)^{-1} \\ &\quad \times [I^i + I^f (E + i\epsilon - H)^{-1} I^i] \varphi_i^{(+)} \\ &= \Phi_i + (E + i\epsilon - H_0)^{-1} t_c^i \\ &\quad \times [1 + (E + i\epsilon - H_0 - t_c^i)^{-1} t_c^i] \Phi_i \\ &\quad + (E + i\epsilon - H_0)^{-1} [1 + t_c^f (E + i\epsilon - H_0 - t_c^f)^{-1}] \end{aligned}$$

$$\begin{aligned} & \times [I^i + I^f (E + i\epsilon - H)^{-1} I^i] \\ & \times [1 + (E + i\epsilon - H_0 - t_c^i)^{-1} t_c^i] \Phi_i. \end{aligned} \quad (8)$$

Therefore, the transition matrix element between Φ_i and Φ_f becomes

$$\begin{aligned} T_{if} = & (\Phi_f, t_c^i [1 + (E + i\epsilon - H_0 - t_c^i)^{-1} t_c^i] \Phi_i) \\ & + (\Phi_f, [1 + t_c^f (E + i\epsilon - H_0 - t_c^f)^{-1}] \\ & \times [I^i + I^f (E + i\epsilon - H)^{-1} I^i] \\ & \times [1 + (E + i\epsilon - H_0 - t_c^i)^{-1} t_c^i] \Phi_i). \end{aligned} \quad (9)$$

The first term which appears only for $|k| = |k'|$ represents the potential scattering and may correspond to the shape elastic scattering of Feshbach, Porter and Weisskopf.⁹⁾ The second term can be written as

$$(\varphi_f^{(-)}, [I^i + I^f (E + i\epsilon - H)^{-1} I^i] \varphi_i^{(+)}), \quad (10)$$

with

$$\varphi_f^{(-)} = \Phi_f^{(-)} (E - i\epsilon - H_0)^{-1} t_c^{f\dagger} \varphi_f^{(-)},$$

which includes all possible processes connecting the initial state i with the final state f . The transition matrix element of the quasi-elastic scattering by the nucleus is, therefore, given by the first term of (10),

$$T_{if}^{\text{quasi}} = (\varphi_f^{(-)}, I^i \varphi_i^{(+)}) = (\varphi_f^{(-)}, I^f \varphi_i^{(+)}). \quad (11)$$

The last equality can easily be verified.

The equation (11) may be interpreted to show the physical situation that an incident particle suffering the forward scatterings by nucleons in a nucleus without momentum transfer makes a collision with momentum transfer and then it leaves the nucleus suffering the succession of the forward scatterings again.

t_c^a is proportional to the forward scattering amplitude $f(0^\circ)$ of an incident nucleon by a bound nucleon in the nucleus in a state a , and it can be represented, as is well known, in terms of a complex optical potential, since the wave function of the nucleus in any state is anti-symmetrical.²⁾ Therefore, t_c^i can be replaced by a complex optical potential which describes the elastic scattering of the incident nucleon. But, for t_c^f , the situation may be slightly complicated. If the excitation is distributed rapidly over the whole nucleus, namely, the struck nucleon which has received an energy from the incident one shares its energy among other nucleons within the nucleus during the collision time, the imaginary part of the potential of the final state may increase appreciably, because the effect of "the Pauli

principle" which diminishes the imaginary part of $f(0^\circ)$ i.e. $k/4\pi$ times total cross section for scattering of the nucleon by the bound nucleon, may be considered to be less important than in the case of the ground state. If one may ignore such circumstances, t_c^f will also be replaced by an empirical one, determined by the observations of the elastic scattering of an incident nucleon which has the equal energy to that of the finally emitted nucleon.

The optical potential presented here is only an approximate one, since the coherent parts are also contained in the term, $I(E + i\epsilon - H)^{-1} I$. Strictly speaking, the calculation of T_{if} by means of (11) using the conventional optical potential takes the collision doubly into account, for the imaginary part of the optical potential implies also the reduction of the incident wave due to the collision process under consideration. This effect, however, is considered to be negligible for large A . Details connected with the concerned problems will be published later.

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The Magnetic Field in the Vicinity of the Solar System

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The purpose of this note is to examine briefly one of the consequences of the assumption that there exists, in the vicinity of the solar system, a general relatively uniform magnetic field such as has been suspected^{1,2)} to lie along the galactic spiral arms. Davis³⁾ has discussed this matter with relation to the sidereal time variation of the intensity of cosmic-ray particles of galactic origin. In the present note we attempt to set an upper limit on the magnitude of the field from considerations of the behavior of low energy cosmic-ray particles of solar origin. Treiman⁴⁾, from similar considerations, has deduced an upper limit for the solar magnetic dipole moment.

Firor⁵⁾ has re-examined some of the trajectories of charged particles incident upon the magnetic dipole field of the earth from the direction of the sun. He has shown that cosmic-ray intensity-increases, associated with solar flares, are in general agreement with his analysis under the assumptions that the particles, if positive and singly charged, have momenta in the region 1-10 Bev/c, and approach the earth from a region if the sky not larger than 60 degrees in longitude and 30 degrees in latitude centered at the sun. The galactic magnetic field cannot be so large as to deflect the trajectories out of this region, if these particles do, indeed, originate at or near the sun. Astronomical evidence, based mostly upon the polarization of starlight, suggests that the spiral arm which contains or is near to the solar system, lies roughly in the direction $\delta=35$ degrees, $\alpha=20$ hours, where δ is the declination and α is the right ascension. This is the direction assumed by Davis³⁾. If we take B , the field, to lie along the spiral arm, about February 1 and August 1 B lies nearly in a meridian plane of the earth containing R , the earth-sun line, and makes an angle of 53 degrees (or 127 degrees) with R . At other

times of the year B lies more nearly perpendicular to R . A simple calculation which takes account of the deflection by B as well as the deflection⁵⁾ in the earth's field shows that a field of $3 \cdot 10^{-6}$ gauss is sufficient to displace the appropriate (4 A.M.) impact zone, during the months of February or August, 45 and 30 degrees west, respectively. If B has a direction opposite to that assumed, these shifts are 30 and 45 degrees east, respectively. At other times of the year B is more nearly perpendicular to R and these shifts are greater. The value $3 \cdot 10^{-6}$ gauss, then, we regard as a conservatively-set upper limit.

This upper limit cannot be raised much by assuming other directions for B . If B lies more nearly perpendicular to the solar plane, the upper limit becomes smaller. If B lies nearer the solar plane, particles can approach the earth practically undeflected for a few weeks twice a year, but during the remaining months they would come from somewhat displaced longitudes and more importantly from displaced latitudes. For the 4 A.M. impact zone, a 15 degree shift in the latitude of incidence produces a shift of about 30 degrees in the longitude of the impact zone. In summary, it seems impossible for B to be as large as 10^{-5} gauss, and it is very difficult to see how it can be more than a few times 10^{-6} gauss, regardless of its orientation.

We have, of course, taken no account of a possible solar dipole field, and must remark that it is possible that the galactic field and that from a solar dipole are orientated in such a way that their combined effect is small. It is perhaps conceivable that the galactic field has in some way influenced the formation of the solar dipole field, and so, if these fields do tend to cancel, the tendency for cancellation would not necessarily be accidental. But it would be accidental for these fields to cancel in just such a way as to produce small net deflections of charged particles approaching the earth from the sun.

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High Latitude Impact Zones for Solar Cosmic Rays

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Firor¹⁾ has pointed out that while his assumption of a simple dipole to represent the earth's magnetic field accounts for most of the features of cosmic-ray intensity-increases accompanying solar flares, this

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assumption does not account for the increases observed at high latitudes, notably those observed at Godhaven (80 degrees N) and Rolute (83 degrees N). We wish to suggest here that a galactic magnetic field, of the type suspected to lie along the galactic spiral arms, can, under certain assumptions, explain the phenomena at least in part.

We shall assume, as in the preceding note, that the field lies along the spiral arm, and further that it lies in the direction that leads to the center of the galaxy. Let Q be a plane perpendicular to the solar plane and along the earth-sun line. Then all (positive) particles at all times of the year arrive from the west side of the plane Q . The trajectories fall into two groups. The first, group I, involves helices with opening angles from 0 to 2π . High momentum particles arrive along $-R$, where R is the earth-sun vector, and along $-(R \times B) \times B$.

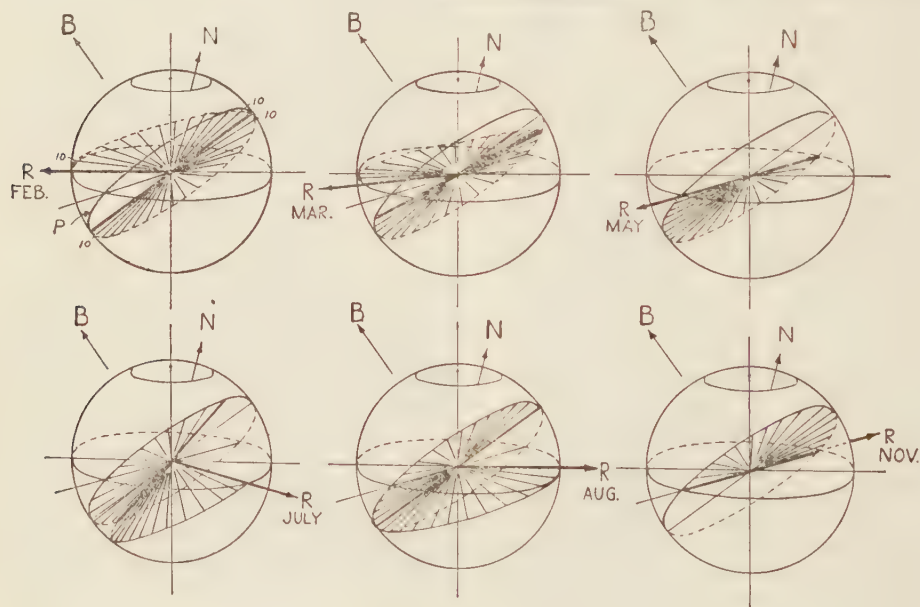


Fig. 1.

Trajectory directions as they approach the earth's dipole field from the sun, having been deflected by a uniform magnetic field, B . The equatorial plane of the coordinate system coincides with the plane of the earth's orbit. P is a plane perpendicular to B , and R is the position vector of the sun at the beginning of the indicated month. Group I trajectories approach from between R and $(R \times B) \times B$, and from the west. Group II trajectories approach from between $-(R \times B) \times B$ and $(R \times B) \times B$ and from the west, and all lie very nearly in the plane P . In general particles of a given momenta can approach from essentially four directions. Approximate directions are shown for 10 Bev/c particles and a field of 10^{-6} gauss.

If the direction of B is reversed from that assumed, the trajectories are all the same, but approach from the east instead of the west.

A second group of trajectories, group II, arrives very nearly in a plane perpendicular to \mathbf{B} , plane P , and involves helices with opening angles greater than 2π . High momentum particles arrive along $\pm(\mathbf{R} \times \mathbf{B}) \times \mathbf{B}$. The general features of these trajectories are shown qualitatively for various times of the year in Figure 1. Now from an inspection of Figures 3 and 4 of Firor's paper, it is clear that particles impinging on the earth's field at latitudes greater than 22 degrees contribute to the (vertical) intensity near the top of the atmosphere only after traversing the type trajectories in the earth's field that do not cross the geomagnetic equator. These trajectories arrive at the earth 20 or 30 degrees west of the longitude of the direction of incidence and in general at latitudes appreciably greater than the latitude of the direction of incidence. Hence, from our Figure 1 and from Firor's Figures 3 and 4, one concludes that if a galactic magnetic field of the type suggested by astronomical evidence exists, *and if particles can be emitted from the sun over a wide range of angles*, there should be high latitude impact zones. One would expect these impact zones to undergo rather complicated seasonal effects, and these seasonal effects should depend strongly upon the direction of the magnetic field.

Professor S. Hayakawa has brought to the author's attention the suggestion made by K. Murakami⁽²⁾ that the time delays which are perhaps apparent in the arrival of particles at high latitudes⁽³⁾ could be due to these particles having traveled long distances in a galactic field. Indeed, delays of 20 minutes to an hour or more are to be expected for

10 BeV/c particles traveling over trajectories of the sort we have discussed here.

Here, as in the preceding note, we have taken no account of a possible solar dipole field. Further, we have had to assume that at the time of a solar flare particles are emitted over a wide range of angles from the sun. (The sketches in Figure 1 may be taken either as trajectories arriving at the earth or leaving the sun.) The extent to which this latter assumption is correct is not known. One large cosmic-ray-increase (March 7, 1942), which occurred during a period of solar activity, was not preceded by a visually observed flare, although it was accompanied by the usual radio fade-out. It is possible that the flare occurred on the hidden side of the sun, in which case one could conjecture that particles can be emitted over a wide range of directions. On the other hand it is also possible that the flare was present but unobserved, and we cannot rule out the possibility that the solar emission of cosmic rays is not always preceded by a flare.

The author wishes to express his appreciation for the kind hospitality of Professor Y. Watase and other members of the cosmic ray group at Osaka City University where the author has been a visiting Fulbright professor during the academic year 1954-55.

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On the Gamma-Transition of Nuclei

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Using A. Bohr's collective model, the gamma-transition of nuclei is studied by the approximations of weak and strong couplings. In the $M4$ -transition, the agreement of experimental data with theoretical results is in general much better on this model than one predicted on other nuclear models, if the strong coupling approximation is adopted.

The results of these two approximations are compared with one another. The difference between them comes from contributions due to a deformed core. These contributions, and consequently the difference between the weak and strong coupling approximations, increase with the order of multipolarity of radiation.

In $E3$ -transition (j -forbiddenness), such as those between $p_{1/2}$ and $7/2+$ states, it is necessary to take into account the nuclear configuration mixing due to the surface coupling. Especially, the coupling with deformations of order three is important for these transitions.

§ 1. Introduction

The present knowledge of the lower energy levels of nuclei has been obtained mostly from the study of the gamma-ray emitted when nuclei make transitions from excited states to lower states.

Especially, the gamma-transition between isomeric states has been investigated by many authors, and various attempts to explain an observed life time have been made. The probability of the gamma-ray emission depends not only on its multipolarity and transition energy, but also on the detailed structure of the nucleus concerned. Accordingly, if one knows the order of multipolarity of the transition from the internal conversion, the angular correlation and other phenomena, one can get some knowledge concerning the nuclear structure as well as the transition mechanism by comparing the experimental value of the life time with the theoretical one.

Weisskopf⁽¹⁾ has given an expression for the matrix elements of gamma transitions in nuclei, under the assumption that only one particular nucleon in a nucleus takes part in the transition and the presence of the other nucleons is ignored. Consequently, the transition matrix element of a given multipole order is predicted to be the same for all nuclei, except for a slight dependence upon the nuclear size, so that the variation of the transition probability from nucleus to nucleus can not be explained. To explain this variation, it does not seem suitable to treat only one particular nucleon in a nucleus. Moszkowski⁽²⁾ has also examined the gamma transition in nuclei fairly in detail on the

basis of the individual particle model. However, there is a considerable disagreement between the results obtained by such a $j-j$ coupling shell model as adopted by him and experimental data. Especially, it is difficult to explain the following points: Firstly, in $E2$ -transition the experimental transition probabilities are much larger than the theoretical ones. This difference becomes larger, the farther lies the nucleus in question from closed shell nuclei. Secondly, in $E3$ -transition, the agreement between experimental and theoretical values is not generally good. The transition $p_{1/2} \leftrightarrow (7/2)^+$ is forbidden by the $j-j$ coupling shell model, because $(7/2)^+$ state is assigned to be of the configuration $((f_{9/2})^3_{7/2})^{3,5,7}$ according to this model. These disagreements between experiments and the theory seem to show the drawback of the individual particle model.

The defects of the above two models in explaining the gamma-transition may be remedied by A. Bohr's collective model.³⁾ The characteristic features of this collective model are to introduce the rotational states and to take into account the deformation of a nuclear core.

The excellent success of this model seems to lie in the fact that it explains low lying energy levels and $E2$ -transitions. The explanation of low lying energy levels of an even-even nucleus becomes possible by its introduction of rotational states, while it has been considered to be difficult with the other models. The $E2$ -transition is explained as the transition between these low lying rotational levels.

It is the main purpose of this work to examine how much better the collective model is than the other ones, by examining the nuclear gamma-transition other than $E2$.

In A. Bohr's model, it is not clear how to define the core. In his early study of odd nuclei he has treated an odd nucleon as the extra nucleon and the remaining even number of nucleons as the core. However, even when there are several nucleons outside the core, A. Bohr's formulation may be extended without loss of its essential features and it has actually been done by Ford¹⁾ to examine the energy levels of excited states.

In the case of treating nuclear transitions in odd-odd nuclei we take the odd number of extra protons and the odd number of extra neutrons as the nucleons outside the core and in even-even nuclei the even number of protons and/or neutrons, as the nucleons outside the core, the core consisting of the even number of protons and neutrons in both cases. However, nucleons in an unfilled subshell may be considered as the extra nucleons. Such a treatment will be stated in § 2.

The strong coupling model is much better than the weak coupling model and the $j-j$ coupling shell model in qualitative aspects. In the case of higher multipole radiations, like $M4$ -transition, when the spin change between the initial and the final states is large, there appears clearly a large difference between the reduced widths predicted by the two approximations.

In $E3$ -transitions (j -forbiddenness), such as those between $p_{1/2}$ and $7/2^+$ states, it is necessary to take into account the nuclear configuration mixing due to the surface coupling. Especially, the coupling with deformations of order three, which induces an $E3$ moment in the surface, is important for these transitions. Detailed results will be stated in § 3.

§ 2. Matrix element for gamma-transition in the collective model

Strong coupling approximation

In the strong coupling approximation of A. Bohr's collective model, the direct interaction between nucleons outside the core is ignored and individual extra nucleons are considered to interact only with the core. The principal assumption is that the period of nucleon motion is so short, compared to that of surface oscillation of the core, that it is a good approximation to treat the nucleon motion as if a nuclear axis is fixed in space.

The deformed core is expressed by sets of coordinates of the distortion relative to the nuclear axis (β, γ) and of coordinates of the nuclear axis relative to a space axis (Euler angles $\theta_1, \theta_2, \theta_3$). Extra nucleons outside the nuclear core are described by coordinates relative to the nuclear axis. If the interaction between a nucleon and the nuclear surface is in fact very strong, the j - j coupling for the extra nucleons will break down. However, if the spin-orbit interaction of a nucleon is stronger than its interaction with the nuclear surface, the total angular momentum j_i of each nucleon will be considered to be a good quantum number, as assumed in this paper.

The wave function of the system in the strong coupling approximation is, according to A. Bohr³⁾

$$|\Omega; n_\beta n_\gamma; IKM\rangle = \sqrt{\frac{2I+1}{16\pi^2}} \varphi_{n_\beta n_\gamma}(\beta, \gamma) \{ \chi_\Omega^j \mathfrak{D}_{MK}^I(\theta_i) + (-)^{I-j} \chi_{-\Omega}^j \mathfrak{D}_{M, -K}^I(\theta_i) \}, \quad (1)$$

where χ_Ω^j is the wave function of the nucleons outside the core with their total angular momentum j and its z -component in the nuclear coordinate system Ω . This χ_Ω^j must be antisymmetric with respect to all exchanges of the nucleons, as in the case of the individual particle model. Ω is described as the sum of the z -components of the angular momenta of the individual extra nucleons. $\varphi(\beta, \gamma)$ represents a vibrational state. \mathfrak{D}_{MK}^I is the wave function for a nuclear rotational state which is normalized so as to give the unitary transformation from the spatial coordinate system to the nuclear coordinate system under consideration. M and K are the z -components of spin I in the spatial and the nuclear coordinate systems respectively. The second term of the right-hand side of eq. (1) is introduced for the total system to have an appropriate symmetry character for the exchange of the coordinate axes. This symmetry character implies that

$$K - \Omega = 0, \pm 2, \pm 4, \dots \quad (2)$$

If the multipole transition operators are taken as $\mathfrak{S}_\mu^{(\lambda)}$ and $H_\nu^{(\lambda)}$ in the spatial and the nuclear coordinate systems respectively, these two transition operators are related to one another through the unitary transformation as

$$\mathfrak{S}_\mu^{(\lambda)} = \sum_\nu \mathfrak{D}_{\mu\nu}^\lambda H_\nu^{(\lambda)}, \quad (3)$$

where λ is the order of multipolarity of the radiation. In the collective model, the transition operator is described by the sum of the part representing the nucleon transition and the part representing the so-called core transition, the latter being due to the degree

of freedom of the collective motion. The transition matrix element can be described by the product of two factors, one being connected with the nucleon state and the other with the collective motion :

$$\begin{aligned}
 \langle I_i M_i | \mathfrak{S}_{\mu}^{(\lambda)} | I_f M_f \rangle = & \frac{\sqrt{(2I_f+1)(2I_i+1)}}{16\pi^2} \langle \varphi^{I_i} | \varphi^{I_f} \rangle \sum_{\nu} \left\{ \int \chi_{\Omega_i}^{j_i} \mathfrak{D}_{M_i K_i}^{I_i} \mathfrak{D}_{\mu \nu}^{\lambda} H_{\nu}^{(\lambda)} \chi_{\Omega_f}^{j_f} \mathfrak{D}_{M_f K_f}^{I_f} \right. \\
 & + (-)^{I_i+I_f-j_i-j_f} \int \chi_{-\Omega_i}^{j_i} \mathfrak{D}_{M_i, -K_i}^{I_i} \mathfrak{D}_{\mu \nu}^{\lambda} H_{\nu}^{(\lambda)} \chi_{-\Omega_f}^{j_f} \mathfrak{D}_{M_f, -K_f}^{I_f} \\
 & + (-)^{I_f-j_f} \int \chi_{\Omega_i}^{j_i} \mathfrak{D}_{M_i K_i}^{I_i} \mathfrak{D}_{\mu \nu}^{\lambda} H_{\nu}^{(\lambda)} \chi_{-\Omega_f}^{j_f} \mathfrak{D}_{M_f, -K_f}^{I_f} \\
 & \left. + (-)^{I_i-j_i} \int \chi_{-\Omega_i}^{j_i} \mathfrak{D}_{M_i, -K_i}^{I_i} \mathfrak{D}_{\mu \nu}^{\lambda} H_{\nu}^{(\lambda)} \chi_{\Omega_f}^{j_f} \mathfrak{D}_{M_f K_f}^{I_f} \right\}. \quad (4)
 \end{aligned}$$

This equation can be reduced to a simpler form by using the formula of rotational group representation,

$$\int \mathfrak{D}_{MK}^{I'} \mathfrak{D}_{\mu \nu}^{\lambda} \mathfrak{D}_{M'K'}^{I''} dR = \frac{8\pi^2}{2I+1} \delta(\nu+K', K) (I' \lambda M' \mu | IM) (I' \lambda K' \nu | IK), \quad (5)$$

where $(I' \lambda M' \mu | IM)$'s are the Clebsch-Gordan coefficients and satisfy the relation

$$(I' \lambda K' \nu | IK) = (-)^{I'+\lambda-I} (I' \lambda -K' -\nu | I-K). \quad (6)$$

Using eqs. (5) and (6), eq. (4) is

$$\begin{aligned}
 \langle I_i M_i | \mathfrak{S}_{\mu}^{(\lambda)} | I_f M_f \rangle = & \frac{1}{2} \sqrt{\frac{2I_f+1}{2I_i+1}} \langle \varphi^{I_i} | \varphi^{I_f} \rangle (I_f \lambda M_f \mu | I_i M_i) \\
 & \cdot \left[\langle j_i \Omega_i | H_{K_i-K_f}^{(\lambda)} | j_f \Omega_f \rangle + (-)^{I_f+\lambda-I-j_f} \langle j_i -\Omega_i | H_{K_f-K_i}^{(\lambda)} | j_f -\Omega_f \rangle \right] (I_f \lambda K_f K_i - K_f | I_i K_i) \\
 & + \left\{ (-)^{I_f-j_f} \langle j_i \Omega_i | H_{K_i+K_f}^{(\lambda)} | j_f -\Omega_f \rangle + (-)^{I_f+\lambda-j_f} \langle j_i -\Omega_i | H_{-K_i-K_f}^{(\lambda)} | j_f \Omega_f \rangle \right\} \\
 & \cdot (I_f \lambda -K_f K_i + K_f | I_i K_i). \quad (7)
 \end{aligned}$$

$\langle j_i \Omega_i | H_{K_i-K_f}^{(\lambda)} | j_f \Omega_f \rangle$'s are the matrix elements related only to extra nucleon coordinates in the nuclear coordinate system and differ from each other only by their angular momentum components parallel to the nuclear axis. For any vector or tensor operator, its dependence upon the magnetic quantum number in the matrix element can be factorized as follows :

$$\langle j \Omega | H_{\epsilon}^{(\lambda)} | j' \Omega' \rangle = \langle j || H^{(\lambda)} || j' \rangle (j' \lambda \Omega' \epsilon | j \Omega), \quad (8)$$

where the first factor in the right hand side is the reduced matrix element, and we use here for the sake of convenience $(2j+1)^{-1/2}$ times of Racah's definition.⁷⁾

Substituting (8) into (7) and using relation (6), we have

$$\begin{aligned}
 \langle I_i || \mathfrak{S}_{\mu}^{(\lambda)} || I_f \rangle = & \sqrt{\frac{2I_f+1}{2I_i+1}} \langle \varphi^{I_i} | \varphi^{I_f} \rangle \langle j || H^{(\lambda)} || j \rangle \cdot \\
 & \cdot \{ (I_f \lambda K_f K_i - K_f | I_i K_i) (j_f \lambda \Omega_f K_i - K_f | j_i \Omega_i) + (-)^{I_f-j_f} (I_f \lambda -K_f K_i + K_f | I_i K_i) \\
 & \times (j_f \lambda -\Omega_f K_i + K_f | j_i \Omega_i) \}. \quad (9)
 \end{aligned}$$

This expression shows that in the case of the strong coupling model, the matrix element for the gamma-transition consists of the core part and the reduced matrix element relating to the nucleons outside the core.

$\langle \varphi'_{i'} | \varphi'_{f'} \rangle$ can be evaluated from A. Bohr's formula, if one knows nuclear spin I and extra nucleon spin j of the initial and final states, as well as the surface deformation β and γ . Now, only the reduced matrix element is left to be evaluated.

Multipole transition operator $H_{\nu}^{(\lambda)}$ is decomposed into the contributions from the individual extra nucleons,

$$H_{\nu}^{(\lambda)} = \sum_i H_{\nu i}^{(\lambda)}. \quad (10)$$

$H_{\nu}^{(\lambda)}$ operates on the i -th nucleon only, and the sum of these operators is symmetric with respect to all the nucleons. As an immediate consequence of the form of the transition operator, a transition which requires the change of orbits of several nucleons can not occur. Noticing this fact, we shall obtain the expression of reduced matrix elements in the case of one, two and several nucleons outside the core successively.

(A) One extra nucleon

We can obtain the expression for the reduced matrix element using eq. (8). In this case, the absolute square of the reduced matrix element of the electric 2^{λ} -pole transition is

$$|\langle j_i || H_E^{(\lambda)} || j_f \rangle|^2 = \frac{e'^2}{4\pi} |\langle i | r^{\lambda} | f \rangle|^2 S_{E\lambda}, \quad (11)$$

where

$$S_{E\lambda} = (2I_i + 1)(2I_f + 1)(2j_f + 1)(2\lambda + 1) |W(l_i j_i l_f j_f; s\lambda)|^2 \cdot |V(l_i l_f \lambda; 000)|^2 \quad (12)$$

is the so-called statistical factor that comes from the angular part of the electric 2^{λ} -pole matrix element. W is the Racah coefficient and V is a function related to the Clebsch-Gordan coefficient.⁷⁾ $\langle i | r^{\lambda} | f \rangle$ is the radial matrix element. e' is the effective charge. s is the intrinsic spin of the nucleon.

A special case, when the difference between the initial and final nucleon spins Δj , and that between the orbital angular momenta Δl , are both equal to the multipole order λ , eq. (12) is reduced simply to

$$S_{E\lambda} = \frac{(j_i - 1/2)!(2\lambda + 1)!!(2j_f)!!}{(2j_i)!!\lambda!(j_f - 1/2)!}. \quad (13)$$

When $\Delta j = \lambda$ and $\Delta l = \lambda - 1$, that is, when the intrinsic nucleon spin is flipped due to the radiative transition, the reduced matrix element of the $M\lambda$ -transition can be expressed in a simple form. The statistical factor $S_{M\lambda}$ resulting from the integration of the angular part of the matrix element is of the same form as (13) and the absolute square of the reduced matrix element is

$$|\langle j_i || H_M^{(\lambda)} || j_f \rangle|^2 = \frac{1}{4\pi} \left(\frac{e\hbar}{2Mc} \right)^2 \left[g_s \lambda - \frac{2\lambda}{\lambda + 1} g_l \right]^2 \cdot |\langle i | r^{\lambda-1} | f \rangle|^2 S_{M\lambda}, \quad (14)$$

where g_l and g_s are the orbital and spin g -factors of the nucleon respectively.

(B) *Two extra nucleons*

In the case of two nucleons with spins j_1 and j_2 outside the core, the allowed transition must be such as

$$(j_1 j_2) j_i \rightarrow (j_1' j_2) j_f,$$

for the transition operator is the sum of single nucleon operators. Of course, as a special case, any two of j_1 , j_2 and j_1' or all of them may be equal. These two extra nucleons may be like particles or unlike particles.

The reduced matrix element is

$$\langle j_i \| H^{(\lambda)} \| j_f \rangle = (-)^{j_2 + \lambda - j_1' - j_2 u} \langle \alpha j_1 \| H_1^{(\lambda)} \| \alpha' j_1' \rangle [(2j_1 + 1)(2j_f + 1)]^{1/2} W(j_1 j_1' j_f; j_2 \lambda), \quad (15)$$

where α denotes any other quantum numbers necessary to specify the state. $\langle \alpha j_1 \| H_1^{(\lambda)} \| \alpha' j_1' \rangle$ is the reduced matrix element for the transition of the nucleon 1 and given by eq. (11) or (14). For like particles, $u=1$ if j_1 , j_1' and j_2 are all different, $u=(2)^{1/2}$ if $j_1=j_2$ or $j_1'=j_2$, and $u=2$ if $j_1=j_1'=j_2$. For unlike particles $u=1$.

(C) *Several extra nucleons*

In the general case where n nucleons are outside the core, the transition scheme most commonly considered is

$$(j_0^{n-1} j_1) j_i \rightarrow (j_0^{n-1} j_1') j_f.$$

The nucleon wave function χ_{Ω}^j must be antisymmetric for all exchanges of nucleons, as in the case of the individual particle model. It is expressed in terms of the fractional parentage coefficients as follows;

$$\chi_{\Omega}^j = \sum_{J_P} \sum_{\Omega_1} \chi(j_0^{n-1}, \alpha_P J_P) \chi(j_1 \Omega_1) (J_P j_1 \Omega_P \Omega_1 | j \Omega) (j_0^{n-1} (\alpha_P J_P) j_1 | j_0^{n-1} j_1, \alpha j), \quad (16)$$

where \sum_{J_P} means the sum over all the parent states $(\alpha_P J_P)$ of j_0^{n-1} and $\chi(j_0^{n-1}, \alpha_P J_P)$ is the wave function of the parent state. In this case, the reduced matrix element is

$$\begin{aligned} \langle (j_0^{n-1} j_1) j_i \| H^{(\lambda)} \| (j_0^{n-1} j_1') j_f \rangle &= n \langle \alpha j_1 \| H_1^{(\lambda)} \| \alpha' j_1' \rangle \sum_{J_P} [(2j_1 + 1)(2j_f + 1)]^{1/2} \\ &\cdot W(j_1 j_1' j_f; j_1 j_f) \cdot (j_0^{n-1} (\alpha_P J_P) j_1 | j_0^{n-1} j_1, \alpha j_i) (j_0^{n-1} (\alpha_P J_P) j_1' | j_0^{n-1} j_1', \alpha' j_f), \end{aligned} \quad (17)$$

where $\langle \alpha j_1 \| H_1^{(\lambda)} \| \alpha' j_1' \rangle$ is the reduced matrix element for the transition of the nucleon 1 and given by eq. (11) or (14).

More generally, in the case of transition such as $j_1^{n_1+1} j_2^{n_2} \rightarrow j_1^{n_1} j_2^{n_2+1}$ we can also obtain the expression of the reduced matrix element for the transition as in the case of eq. (17), though it is complicated. We do not give here the expression for the transition of this kind, as the most common type of particle transition that actually occurs will be

$$(j_0^{n-1}j_1)j_i \rightarrow (j_0^{n-1}j_1')j_f.$$

Weak coupling approximation

(D) *One extra nucleon*

The wave function is given, according to Bohr and Mottelson³⁾, by

$$\Psi = N[|j; 00; I(=j)M\rangle + \sum_{j'} |j'; 12; IM\rangle \langle j'; 12; I|], \quad (18)$$

where

$$N^{-2} = 1 + \sum_{j'} |\langle j'; 12; I| |^2.$$

The first term in the right hand side represents the state in which a nucleon has total angular momentum j and the core is not excited. The second term represents the state in which a nucleon with total angular momentum j' couples with the core and one phonon of angular momentum 2 is excited so that the resultant spin is I .

The expansion coefficient $\langle j'; 12; I|$ is given by Bohr and Mottelson (Ap. II. 3) as

$$\langle j'; 12; I| = k \sqrt{\frac{\hbar\omega}{2C}} \frac{\langle j|b|j'\rangle}{\hbar\omega + \mathcal{A}_{jj'}},$$

where C is the deformability of the core, $\hbar\omega$ the energy of the phonon, k the coupling constant, and $\mathcal{A}_{jj'}$ the separation between two particle levels of angular momenta j and j' . $\langle j|b|j'\rangle$ is the angular part of the matrix element of the interaction between the core and the nucleon, and is given by Bohr and Mottelson³⁾ (Ap. II. 2).

For a particle transition, its reduced matrix element is

$$\begin{aligned} \langle I_i || H^{(\lambda)} || I_f \rangle &= N_i N_f [\langle j_i (=I_i) || H^{(\lambda)} || j_f (=I_f) \rangle + \sum_{j_i' j_f'} (-)^{\lambda-j_f'-I_i} \langle j_i' || H^{(\lambda)} || j_f' \rangle \\ &\cdot [(2j_i'+1)(2I_f+1)]^{1/2} W(j_i' I_i j_f' I_f; 2\lambda) \langle j_i'; 12; I_i | \rangle \langle j_f'; 12; I_f | \rangle]. \end{aligned} \quad (19)$$

The explicit expressions of the matrix elements $\langle j_i || H^{(\lambda)} || j_f \rangle$ and $\langle j_i' || H^{(\lambda)} || j_f' \rangle$ are given by eqs. (11) and (14).

(E) *Two extra nucleons*

The wave function is obtained in a similar way to the case of one extra nucleon.

$$\begin{aligned} \Psi &= N[|j_1 j_2; 00; IM\rangle + \sum_{j_1' j_2'} \sum_J |j_1' j_2'(J); 12; IM\rangle \langle j_1' j_2'(J); 12; I|], \\ N^{-2} &= 1 + \sum_{j_1' j_2'} \sum_J |\langle j_1' j_2'(J); 12; I| |^2. \end{aligned} \quad (20)$$

The expansion coefficient is expressed as

$$\begin{aligned} \langle j_1' j_2'(J); 12; I| &= k \sqrt{\frac{\hbar\omega}{2C}} \left[\delta_{i_2 j_2'} \frac{\langle j_1 | b | j_1' \rangle}{\hbar\omega + \mathcal{A}_{j_1 j_1'}} (-)^{I+j_1'-J-i_1} [(2J+1)(2j_1+1)]^{1/2} \cdot \right. \\ &\cdot W(j_2 j_1' I 2; J j_1) + \delta_{j_1 j_1'} \frac{\langle j_2 | b | j_2' \rangle}{\hbar\omega + \mathcal{A}_{j_2 j_2'}} [(2J+1)(2j_2+1)]^{1/2} W(j_1 j_2' I 2; J j_2) \left. \right]. \end{aligned}$$

The reduced matrix element for a particle transition is

$$\begin{aligned} \langle I_i \| H^{(\lambda)} \| I_f \rangle = & N_i N_f [\langle (j_{1i} j_2) I_i \| H^{(\lambda)} \| (j_{1f} j_2) I_f \rangle + \sum_{j_{1i} j_2' j_{1f}'} \sum_{I_i' I_f'} (-)^{\lambda - I_f - I_i} \\ & \cdot \langle (j_{1i} j_2') J_i \| H^{(\lambda)} \| (j_{1f} j_2') J_f \rangle \cdot \\ & \cdot [(2J_i + 1) (2I_f + 1)]^{1/2} W(J_i I_i J_f I_f; 2\lambda) \langle j_{1i} j_2' (J_i); 12; I_i | \rangle \langle j_{1f} j_2' (J_f); 12; I_f | \rangle]. \end{aligned} \quad (21)$$

The explicit expressions of the matrix elements $\langle (j_{1i} j_2) I_i \| H^{(\lambda)} \| (j_{1f} j_2) I_f \rangle$ and $\langle (j_{1i} j_2') J_i \| H^{(\lambda)} \| (j_{1f} j_2') J_f \rangle$ are given by eq. (15).

(F) *Several extra nucleons.*

We consider such cases where the configuration $j_0^{n-1} j'$ mixes with the configuration $j_0^{n-1} j$ by the surface coupling. The wave function is given by

$$\begin{aligned} \Psi = & N [| j_0^{n-1} j; 00; IM \rangle + \sum_{j'} \sum_{J'} | j_0^{n-1} j' (J'); 12; IM \rangle \langle j_0^{n-1} j' (J'); 12; I |], \\ N^{-2} = & 1 + \sum_{j'} \sum_{J'} | \langle j_0^{n-1} j' (J'); 12; I | \rangle |^2. \end{aligned} \quad (22)$$

The expansion coefficients are expressed as

$$\begin{aligned} \langle j_0^{n-1} j' (J'); 12; I | = & \left[\sum_{j_0} \langle j_0^{n-1} j' (J') 2I | j' 2(j) j_0^{n-1} (J_0) I \rangle \frac{\langle j' | h | j \rangle}{\hbar \omega + A_{jj'}} \right. \\ & \left. + (n-1) \sum_{j_0} \langle j_0^{n-1} j' (J') 2I | j_0 2(j_0) j_0^{n-2} (J) I \rangle \delta_{jj'} \frac{\langle j | h | j \rangle}{\hbar \omega} \right] k \sqrt{\hbar \omega / 2C}. \end{aligned}$$

The reduced matrix element for a particle transition is

$$\begin{aligned} \langle I_i \| H^{(\lambda)} \| I_f \rangle = & N_i N_f [\langle j_0^{n-1} j_i; I_i \| H^{(\lambda)} \| j_0^{n-1} j_f; I_f \rangle + \sum_{j_{1i} j_{1f}} \sum_{I_i' I_f'} (-)^{I_f + J_f' - I_i - J_i'} \\ & \cdot \langle j_0^{n-1} j_i'; J_i \| H^{(\lambda)} \| j_0^{n-1} j_f'; J_f' \rangle \cdot \\ & \cdot [(2J_i' + 1) (2J_f' + 1)]^{1/2} W(\lambda J_f' I_i 2; J_i' I_f) \langle j_0^{n-1} j_i' (J_i'); 12; I_i | \rangle \langle j_0^{n-1} j_f' (J_f'); 12; I_f | \rangle]. \end{aligned} \quad (23)$$

The reduced matrix elements in the right hand side are given by eq. (17).

§ 3. Comparison with experimental data

In order to compare the experimental data with the theoretical predictions, we define the ratio of the experimental value of a radiation width Γ'_{exp} to the corresponding theoretical value Γ_{th} as

$$F = \Gamma_{exp} / \Gamma_{th}. \quad (24)$$

The theoretical radiation width Γ_{th} of multipolarity λ is

$$\Gamma_{th}^{(\lambda)} = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} (E_\gamma / \hbar c)^{2\lambda+1} \cdot | \langle I_i \| \hat{\mathfrak{H}}^{(\lambda)} \text{ (or } H^{(\lambda)}) \| I_f \rangle |^2, \quad (25)$$

where the reduced matrix element is given by eqs. (9) and (19) in the strong and weak coupling approximations respectively. Errors for I'_{th} come from the uncertainty in the potential shape and the nuclear size. When F deviates from unity even if these errors are taken into account, this deviation should be attributed to the defect of our nuclear model.

The radial part of the matrix element depends upon the potential shape and the nuclear size. However, the difference between the matrix elements evaluated with a wave function of a square well potential and with a constant wave function is about 10 to 15 percent. Moreover, this difference is known to be insensitive to the depth of the square well potential. Thus, in calculating the radial part of the matrix element, we use the constant wave function. It is given by

$$\langle i|r^\lambda|f\rangle = 3/(3+\lambda) \cdot R^\lambda. \quad (26)$$

As the nuclear radii, we take $R=1.45 \times 10^{-13} A^{1/3}$ cm. The magnitude of the matrix element is sensitive to the radius and the uncertainty caused by it becomes larger, the higher the multipole order is. For example, in the case of the electric transition, the uncertainty of 10% in the radius R gives the uncertainties of the matrix elements of about 10%, 20%, 30%,... as the multipole order λ 's are 1, 2, 3,... respectively.

As the magnitudes of parameters B , C etc. appearing in the collective model, we use those estimated on a hydrodynamical model by A. Bohr and Mottelson³⁾ and by Ford¹⁾. (Table I.)

Table I. Energy parameters of hydrodynamical model.

A	30	50	100	200
$C(\text{MeV})$	41.46	53.93	65.6	62.9
$\hbar^2/B(\text{MeV})$	0.590	0.252	0.0794	0.0249

The values thus obtained are not always good, for example, the quadrupole moments and the excited levels of even-even nuclei based on them do not always agree with experimental data. The experimental value of the parameter β describing the core deformation, estimated from the quadrupole moments, is slightly smaller than the value estimated on the hydrodynamical model. However, we shall refer to the latter β since its order of magnitude seems to be correct.

The values of $\langle \varphi^{I_i} | \varphi^{I_f} \rangle$ for the transition of an extra nucleon based on the assumption of $I=K=\Omega=j$ are listed in Table II.

Table II. The values of $\langle \varphi^{I_i} | \varphi^{I_f} \rangle$ for one extra nucleon transition.

ΔI	A	30	50	100	200
$3/2 \leftarrow 5/2$		0.954	0.954	0.953	0.948
$3/2 \leftarrow 7/2$		0.894	0.893	0.888	0.879
$5/2 \leftarrow 7/2$		0.984	0.984	0.983	0.983

(a) *M4-transition*

In the case of gamma-transitions for low lying isomeric states, the spin change between two states is generally large, so that they are high multipole transitions. Especially, *M4*-transitions of odd-*A* nuclei are known very well,^{5),6)} and also investigated¹⁾²⁾. Most of these nuclear isomers are pre-closed shell nuclei of the magic numbers 50, 82, 126, and their transitions are respectively $g_{9/2} \longleftrightarrow p_{1/2}$, $h_{11/2} \longleftrightarrow d_{3/2}$, $i_{13/2} \longleftrightarrow f_{5/2}$. We list some of these transitions in Table III, except for $g_{9/2} \longleftrightarrow p_{1/2}$ transition. For $j=1/2$ there is no coupling between the particle and the nuclear surface. We must take care of the fact that in this case the strong coupling wave function (1) reduces to the uncoupled wave function. In this paper, however, we do not proceed into this point.

Table III. *M4*-transitions in odd-*A* nuclei. *F* gives the ratio of the experimental value of a radiation width to the corresponding calculated value.

Nucleus	E_T (KeV)	particle configuration		$\langle \varphi^I i \varphi^I j \rangle$	F_{ind}	F_{weak}	F_{strong}
		initial	final				
Te ¹²⁹	106	$h_{11/2}$	$d_{3/2}$	0.78	0.154	0.203	0.755
Te ¹³¹	183	$(d_{3/2})^2 h_{11/2}$	$(d_{3/2})^3$	0.95	0.472	0.612	1.568
Xe ¹³¹	163	$h_{11/2}$	$d_{3/2}$	0.78	0.085	0.113	0.416
Xe ¹³³	232	$(d_{3/2})^2 h_{11/2}$	$(d_{3/2})^3$	0.95	0.210	0.262	0.698
Xe ¹³⁵	520	$h_{11/2}$	$d_{3/2}$	0.78	0.107	0.134	0.527
Ba ¹³³	275	$h_{11/2}$	$d_{3/2}$	0.78	0.058	0.069	0.259
Ba ¹³⁵	269	$(d_{3/2})^2 h_{11/2}$	$(d_{3/2})^3$	0.95	0.242	0.302	0.804
Ba ¹³⁷	661	$h_{11/2}$	$d_{3/2}$	0.78	0.079	0.103	0.389
Pb ²⁰⁷	1063	$i_{13/2}$	$f_{5/2}$	0.73	0.077	0.101	0.291

In the weak coupling approximation, the matrix element (19) is reduced to

$$\langle I_i || H^{(\lambda)} || I_f \rangle = N_i N_f \left[1 + (-)^{\lambda - j_i - j_f} \frac{5}{64\pi^2 C \hbar \omega} k^2 \frac{[(2j_i + 1)(2j_f + 1)]^{1/2} W(j_i j_i j_f j_f; 2\lambda)}{j_i(j_i + 1)} \right]^{1/2} \left[\frac{(2j_f - 1)(2j_f + 3)}{j_f(j_f + 1)} \right]^{1/2} \cdot \langle j_i || H^{(\lambda)} || j_f \rangle \quad (27)$$

for a particular case of $j_i' = j_i$ and $j_f' = j_f$. The reduced matrix element in the right hand side is given by eq. (14). The transition matrix elements (21) and (23) are reduced also to simple forms under the assumption that $j_i = j_i' = J_i'$ and $j_f = j_f' = J_f'$ as before. This corresponds to the fact that we take only a particular configuration mixing in which one phonon of angular momentum 2 is excited due to the surface coupling. Of course, the values of *F* in the weak coupling approximation show only a qualitative tendency because of the particular configuration adopted.

In the strong coupling approximation, it is assumed that $I=K=Q=j$ for any excited state and for $j=3/2$ state. The values of F in this approximation are listed in Table III. This assumption for the excited state may not always be good. The case of $j=3/2$ requires a special attention, since the configuration ($\gamma=\pi$, $Q=3/2$) and ($\gamma=0$, $Q=1/2$) are degenerate³⁾. In this case there is no exact limiting solution of type (1). The strong coupling wave function in this case is given by Bohr and Mottelson (Ap. III. 3)³⁾, which is a solution of a set of coupled differential equations. We do not obtain the exact solution of such differential equations but make an approximate estimate. Neglecting the overlap of the vibrations about $\gamma=0$ and $\gamma=\pi$, we obtain two degenerate solutions. By the effect of the degeneracy of the γ -oscillation, the values of F is found to increase by a few percent in comparison with the state $\gamma=\pi$, $Q=3/2$.

The strong coupling model gives considerably large values of F compared with the weak coupling model and with the individual particle model, and shows a good tendency in agreement with the experimental data.

(b) $E3$ -transition

It is known that there is a type of transition $(7/2) + \longleftrightarrow (1/2) -$, which occurs for nuclei of nucleon numbers 43, 45, 47. The $(7/2) +$ state is assigned⁵⁾ on the basis of the shell model as $((f_{7/2})_{1/2})_{3,5,7}$. For the pure configuration of this type, the $E3$ -transition would be forbidden, because several nucleons would have to change their orbits in the course of the transition. The smallness of empirical radiative width $\Gamma'_{e,p}$ for $E3$ -transitions can be interpreted as the result of a small deviation from the individual particle model. The transitions, such as those between $p_{1/2}$ and $7/2 +$ states, are assumed to occur by the mixing of a small amount of the $p_{3/2}$ state with the former and $(f_{7/2})$ state with the latter, respectively.

The values of F_{int} in Table IV are based on Γ'_{th} taking the configuration mixing into account by means of the simple perturbation theory and the values of the deviations are estimated by adopting the two-body interaction strengths and the integrals which are determined from the empirical data of pairing energies.⁸⁾ For odd neutron nuclei, the values of Γ'_{th} are of the order of $(Z/A^{1/2})^2$ because of the recoil of the core. If a proton configuration is taken into account, the large deviations of F -value from unity may be removed.*

In the collective model, the surface coupling will mix the particle states and, furthermore, the coupling with a deformation of order three induces an $E3$ moment in the surface.³⁾ The contribution from the $E3$ moment in the surface is somewhat larger than those from the ordinary electric moment of a particle, due to the large value of charge involved in the surface motion.

In the weak coupling approximation, the first term in the right hand side of (23) vanishes, because this term corresponds to the transition between the pure configurational

* We shall discuss this problem in a forthcoming paper.

states. In the particular case where we take into account only the mixing of the $p_{3/2}$ and $g_{7/2}$ configurations in the initial and final states, the transition matrix element is reduced to

$$\begin{aligned} \langle I_i || H^{(\lambda)} || I_f \rangle = & N_i N_f \sum_{J_i J_f J_P} \sum_{j_i' j_f'} (-)^{J_f + J_f' - I_i - J_i'} n \langle j_i' || H_1^{(\lambda)} || j_f' \rangle \cdot \\ & [(2j_i' + 1)(2j_i' + 1)(2I_f + 1)(2J_f' + 1)]^{1/2} \cdot W(\lambda j_f' J_i' J_P : j_i' J_f') W(\lambda J_f' I_i 2 : J_i' J_f) \cdot \\ & \cdot (j_0^{n-1}(J_P) j_i' J_i' || j_0^{n-1} j_i' J_i') \cdot (j_0^{n-1}(J_P) j_f' J_f' || j_0^{n-1} j_f' J_f') \cdot \langle j_0^{n-1} j_i(J_i'); 12; I_i \rangle \cdot \\ & \cdot \langle j_0^{n-1} j_f(J_f'); 12; I_f \rangle, \end{aligned} \quad (28)$$

where

$$\begin{aligned} \langle j_0^{n-1} j'(J'); 12; I \rangle = & k \sqrt{\hbar\omega/2C} [(2J' + 1)(2j + 1)]^{1/2} W(J_P j' I 2; J' j) \cdot \\ & \cdot \langle j | b | j' \rangle / (\hbar\omega + A_{jj'}), \\ \langle j | b | j' \rangle = & -\sqrt{5/64\pi} (1/j) \sqrt{3j(2j+3)/(j+1)(j+2)} \quad (j' = j+1), \\ = & -\sqrt{5/64\pi} \sqrt{(2j-1)(2j+3)/j(j+1)} \quad (j' = j), \\ = & -\sqrt{5/64\pi} (1/j) \sqrt{3j(2j-1)/(j-1)(j+1)} \quad (j' = j-1). \end{aligned}$$

The energy differences A between $p_{1/2}$ and $p_{3/2}$ states, and between $g_{7/2}$ and $g_{9/2}$ states, correspond to the doublet splitting due to the spin-orbit interaction. We put

$$A = a(2l+1)A^{-2/3}. \quad (29)$$

The constant a is determined from the splitting energy A between $d_{3/2}$ and $d_{5/2}$ levels of $^{17}_{8}\text{O}$, which is 5 Mev. For the $E3$ moment in the surface⁽³⁾ which is induced by the particle transition, the reduced matrix element in the right hand side of (28) is

$$\langle j_i' || H_{\text{surface moment}}^{(3)} || j_f' \rangle = \frac{3}{4\pi} Z e \frac{k}{C_3} \frac{(\hbar\omega_s)^2}{(\hbar\omega_s)^2 - (E_i - E_f)^2} R^3 I S_{E3}^{1/2}, \quad (30)$$

where

$$I = \int R_{n_i i} R_{n_f j} r^2 dr,$$

E_i and E_f are the energies of the particle in the initial and final states. The values of $\hbar\omega_s$ and C_3 are about 5 Mev and 200 Mev for $A=100$ respectively.

The results are shown in the eighth column of Table IV inclusive of the contribution from the $E3$ -moment by the surface deformation which is induced by the particle transition. Some of them deviate greatly from unity. It may be attributed to an uncertainty of the hydrodynamical parameters and to choice of the particular configuration mixing in the nuclear states.

It is worthy of our notice that the coupling with the deformations of order three, which induces the $E3$ moment in the surface, is important for these transitions. In the case of the strong coupling approximation, I'_{th} -values depend entirely on the mixing of the $g_{7/2}$ -state due to the surface coupling and on the selection rules associated with the Q and K quantum numbers. We shall study these points in detail in the forthcoming paper.

Table IV. $E3$ -transitions of $(7/2+) \longleftrightarrow (1/2-)$ (j -forbiddenness). The values of F_{ind} are based on F_{th} taking the configuration mixing into account by means of the simple perturbation theory.

Nucleus	E_{γ} (KeV)	pure particle configuration		F_{ind}	F_{weak}
		initial	final		
Se ⁷⁷	162	$(g_{9/2})^3_{7/2}$	$(g_{9/2})^2p_{1/2}$	$2.24 \cdot 10^7$	61.90
Se ⁷⁹	80	$(g_{9/2})^4p_{1/2}$	$(g_{9/2})^5_{7/2}$	$2.16 \cdot 10^6$	11.90
Se ⁸¹	98	$(g_{9/2})^7_{7/2}$	$(g_{9/2})^6p_{1/2}$	$2.50 \cdot 10^5$	1.904
Kr ⁷⁹	127	$(g_{9/2})^2p_{1/2}$	$(g_{9/2})^3_{7/2}$	$4.20 \cdot 10^6$	11.55
Kr ⁸¹	187	$(g_{9/2})^4p_{1/2}$	$(g_{9/2})^5_{7/2}$	$1.42 \cdot 10^6$	7.555
Kr ⁸³	32.2	$(g_{9/2})^6p_{1/2}$	$(g_{9/2})^7_{7/2}$	$1.99 \cdot 10^5$	1.777
Rh ¹⁰³	40	$(g_{9/2})^5_{7/2}$	$(g_{9/2})^4p_{1/2}$	0.013	2.057
Rh ¹⁰⁵	130	$(g_{9/2})^5_{7/2}$	$(g_{9/2})^4p_{1/2}$	0.018	2.936
Ag ¹⁰⁷	93.9	$(g_{9/2})^7_{7/2}$	$(g_{9/2})^6p_{1/2}$	0.169	18.17
Ag ¹⁰⁹	89	$(g_{9/2})^7_{7/2}$	$(g_{9/2})^6p_{1/2}$	0.272	28.33

§ 4. Conclusions

Static phenomena, such as energy levels, magnetic moments, quadrupole moments etc., have been investigated by many authors in detail on various models. Among them, A. Bohr's collective model has been especially successful in explaining many phenomena. In this model the $E2$ -transition is explained as a collective transition of a nucleus³⁾.

The first aim of this paper is to investigate how good this collective model is in other modes of gamma-transitions compared with the single particle model or the individual particle model.

The single particle model, in most cases, over-estimates matrix elements by a factor ten or more, but under-estimates them in some cases, owing to its oversimplification. Moszkowski has examined the gamma transition in detail on the basis of the individual particle model. The results obtained by such a model do not always show a good agreement with the experimental data. However, these situations can be made considerably better, if one takes the strong coupling approximation in the collective model into account.

The second aim is to compare the strong and the weak coupling models. The difference between these two approximations comes from the contributions of the deformation of a core and becomes larger as the multipole order of the transition increases.

The third aim is to explain the $E3$ -transitions (j -forbiddenness), such as those between $p_{1/2}$ and $(7/2)+$ states. The fact that the magnitudes of the experimental radiative widths for odd-proton nuclei and for odd-neutron nuclei are comparable indicates that the coupling with the deformations of order three may be important for these transitions as suggested by Bohr and Mottelson³⁾. In fact, these transitions are explained by

the consideration of the $E3$ moment in the surface which is induced by the particle transition.

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Nuclear Structure as Revealed by Electric Excitation

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Electric excitation of low-lying levels in medium weight and heavy nuclei have been made from which the energy ΔE of a given level and the cross section σ_{ex} for its excitation can be deduced. Electric quadrupole moments, $Q_0(\Delta E)$ and $Q_0(\sigma_{ex})$ respectively, based on these two sets of data show markedly different values and markedly different trends as the closed proton shells at 50 and 82 are approached. If it is assumed that the $Q_0(\Delta E)$ is indicative of the deformation of mass and $Q_0(\sigma_{ex})$ is indicative of the deformation of charge, the results suggest that a) the protons and neutrons are not uniformly distributed throughout the nucleus, b) the spherical symmetry of the protons is greater than that of the neutrons near the proton shells, c) the protons are concentrated closer to the center of the nucleus than the neutrons, and/or d) the flow within the nucleus is not purely irrotational.

§ 1. Collective model

The collective nuclear model began a few years ago as an attempt to understand the observed large intrinsic deformations which are ten times as large as can be explained in terms of shell theory. This model has since been able to explain these and a whole series of other interesting nuclear properties. The first treatise on this new model, by its originator Aage Bohr, appeared in 1952 and has since been improved by Wheeler and Hill and refined by Bohr and Mottelson. The principal idea is that the potential in which the particle moves is deformable. Hence the core which had been considered as rigid in the shell model also contributes to the properties of the nucleus. The centrifugal pressure of the particles moving in their orbits causes the wall of the nucleus to bulge which in turn causes the large observed deformations. Near closed shells, the walls of the nucleus are quite rigid (i.e. the liquid-drop has a large surface tension) and there are only a few particles outside the closed shell which could cause deformation of the potential. Hence near closed shells large deformations are not expected. Away from closed shells, the potential is not rigid and many particles may cooperate to deform the potential. One would, therefore, expect large nuclear deformations in the region midway between closed shells as observed in experiments discussed below.

In the collective model we write the Hamiltonian of the system as the sum of three terms :

*) This paper was presented at a physics colloquium in Yukawa Hall, Kyoto University on February 8, 1955. At that time the author was Fulbright Lecturer at Osaka University.

$$H = H_{sp}(\chi_i) + H_s(\alpha) + H_{int}(\chi_i, \alpha), \quad (1)$$

where $H_{sp}(\chi_i)$ is essentially the shell-structure Hamiltonian, $H_s(\alpha)$ is the Hamiltonian of the surface, which is a function of parameters describing the surface (shape, orientation, etc.) and the interaction between the surface and the core. The collective portion of $H_s(\alpha)$ has been calculated by Bohr assuming that all collective motions of the nucleus are like the irrotational flow of a fluid (shades of the liquid-drop!). The interaction term H_{int} determines the nuclear deformation since it governs the interaction between the particles and the surface (*i.e.* determines the centrifugal pressure). If H_{int} is small, then the nucleus is spherically symmetric and $H_s(\alpha)$ describes the surface oscillations of a spherical liquid-drop. If H_{int} is large, then the nucleus is in general an ellipsoid of rotation and has another mode of motion—the rotation of the surface of the ellipsoid in space without change of shape. In this case, the collective part of the Hamiltonian can be written as:

$$H_s(\alpha) = H_{rot} + H_{vib}, \quad (2)$$

where H_{vib} is the portion of the Hamiltonian governing ordinary surface vibrations and

$$H_{rot} = 1/2 \mathcal{I} \bar{\omega}^2, \quad (3)$$

where \mathcal{I} is the *effective* moment of inertia and $\bar{\omega}$ is the angular velocity of the motion. \mathcal{I} depends upon the dynamics of the motion. For irrotational flow, the motion is essentially a surface wave and:

$$\mathcal{I} = \mathcal{I}_0 \beta^2, \quad (4)$$

where \mathcal{I}_0 is the actual mechanical moment of inertia of the nucleus. The parameter β

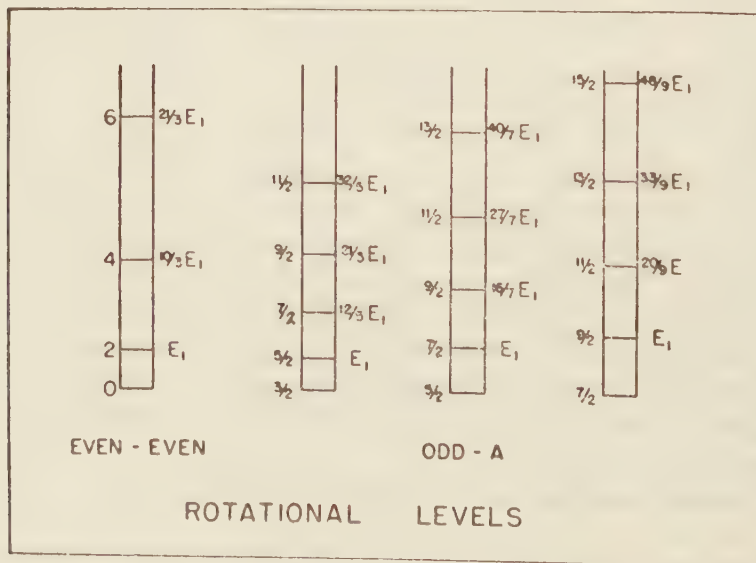


Fig. 1 Rotational levels in nuclei. From "symmetry" considerations the sequence is $I=2, 4, 6$, etc. in even A nuclei and $I=I_0+1, I_0+2$, etc. in odd A nuclei. Nuclei with $I_0=1/2$ do not follow the odd A sequence.

describes the deformation and is related to the eccentricity of the nucleus. For most nuclei, β is very small and hence the effective moment of inertia \mathcal{I} is much smaller than \mathcal{I}_0 . The energy associated with the motion is :

$$E = \frac{\hbar^2}{2\mathcal{I}} [I(I+1) - I_0(I_0+1)], \quad (5)$$

where I is the spin of excited level and I_0 the spin of the ground level. We should therefore expect from (5) a sequence of energy levels as shown in Fig. 1.

§ 2. Transition probabilities

Another important property which these "rotational" levels possess is that the transition probabilities can be calculated using the "collective" model wave functions and compared with experimental half lives. The fact that the Hamiltonian can be written as the sum (1) means that the wave functions for such rotational levels can be written as the product of two functions, one describing the internal state of the nucleus and one describing the collective rotational motion. From the properties of the rotational wave functions only transitions preserving the parity of the system need be considered. In particular, we shall be interested in electric quadrupole transitions. Since the internal nuclear configuration does not change in a rotational excitation, the intrinsic ground state electric quadrupole moment comes out as a factor when $E2$ matrix elements are calculated. The reduced transition probability is thus :

$$B(E2) = K(I, I_0) Q_0^2, \quad (6)$$

where $K(I, I_0)$ is a numerical factor depending on the spin and Q_0 is the intrinsic electric quadrupole moment.

Experimental results

Experiments yield the energies of low-lying levels and transition probabilities between these levels. Energies can be measured in many ways—cascades after alpha, beta or isomeric gamma-decay, nuclear reactions, electric excitation, and others. It is more difficult to measure half-lives, especially the short lives predicted by the rotational hypothesis. Fortunately many of those levels which have short half-lives have easily measurable electric excitation cross sections.

The basic process of electric excitation is quite simple. A proton (or other charged particle) approaches a nucleus and exchanges kinetic energy through the mutual electrostatic field. The proton never reaches the nucleus. The process is entirely electromagnetic, hence the cross section is easily calculated to be the product of two factors :

$$\sigma_{ex} = A(E, \Delta E, Z, m) B(E2). \quad (7)$$

$A(E, \Delta E, Z, m)$ is obtained from electromagnetic theory. $B(E2)$ is the square of the nuclear matrix element, which of course is essentially the quantity which governs the half-life of

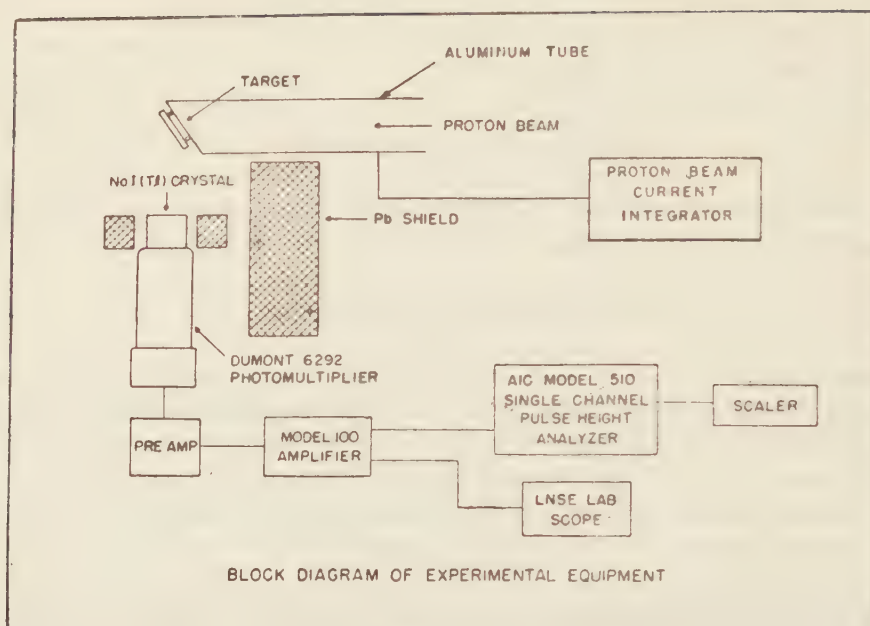


Fig. 2

Experimental equipment

The simplicity of the experimental equipment is shown in the figure. The crystal was placed perpendicular to the proton beam so that it could be effectively shielded from γ -rays coming from the slit system of the generator. The aluminum tube was lined with sheet Sn to reduce the background of γ -rays from scattered protons.

the de-excitation process, see (6). The usefulness of electric excitation now becomes apparent. If, for example, the gamma-rays following proton bombardment are measured, both the energy ΔE and the intensity, proportional to σ_{ex} , are significant quantities.

The experimental method used in the present work is illustrated in Fig. 2. In all cases gamma-rays arising from the de-excitation of nuclear levels were measured. The overall radiations observed fall into three categories:

1. A line spectrum of K X-rays arising from the ionization of the atoms in the target by proton bombardment.
2. A line spectrum of gamma-rays from electric excitation of the nuclei.
3. A continuous bremsstrahlung emitted when the protons are decelerated in the target.

The yield of K X-rays from this type of experiment has been studied by Merzbacher *et al*⁽¹⁾. The bremsstrahlung observed has the spectral distribution, intensity and cross section predicted by Sommerfeld⁽²⁾. These two radiations, however, give rise to rather unpleasant background effects which often make it difficult to identify the nuclear gamma-rays.

After separating out the background effect, the next experimental problem is to determine whether a given nuclear gamma-ray results from electric excitation or from a process involving penetration of the Coulomb barrier. It is clear that (7) applies only if the proton

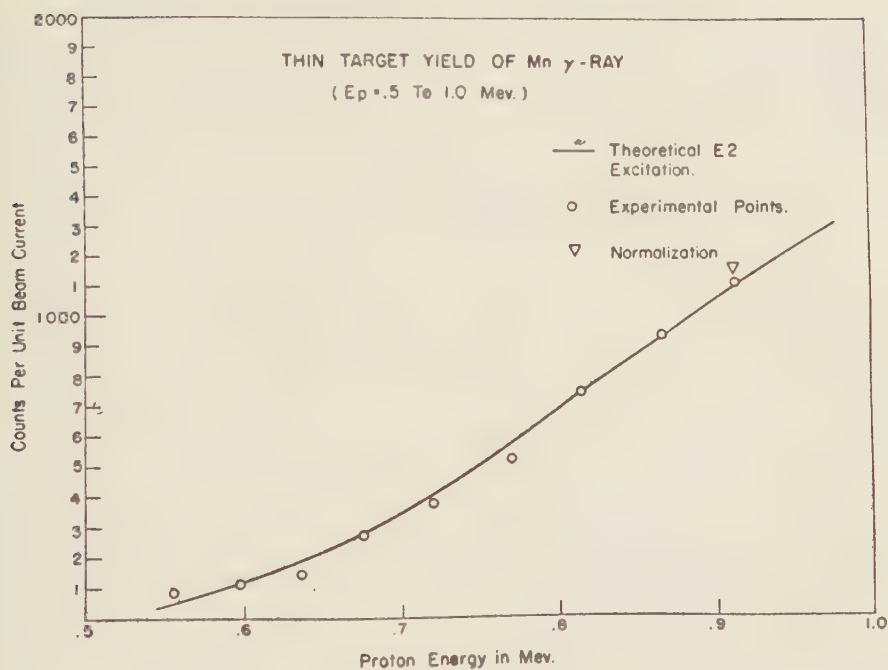


Fig. 3a

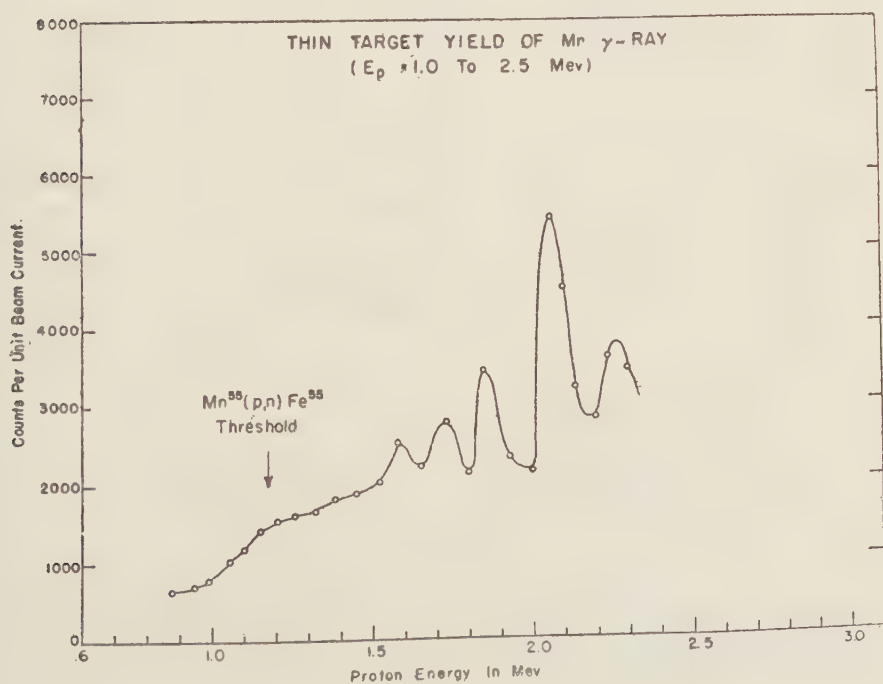


Fig. 3b

never enters the nucleus. In the case of heavy nuclei, say $Z > 50$, the Coulomb barrier is so high that compound nucleus formation is highly improbable for protons in the energy range available to us (< 4 Mev). For the lighter elements this is not the case, and a measurement of the yield of the gamma-ray as a function of proton energy is made in order to determine the process by which it originates. The yield function from electric

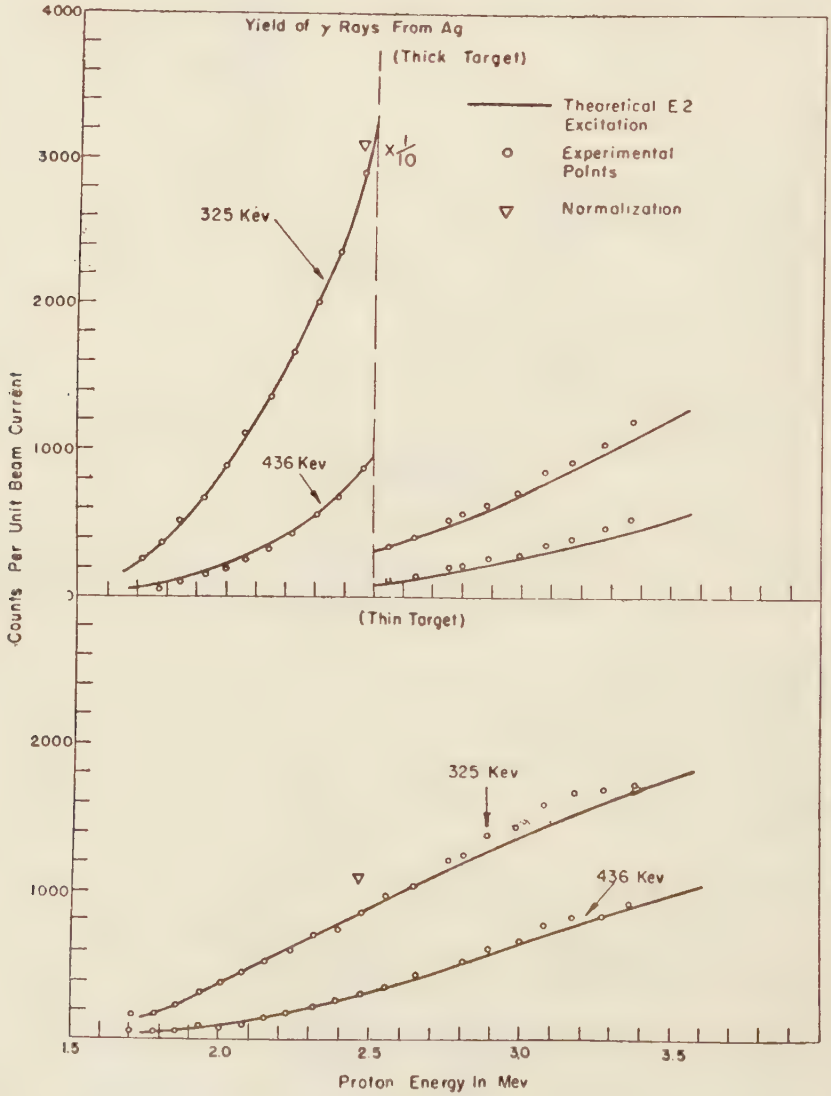


Fig. 4

γ -Ray yields from thick and thin Ag targets

The yield of γ -rays as a function of proton energy from thick and thin Ag targets is shown in this figure. The thin target used in these measurements was roughly 80 kev thick to protons at 2 Mev.

excitation is characteristically different from the yield function obtained when a compound nucleus is formed. The yield of the 131 kev gammas from Mn^{55} is a good example. At low energies, for which it is not probable for the proton to penetrate the Coulomb barrier, the yield function observed, see Fig. 3a, agrees well with that expected from electric ($E2$) excitation. Above 1 Mev barrier penetration becomes important and the yield function, Fig. 3b, shows the characteristic compound nucleus formation resonance structure. Thus the cross section for electric excitation of this level should be measured at energies below 1 Mev.

Fig. 4 shows the yield of two strong gammas from silver ($Z=47$) as a function of proton energy. It is clear that both experimental curves fit the $E2$ excitation function quite well up to about 2.7 Mev. For $Z > 47$, therefore, barrier penetration is not important below this proton energy.

This process has been used to study fairly thoroughly the heavy elements midway between nuclear shells, around $A=180$, and nuclei in the region near the 50 closed proton and neutron shells. In both regions there are many excited levels which exhibit large cross sections. Figures 5, 6 and 7 illustrate the type of data obtained. In Fig. 5 the radiation from the proton bombardment of V_2O_5 is apparently pure bremsstrahlung; the characteristic X-rays have been strongly attenuated by the iron absorber. The shape and peak energy of the bremsstrahlung spectrum depend on the absorber and are essentially independent of the material being bombarded as can be seen by comparison with the bremsstrahlung, lower pulse height portion, of the spectrum from manganese also shown in Fig. 5. The 131 kev gammas from electric excitation of Mn^{55} are well resolved from the bremsstrahlung. This energy level in Mn^{55} has been observed previously by inelastic scattering of protons³⁾ and by electric excitation with alpha particles⁴⁾. The energy of the Mn gamma-ray was determined as 131 ± 3 kev by comparing it with the energy of the K X-rays from lead and tantalum and the 137 kev gamma-ray from Ta^{181} under proton bombardment, for which the energies are well established. The cross section for the production of the 131 kev gammas at a proton energy of 0.94 Mev (only slightly different than that in Fig. 5) was determined to be 0.29 ± 0.11 mb. This measurement required the production of thin Mn targets of known thickness and the calibration of the detection equipment with gamma-rays of essentially the same energy (122 kev) from a standardized source of radioactive Co^{57} placed at the target where the protons strike.

The broad low-energy peak shown in Fig. 6 for silver and palladium (thick targets) is due to proton bremsstrahlung. The characteristic X-rays in both cases have been almost entirely attenuated by the cadmium absorber. Gamma-rays at 90, 325 and 436 kev are clearly resolved for the Ag target. Gamma-rays at 350 and 430 kev are clearly seen from Pd but are not as well resolved because they are considerably lower in intensity and are somewhat closer in energy. The energy and cross section measurements on these two elements are summarized in Table 1.

Fig. 7 shows the superposed pulse spectra observed⁵⁾ during bombardment of targets enriched in Re-185 and Re-187 with 2.5 Mev protons. In each cases 0.1 inch of copper was used to reduce the K X-rays, which comprise the major source of the dominant peak

Table 1. Electric excitation measurements on Ag and Pd

Element	Gamma Energy (kev)	Cross Section in mb at indicated E_p in Mev	
		Thick Target	Thin Target (85 kev)
Ag	325	0.25 ± 0.10 ($E_p = 2.68$)	0.27 ± 0.11 ($E_p = 2.68$)
Ag	436	0.15 ± 0.07 ($E_p = 2.68$)	0.20 ± 0.08 ($E_p = 2.68$)
Pd	350	0.033 ± 0.013 ($E_p = 2.46$)	—
Pd	430	0.078 ± 0.031 ($E_p = 2.46$)	—

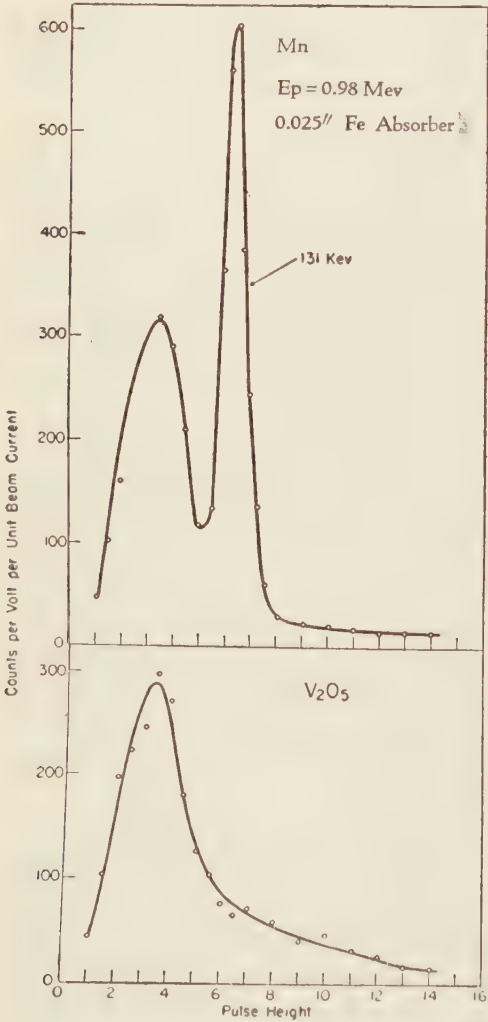


Fig. 5. γ -Spectra from thick Mn and V_2O_5 targets. The broad low energy peak observed from both targets is due to proton bremsstrahlung. Characteristic X-rays from Mn and V are almost entirely attenuated by the Fe absorber. A strong 131 kev γ -ray is observed from the Mn target while no γ -rays are seen in V.

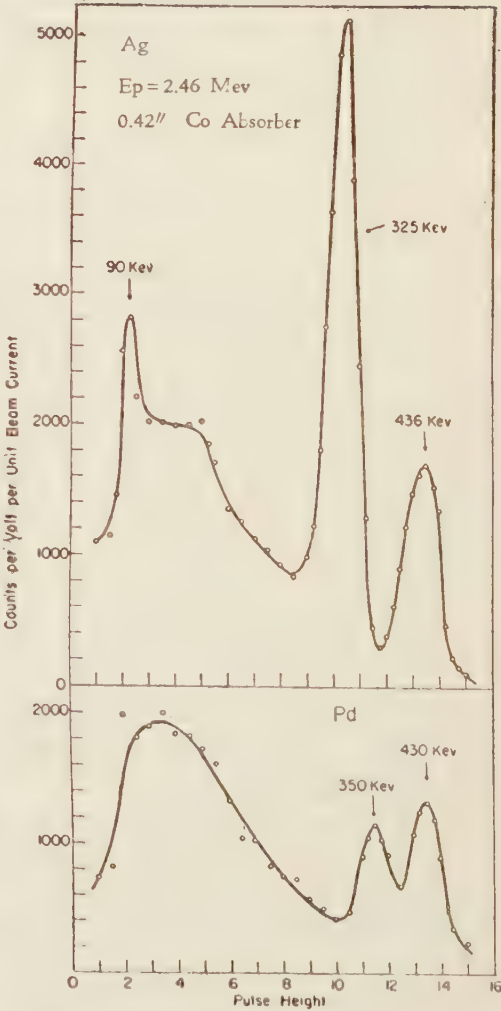


Fig. 6. γ -Spectra from thick Ag and Pd targets. The broad low energy peak observed from both targets is due to proton bremsstrahlung. Characteristic X-rays from Ag and Pd are almost entirely attenuated by the Cd absorber. γ -rays at 90, 325, and 436 kev are observed from the Ag target, and γ -rays at 350 and 430 kev are observed from the Pd target.

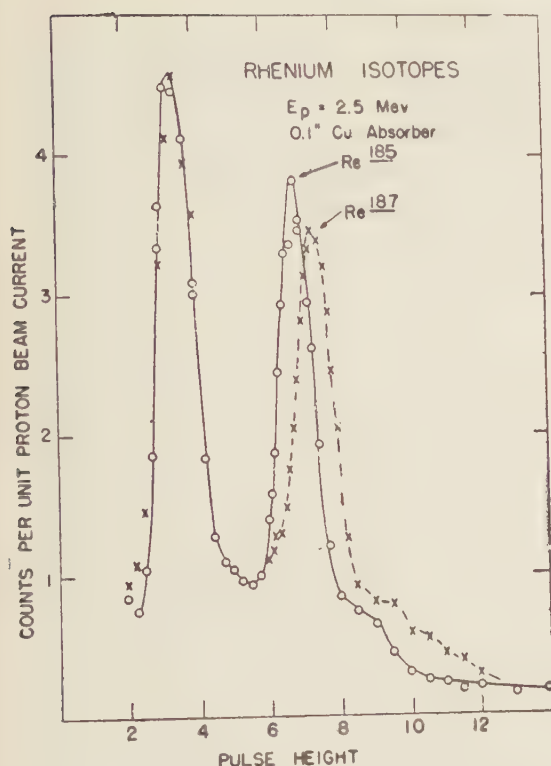


Fig. 7. Electric excitation of rhenium isotopes. Superposition of the pulse spectra observed during bombardment of targets enriched in Re-185 and Re-187 with 2.5-Mev protons. In each case, a 0.1 inch copper absorber was used to reduce the X-ray background. No correction for attenuation in the copper has been made in the figure.

Energies of these gammas, considered correct to ± 3 percent are: Re¹⁸⁵.....130 kev, Re¹⁸⁷.....139 kev.

The irregular pulse distribution at the right side of the figure may be interpreted as low-intensity cascade gammas from the second level in these isotopes, which is also excited by electric excitation at this proton energy.

at the left. In heavy elements such as this, the bremsstrahlung is so small compared to the X-ray intensity that it is hardly noticeable. The energies of the Re gammas, considered correct to about 3 per cent, are: Re¹⁸⁵ 130kev and Re¹⁸⁷ 139 kev. The irregular pulse distribution at the right side of these two peaks may be interpreted as low-intensity cascade gammas from the second level in these Re isotopes, which is also excited by electric excitation at this proton energy.

§ 3. Conclusions

Considerable more data of the foregoing type have been obtained, and we now consider these in terms of the theory previously discussed. The first important information is the ratio of the energies of the first and second excited states for which in a few cases it is possible to obtain a value. The rotational model predicts that the spectrum is given precisely by formula (5). The second excited level is roughly as predicted in terms of the first level in all the odd A nuclei which we have studied and in which the yield is sufficient to allow the energy to be determined. We cannot see the second levels in even A nuclei because of the

small cross sections for octupole excitation. The agreement between theory and experiment here is evidence that the wave functions obtained from the collective model are good approximations to the actual wave functions. However, the energy splitting is a property of the rotational wave function and is therefore essentially dependent on the kinematics of the motion only. Hence about all this tells us is that something rotates but we cannot say precisely what rotates.

In an attempt to answer this question, we must make more detailed comparisons. One of the unique features of the rotational levels is that both the energy and the radiative transition probability of such a level depend upon the same parameter, the ground state

Table 2. Experimental data on ΔE and Q_0

The values of ΔE were obtained for the nuclei listed by means of a scintillation spectrometer or from beta-ray spectra. Values for the electric quadrupole moment Q_0 have been deduced from half-life measurements of the radioactivity (Raa), from the tabulated ΔE 's, from the observed cross section for electric excitation ($\sigma_{ex}(l)$) of the given level and from spectroscopic (Spec.) measurements of hyperfine structure.

*Nucleus	ΔE (kev)	Raa	Q_0 in 10^{-24} cm ²		Spec.
			ΔE	$\sigma_{ex}(l)$	
⁴⁹ In ¹¹⁵	335	—	—	—	2.2
⁵³ I ¹²⁷	57?	—	—	—	—1.3
⁶³ Eu ¹⁵¹	—	—	—	—	3.5
⁶³ Eu ¹⁵³	103	—	—	—	7.0
⁶⁶ Dy ¹⁶⁰	85	7	—	—	—
⁶⁸ Er ¹⁶⁶	80	8	—	—	20(⁶⁸ Er ¹⁶⁷)
⁷⁰ Yb ¹⁷⁰	84	8	—	—	11(⁷⁰ Yb ¹⁷³)
⁷¹ Lu ¹⁷⁵	112	—	17.2	6.5	13
⁷¹ Lu ¹⁷⁶	240	—	17.5	—	9—12
⁷² Hf ¹⁷⁶	87	8	16	22.1	—
⁷² Hf ¹⁷⁷	112	—	13	10.5	—
	235	—	13.9	—	—
⁷² Hf ¹⁷⁸	91	—	15.8	16.8	—
⁷² Hf ¹⁷⁹	122	—	12.6	6.3	—
	250	—	13.6	—	—
⁷² Hf ¹⁸⁰	92	—	15.8	16.3	—
⁷³ Ta ¹⁸¹	138	—	16	6.7	13
	300	—	16.1	5.8	
⁷⁴ W ¹⁸²	101	7	15.1	5.2	—
⁷⁴ W ¹⁸³	103	—	13.9	2.7	—
⁷⁴ W ¹⁸⁴	112	—	14.6	4.5	—
⁷⁴ W ¹⁸⁶	124	—	14.1	4.1	—
⁷⁵ Re ¹⁸⁵	130	—	14.8	4.7	7.8
	290	—	15.0	—	
⁷⁵ Re ¹⁸⁷	139	—	14.4	4.3	7.3
	320	—	14.4	—	
⁷⁶ Os ¹⁸⁶	137	6	—	—	—
⁷⁸ Pt ¹⁹⁴	330	—	8.9	2.4	—
⁷⁸ Pt ¹⁹⁵	210	—	10.3	3.2	—
⁷⁸ Pt ¹⁹⁶	360	—	8.6	1.7	—
⁷⁸ Pt ¹⁹⁸	425	—	8.0	1.4	—
⁸⁰ Hg ¹⁹⁷	165	2	—	—	—
⁸⁰ Hg ¹⁹⁸	411	2	—	—	—
⁸⁰ Hg ¹⁹⁹	368	2	—	—	—
⁸⁰ Hg ²⁰¹	—	—	—	—	—
⁸³ Bi ²⁰⁹	—	—	—	—	2.5
⁸⁴ Po ²¹²	719	2	—	—	—0.72
⁸⁴ Po ²¹⁴	606	2	—	—	—

* M1 radiation assumed for odd A nuclei.

deformation of the nucleus. It is therefore of some interest to obtain the values of the ground state deformations by these two methods and compare them with each other and with the deformations obtained from independent observations. It can be seen from Table 2 that there is a substantial difference between the values of Q_0 (which depends on the nuclear deformation) obtained from the cross section $\sigma_{ex}(l)$ and from the energy of the transition JE . In particular, except for the three even isotopes of hafnium for which the $Q_0(\sigma)$ is slightly larger than $Q_0(JE)$, the $Q_0(JE)$ is substantially greater than $Q_0(\sigma)$. In the middle of the shell, see Hf^{179} , the latter difference is roughly a factor of two whereas near the closed shell at Pb^{208} the difference is about a factor of seven. Similar results have been obtained for the lighter nuclei⁶⁾.

$Q_0(JE)$ depends on the deformation of the mass distribution since it arises from the effective moment of inertia of the nucleus. $Q_0(\sigma)$ depends on the deformation of the charge distribution since this is what determines the radiative properties (and conversely the electric excitability) of the nucleus. It follows therefore that these two experimental values of Q_0 will be equal only if the two types of deformation are equal. It is generally assumed that the charges (protons) and the masses (protons and neutrons) are uniformly distributed throughout the nuclear volume. In which case the deformation of the charge and mass would be expected to be the same.

The larger value of $Q_0(JE)$ as compared to $Q_0(\sigma)$, especially near the closed proton shells at 50 and 82 may be explained as follows:

- a) the protons and neutrons are not uniformly distributed throughout the nucleus.
- b) the spherical symmetry of the protons is greater than the spherical symmetry of the neutrons near the closed proton shells at 50 and 82 despite the fact that at 50 the neutron number is between the neutron shells at 50 and 82 whereas at proton shell 82 the neutron number is equal to the neutron shell at 126.
- c) the protons are concentrated closer to the center of the nucleus than the neutrons thereby resulting in a larger effective moment of inertia for rotation than would be indicated by the electric excitation cross section.
- d) as an alternative or additional factor to (c), the flow within the nucleus is not purely irrotational but some particles circulate with the surface wave thereby enhancing the measured moment of inertia of the nucleus.

Johnson and Teller⁷⁾ have given supporting evidence that the neutrons, which contribute the bulk of the mass to heavy nuclei, may be more on the outside while the charge, *i.e.* the protons, may be more toward the inside. Recent measurements of nuclear radii using neutrons (mass sensitive) give somewhat higher values than those obtained with electrons (charge sensitive). On the other hand, there is evidence to the contrary⁸⁾. If a nuclear core exists in heavy nuclei and retains its "light nuclear character", such as the $Z=50$ and $N=70$ of Sn, the surrounding nuclear matter in doubly magic Pb^{208} would contain a much higher proportion of neutrons, $N=56$, than protons, $Z=32$. If in addition this outer nuclear matter were capable of greater deformation and irrotational flow, the observed differences in Q_0 might be explained. Further theoretical work is needed in order to test this hypothesis.

An additional qualification needs to be introduced in the conclusions drawn from Table

2. Except for Ta^{181} for which the situation is established, the values of $Q_0(\sigma)$ for odd A nuclei are uncertain because of assumed pure $M1$ radiation. However, the amount of $E2$ radiation present is very unlikely to be sufficient to make a significant difference in the relative values of the Q_1 's. Furthermore the values of $Q_0(\sigma)$ for Hf^{177} and Hf^{179} are somewhat uncertain because of the uncertainty in the ground state spins.

So few spectroscopic data are available and those that are have such large experimental uncertainty that a comparison of Q_0 (spec) with the other Q_n 's is not very informative. In general Q_0 (spec) lies between $Q_0(\Delta E)$ and $Q_0(\sigma)$.

The Q_0 (Raa) values are based on measured half-lives of rapidly decaying isomers. For this reason these data are also rather uncertain, though the accuracy of this type of measurement is being improved rapidly. There are too few cases in which a direct comparison can be made to obtain any definite conclusions. However, valuable comparisons of this kind should be possible in the near future.

One way of getting at the answer to the foregoing dilemma is by determining the gyromagnetic ratio of the collective motion. The branching ratios of second rotational levels in odd A nuclei depend essentially on the gyromagnetic ratios. Coincidence measurements now in progress should enable these branching ratios to be determined. If the gyromagnetic ratio is that predicted for a uniform distribution of charge and mass, the observed differences are probably due to irrotational flow. If the gyromagnetic ratio is different than predicted, the effect may be due at least partially to the non-uniform distribution of protons and neutrons.

For years the geophysicists have been using seismic waves to gain some inkling of the composition deep down in the earth's interior. It is encouraging to feel that nuclear physicists now have tools that offer some promise of telling about the not-so-deep-down insides of nuclei.

We wish to express our appreciation for valuable discussions with Professors Satio Hayakawa and Osamu Miyatake. The support of the U. S. Educational Commission in Japan is gratefully acknowledged.

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Separation Energies and Nuclear Structures in Light Nuclei

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The separation energies of light nuclei $10 \leq A \leq 50$ are considered in relation to the nuclear structure. We can grasp the physical meaning of the complicated variation of the separation energies of nucleons from nucleus to nucleus phenomenologically, by comparing them with the contributions to the separation energies, from each term of the semi-empirical mass formula. The systematic deviation of the separation energies from the semi-empirical mass formula disappears by the corrected mass formula by the uniform model which assumes the two-body interactions between symmetric pairs in supermultiplet structure. This fact strongly supports that the symmetry effect of the nuclear binding energy originates from the above mentioned two-body interactions. The deviations of the separation energies from the corrected mass formula of the uniform model show the evidences to support the independent particle model. In this mass region, the nucleus has, besides the dominant supermultiplet structure, the $j-j$ coupling shell structure. The comparison between the mass formula and the above mentioned potential energy of two-body interactions which give the symmetry effects quantitatively, and the requirement that the separation energy is nearly equal to the depth of the energy level of the last nucleon measured from the top of the potential in the independent particle model, lead us to the conclusion that there are other kinds of nuclear potential energies which are not sensitive to the symmetry effect and to the "expansibility" of the nucleus, and bear about a half of the total potential energy of the nucleus. And then, the depth of the average potential for the last nucleon is estimated to be about 50 Mev and is a linearly decreasing function of X ($X=Z-N$ for protons, $X=N-Z$ for neutrons).

§ 1. Introduction and summary

The separation energy of a proton or a neutron from a nucleus is the energy required to remove the most loosely bound proton or neutron adiabatically. The corresponding quantity in the atom is the ionization energy of an outermost electron. Ionization energies give important knowledge about the interaction between the electron and the residual atom and are intimately related to the atomic shell structure. The effects of the nuclear shell structure on the separation energies of nuclei have been studied by many authors, and the separation energies show the discontinuities at each of the shell edges,¹⁾ although the magnitudes of the discontinuities have not been explained in relation to the nuclear structures. The fact that both the electrons in atoms and the nucleons in nuclei are fermions gives a similar character to the ionization and separation energies, but on the other hand, they have quite different characters in many respects. In the atom there is only one kind of fermions, the electrons, moving in the strong electric field of the nucleus, whereas in the nucleus there are two kinds of fermions, the protons and the neutrons, interacting strongly with each other. The force acting on an electron in the atom is, in fairly good approximation,

replaced by the average static force, and when an electron is removed from an atom, the residual atom changes its structure only slightly. As for the nuclei, the validity of the approximation of the single particle model is not clear experimentally as well as theoretically and when a nucleon is removed from a nucleus, the residual nucleus changes its structure appreciably (Section 4). Therefore, the ionization energies vary monotonically with respect to the numbers of electrons of the atoms except at the shell edges, whereas the separation energies vary in a complicated way with the numbers of protons and neutrons. To understand these variations, we need a theory which gives the total binding energies of nuclei as a function of the numbers of protons and neutrons, because we must take into account the change of the nuclear structure when a nucleon is removed. Of course, at present there is no theory except the semi-empirical ones.

The variation of separation energies is large in the region where the mass number is small, and becomes smaller as the mass number becomes large. For the region of mass number $16 \lesssim A \lesssim 50$, the analysis of this variation is tried at first, since the variation is large and regular in the region. The results are summarized as follows.

1. We can grasp the physical meaning of the complicated variation of the separation energies of nucleons from nucleus to nucleus phenomenologically, by comparing them with the contributions to the separation energies, from each term of the semi-empirical mass formula (with the coefficient of Fermi⁽²⁾) (Section 2).

2. The systematic deviations of the separation energies from the values calculated from the semi-empirical mass formula explained excellently well if we use the corrected mass formula by the uniform model⁽³⁾ which assumes the two-body interactions between symmetric pairs in supermultiplet structure. This fact strongly supports the fact that the symmetry effect of the nuclear binding energy originates from the above mentioned two-body interactions (Section 3).

3. The deviations of the separation energies from the corrected mass formula of the uniform model show the evidences to support the independent particle model (I. P. M.). In this mass region, the nucleus has, besides the dominant supermultiplet structure, the $j-j$ coupling shell structure⁽⁴⁾ (Section 3).

4. The comparison between the mass formula and the potential energy of two-body interactions which gives the symmetry effects quantitatively, and the requirement that the separation energy is nearly equal to the depth of the energy level of the last nucleon measured from the top of the potential in the independent particle model, suggest us that there are other kinds of nuclear potential energies which are not sensitive to the symmetry effect and to the "expansibility" of the nucleus which is defined by (28), and bear about a half of the total potential energy of the nucleus. And then, the depth of the average potential for the last nucleon is estimated to be about 50 Mev and is a linearly decreasing function of X (X is defined in Section 3) (Section 4).

§ 2. Separation energies and semi-empirical mass formula

In Figs. 1-4, the separation energies of a proton and a neutron, S_p and S_n , from the

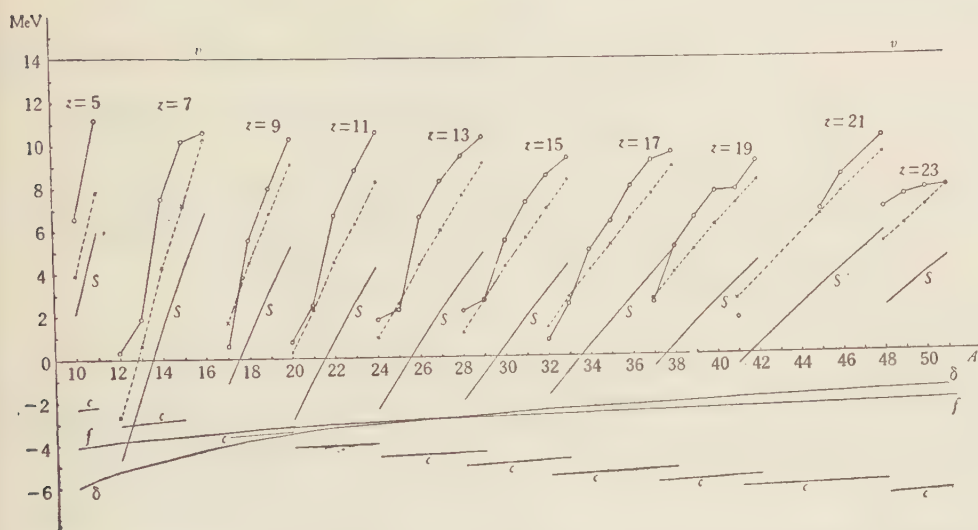


Fig. 1. Plot of odd proton separation energies from nuclei (A, Z). Points of given Z are connected. \circ : Experimental value. \times : Calculated value from the mass formula (1). Included for comparison are curves of contributions of five terms of mass formula (1), v : volume, f : surface, s : symmetry, c : coulomb and δ .

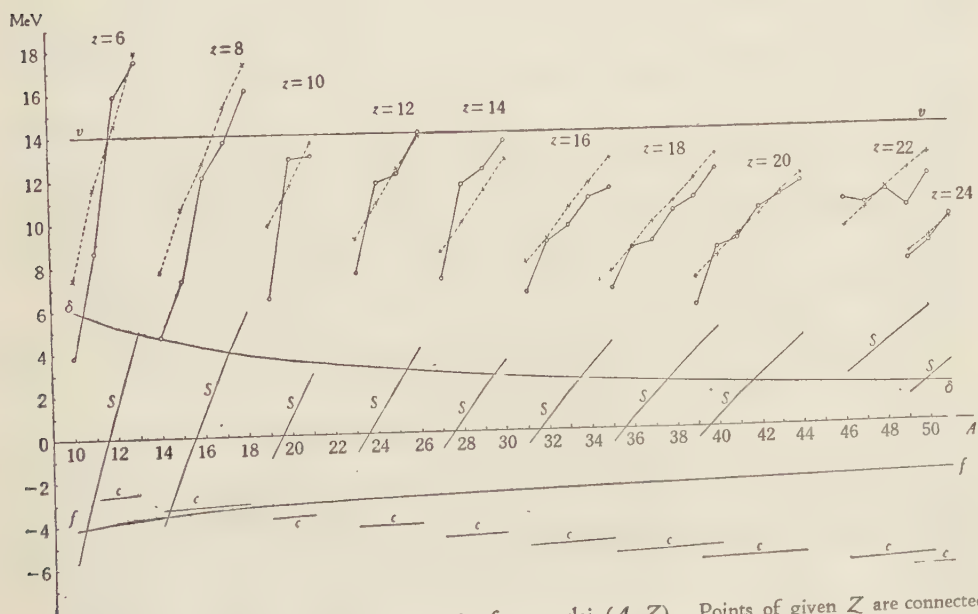


Fig. 2. Plot of even proton separation energies from nuclei (A, Z). Points of given Z are connected. \circ : Experimental value. \times : Calculated value from the mass formula (1). Included for comparison are curves of contributions of five terms of mass formula (1), v : volume, f : surface, s : symmetry, c : coulomb and δ .

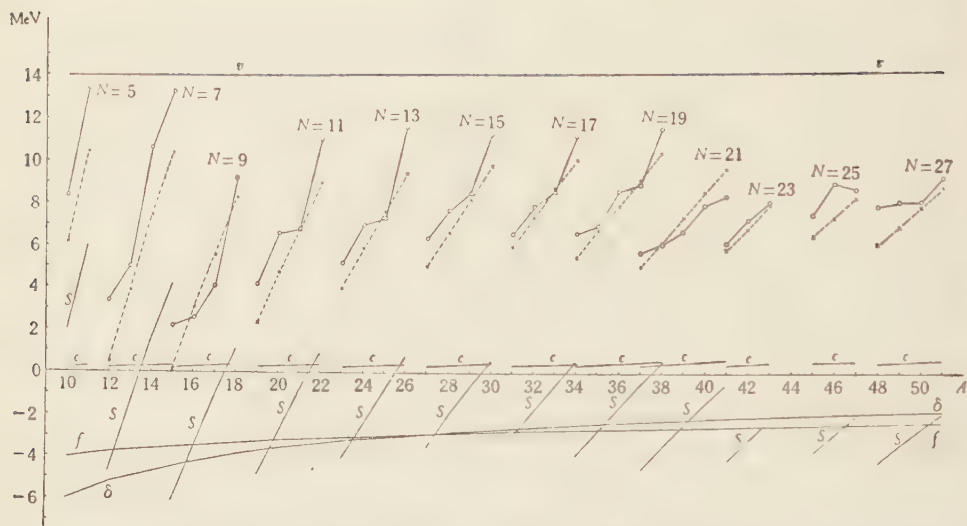


Fig. 3. Plot of odd neutron separation energies from nuclei (A, N). Points of given N are connected. \circ : Experimental value. \times : Calculated value from the mass formula (1). Included for comparison are curves of contributions of five terms of mass formula (1), v : volume, f : surface, s : symmetry, c : coulomb and δ .

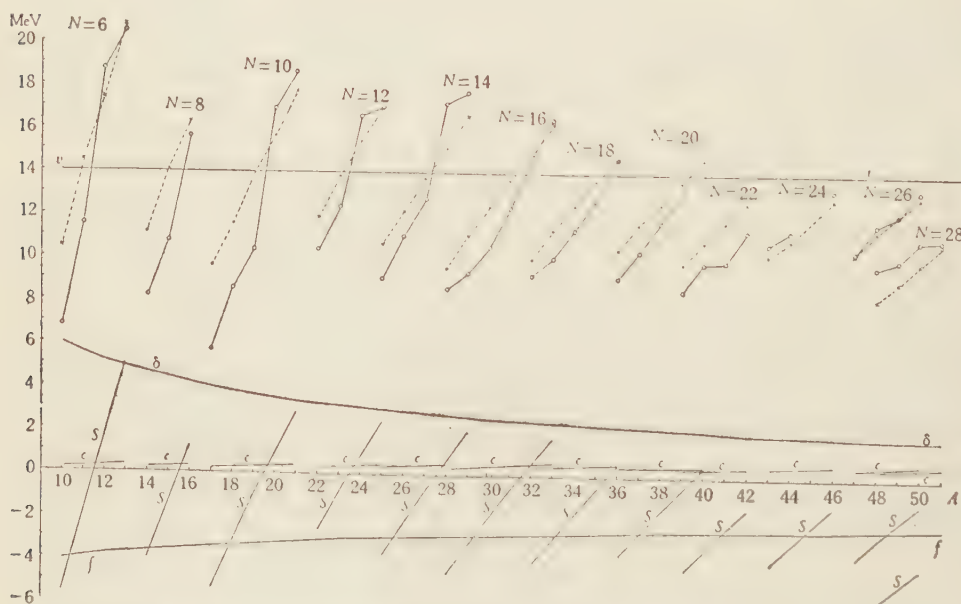


Fig. 4. Plot of even neutron separation energies from nuclei (A, N). Points of given N are connected. \circ : Experimental value. \times : Calculated value from the mass formula (1). Included for comparison are curves of contributions of five terms of mass formula (1), v : volume, f : surface, s : symmetry, c : coulomb and δ .

nuclei (A, Z) and (A, N) are analysed in terms of the contribution of each term of the semi-empirical mass formula with the coefficients of Fermi,²⁾

$$B(A, Z) = 14.03A - 13.03A^{2/3} - 77.27(A/2 - Z)^2/A - 0.5837 Z^2/A^{1/3} - \delta(A, Z), \text{ (in Mev)} \quad (1)$$

where $\delta(A, Z) = 0$ for odd A and $\delta(A, Z) = \pm 33.52A^{-3/4}$ for odd Z / even Z and even A . The reference to the adopted experimental values are summarized in Appendix.

§ 3. Interpretation by the uniform model

We have seen from Figs. 1-4, in the preceding section that the magnitudes of the separation energies of $S_p(Z, A)$ and $S_n(N, A)$ depend mainly on the proton excess $(2Z - A)$ and the neutron excess $(2N - A)$, apart from the charge effects and even-oddness of Z and N . Hence it seems convenient to denote $S_p(Z, A)$ and $S_n(N, A)$ as $S_p(X_p, Z)$ and $S_n(X_n, N)$ where X_p and X_n are $X_p = 2Z - A$, $X_n = 2N - A$. Further, we shall use the word "excess X ." X is defined as

$$\begin{aligned} X &= X_p = 2Z - A \text{ for protons,} \\ X &= X_n = 2N - A \text{ for neutrons,} \end{aligned} \quad (2)$$

where Z , N and A are the proton and neutron numbers and the mass number of the nucleus from which the nucleon is removed. Then the separation energy S is specified by the nucleon kind p or n which we denote with τ , and the nucleon number Z or N which we denote with L , and the excess X . Thus we can write the separation energy as $S_\tau(X, L)$. One can see from Figs. 1-4.

1. $S_\tau(X, L)$'s with given τ and X and given even-oddness of L do not depend strongly on L .

2. $S_\tau(X, L)$'s with given X and given L are nearly equal if the coulomb energy contributions are subtracted from both of them.

3. $S_\tau(X, L)$'s with given τ and given X depend strongly on even-oddness of L .

4. $S_\tau(X, L)$'s with given τ and given L strongly depend on X .

In Fig 5. the deviations of the separation energies $S_\tau(X, L)$ with the given X and odd L from the calculated values with the mass formula (1) are plotted against L . The deviations of two-nucleon separation energies from the nuclei of even $(L+1)$

$$S_\tau(X+1, L+1) + S_\tau(X, L)$$

are also plotted at the same abscissa, to make easy the comparison of one and two-nucleon separation energies which have the same residual nucleus.

The following features *a* and *b* in Fig. 5 show the systematic deviations of S_p and S_n from the mass formula (1), and suggest us to analyse the data by Wigner's uniform model.*

*) T. L. Collins, A. O. Nier and W. H. Johnson¹⁷⁾ and R. E. Halsted¹⁸⁾ analysed the nuclear binding energies by the Wigner's mass formula. They compared with the Wigner's formula the potential energy, calculated from the nuclear mass and the kinetic energy given by the Fermi gas model, and obtained the values for two parameters. But our method of analysis is different from theirs and uses the formula (11).

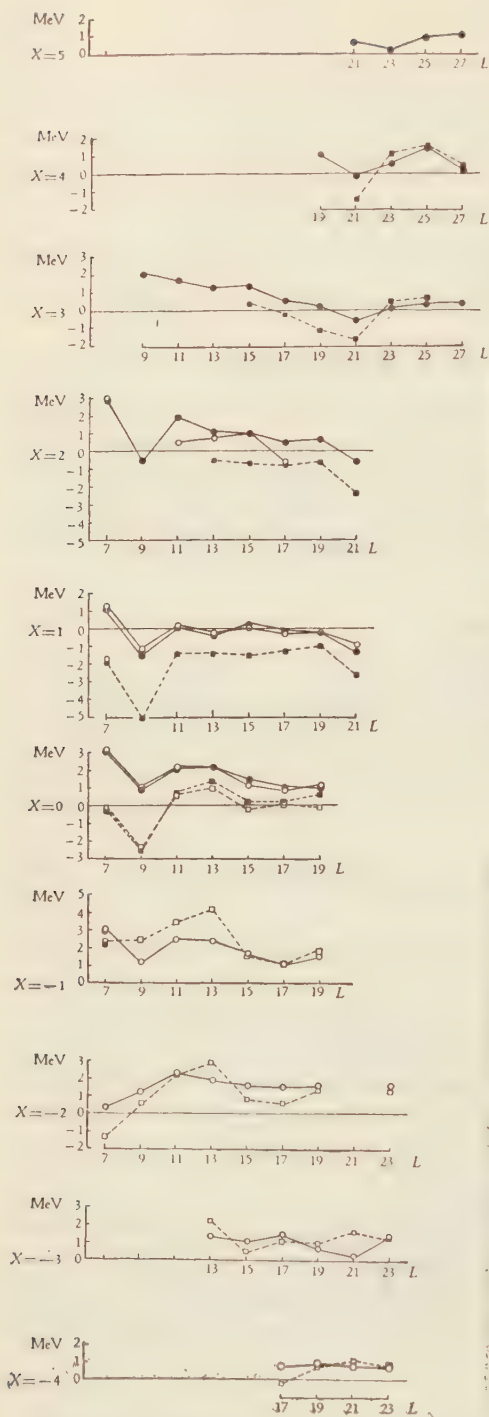


Fig. 5. Plot of $\Delta\tau^{(1)}(X, L)$ and $\Delta\tau^{(2)}(X, L)$, deviations of $S_\tau(X, L)$ and $S_\tau(X+1, L+1) + S_\tau(X, L)$ from the values calculated with the mass formula (1) against odd L . \circ : for one proton, \bullet : for one neutron, \square : for two protons, \blacksquare : for two neutrons.

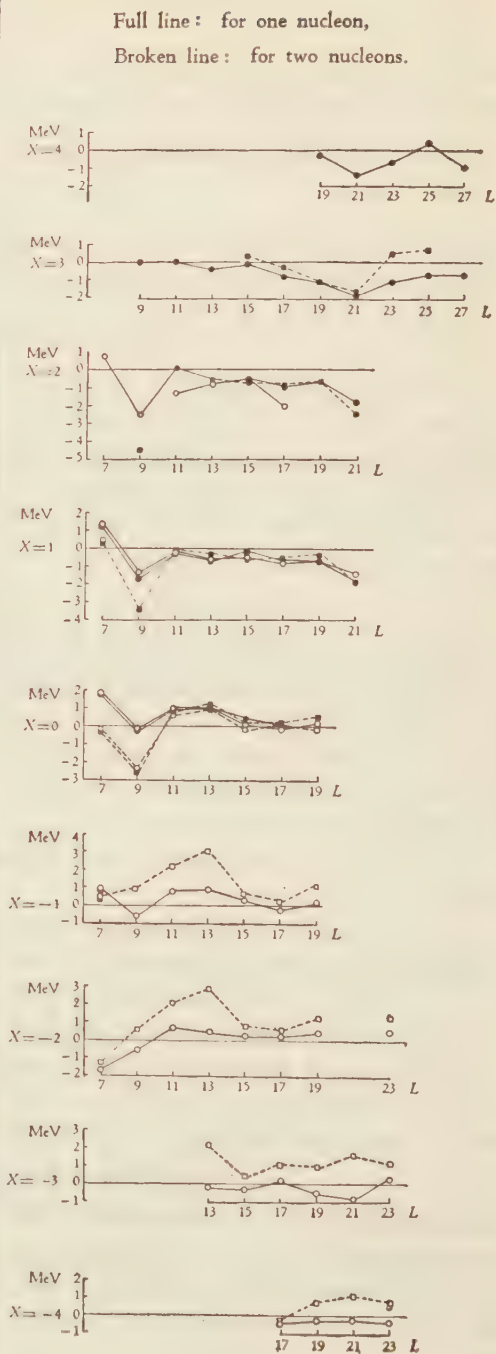


Fig. 6. Plot of $\Delta\tau^{(1)}(X, L)$ and $\Delta\tau^{(2)}(X, L)$, deviations of $S_\tau(X, L)$ and $S_\tau(X+1, L+1) + S_\tau(X, L)$ from the values calculated with the mass formula (11) against odd L . \circ : for one proton, \bullet : for one neutron, \square : for two protons, \blacksquare : for two neutrons.

a. The connected lines for one nucleon are too high by nearly the same amount except for $X=1$, and are higher than those for the two-nucleons except for $X=-1$.

b. The connected line for one nucleon of $X=1$ is lower in comparison with others. The connected line for two nucleons of $X=1$ is lower, and that of $X=-1$ is higher in comparison with others.

Now we derive the corrected mass formula given by (11) by the uniform model⁽³⁾ which assumes that the Hamiltonian of the nucleus (A, Z) is given as

$$H(A, Z) = ZM_p C^2 + (A-Z)M_n C^2 + \sum_{i=1}^A T_i + 1/2 \sum_{i,j=1}^A V(x_i, x_j) + 1/2 \sum_{i,j=1}^Z V_c(x_i, x_j), \quad (3)$$

where $x_1 \cdots x_Z$, and $x_{Z+1} \cdots x_A$ are the co-ordinates of protons and neutrons, T_i is the kinetic energy of the i -th particle, $V_c(x_i, x_j)$ is the coulomb energy between i -th and j -th particle, and $V(x_i, x_j)$ is the potential energy of the two-body nuclear force between the particles i and j . The wave function of the ground state of the nucleus (A, Z) Φ_{0AZ} satisfies

$$H(A, Z) \Phi_{0AZ} = M(A, Z) \Phi_{0AZ}. \quad (4)$$

The uniform model assumes that the expectation values of the potential energies in the state Φ_{0AZ} are given as follows, when the anti-symmetric pairs make no contribution to the potential energy,

$$\langle 1/2 \sum V_c(x_i, x_j) \rangle_{0AZ} = (3/5) (e^2/r_0) Z^2 A^{-1/3}, \quad (5)$$

$$\langle 1/2 \sum_{i,j} V(x_i, x_j) \rangle_{0AZ} = -2n_+(A, Z) L_1 A^{-1} \equiv V_+, \quad (6)$$

where $-2L_1/A$ is the average potential energy per symmetric pair in the nucleus A , and $n_+(A, Z)$ is the number of symmetric pairs in the nucleus (A, Z) and is given by⁽³⁾⁽⁵⁾

$$n_+(A, Z) = (3/16) A^2 + (3/4) A + (5/4) - (1/2) \xi(A, Z), \quad (7)$$

where

$$\begin{aligned} \xi(A, Z) &= (1/2) T_{\frac{1}{2}}^2 + 2|T_{\frac{1}{2}}| + (13/4), \text{ for odd } A \text{ nuclei,} \\ &= (1/2) T_{\frac{1}{2}}^2 + 2|T_{\frac{1}{2}}| + 2(5/2), \text{ for even-even nuclei,} \\ &= (1/2) T_{\frac{1}{2}}^2 + 2|T_{\frac{1}{2}}| + 4, \text{ for odd-odd } N \neq Z \text{ nuclei,} \\ &= 5, \text{ for odd-odd } N = Z \text{ nuclei.} \end{aligned} \quad (8)$$

The above $n_+(A, Z)$ are derived assuming that the spin of each nucleon is a good quantum number. In the light nuclei region, the spin-orbit coupling is not strong and, according to Peaslee⁽⁶⁾, in the nuclei of the mass number region considered here the supermultiplet structure⁽⁷⁾ is dominant.⁽⁸⁾ But in heavier nuclei, the change in $n_+(A, Z)$ resulting from the strong spin-orbit coupling must be scrutinized.

In the case $|T_{\frac{1}{2}}| \leq 2$, $\xi(A, Z)$ in (8) can be expressed almost exactly as

$$\xi(A, Z) = 1.75 T_{\frac{1}{2}}^2 + 3.95 + A + A_0,$$

where

$$\begin{aligned}
 A &= \begin{cases} 0 & \text{for odd } A \text{ nuclei,} \\ 0.81 & \text{for odd-odd nuclei,} \\ -0.81 & \text{for even-even nuclei,} \end{cases} \\
 A_0 &= \begin{cases} 0.25 & \text{for odd-odd } N=Z \text{ nuclei,} \\ -0.63 & \text{for even-even } N=Z \text{ nuclei,} \\ 0 & \text{for } N \neq Z \text{ nuclei.} \end{cases}
 \end{aligned} \tag{9}$$

Then we have in the mass number region considered here,

$$\begin{aligned}
 M(A, Z) &= \langle H(A, Z) \rangle_{0AZ} \\
 &= ZM_p C^2 + (A-Z) M_n C^2 + \langle \sum T_i \rangle_{0AZ} \\
 &\quad + (3e^2/5 r_0) Z^2 A^{-1/3} + V_s^S + V_s^A,
 \end{aligned}$$

where

$$\begin{aligned}
 V_s^A &= -((3/8)A + (3/2) + 1.70 A^{-1}) L_1, \\
 V_s^S &= (1.75 T_{\zeta}^2 + J + J_0) L_1 A^{-1}.
 \end{aligned} \tag{10}$$

V_s^S is the part of the potential energy V_s given by (6) which contributes to the symmetry effect of the binding energy. V_s^A is the part of V_s which depends only on A and does not contribute to the symmetry effect. According to the discussions in the next section, it seems likely that there exist other kinds of potential energies which depend on A only.

Hence, if we assume that the part of potential energy which depend only on A is determined empirically by the first and second term of the mass formula (1), the empirical mass formula corrected by the uniform model gives the following binding energy,

$$\begin{aligned}
 B_0(A, Z) &= 14.03A - 13.03A^{2/3} - (3/5)(e^2/r_0)Z^2 A^{-1/3} \\
 &\quad - (1.75 T_{\zeta}^2 + J + J_0) L_1 A^{-1}.
 \end{aligned} \tag{11}$$

In the formula (11), the symmetry energy which comes from the kinetic energy term $\langle \sum T_i \rangle_{0AZ}$ is omitted to be considered separately. The term $77.27 T_{\zeta}^2/A$ in (1) has explained very well the separation energy variations in Figs. 1-4, so we determine L_1 in (11) by

$$1.75 L_1 T_{\zeta}^2 = 77.27 T_{\zeta}^2$$

and obtain

$$L_1 = 44.15 \text{ Mev.} \tag{12}$$

L_1 in (12) determines the value of $L_1 J/A$ which corresponds to $\delta(A, Z)$ in (1) as

$$L_1 J/A = \pm 35.7/A \text{ Mev.} \tag{13}$$

and the value of $L_1 A_0/A$ which is the energy to be added to the $N=Z$ nuclei as

$$\Delta_0 L_1/A = \begin{cases} 11.05/A \text{ Mev for odd-odd nuclei,} \\ -27.85/A \text{ Mev for even-even nuclei.} \end{cases} \quad (14)$$

The term (13) is appreciably smaller* than $\partial(A, Z)$ in (1) and explains the systematic deviation a , and the term (14) represents the extra energies of $N=Z$ nuclei and explains the systematic deviation b .

In Fig. 6, the deviations of $S_\tau(X, L)$ of odd L from the values calculated with (11) which are denoted by $\Delta_\tau^{(1)}(X, L)$, and the deviations of $S_\tau(X, L) + S_\tau(X+1, L+1)$ from the values calculated with (11), which are denoted by $\Delta_\tau^{(2)}(X, L)$, are plotted against L , in the same way as in Fig. 5.

Fig. 6 shows that the formula (11) represents the binding energies very well, and the systematic deviations in Fig. 5 have disappeared. In this fit the contribution of the last term in (11), which is determined by using only one parameter L_1 , is essential.

The variations of $\Delta_\tau^{(1)}(X, L)$ and $\Delta_\tau^{(2)}(X, L)$ with respect to L are well understood as the variations of the kinetic part of $S_\tau(X, L)$ expected from the I. P. M., and are considered as evidences of the I. P. M. structure.

1. At $L=9$ and $L=21$, the value of $\Delta_\tau^{(1)}(X, L)$ falls. The magnitude of this fall becomes gradually small as X becomes small and negative. The fall of $\Delta_\tau^{(2)}(X, L)$ is deeper than that of $\Delta_\tau^{(1)}(X, L)$. The conspicuous exception is the point for $S_n(3, 9)$, and we want to discuss about $S_n(3, 9)$ in a separate paper.

2. At $L=11, 13$ (and 19), the values of $\Delta_\tau^{(1)}(X, L)$ and $\Delta_\tau^{(2)}(X, L)$ rise, when, X is negative. These rises become obscure at $X=0$, and disappear when X becomes positive. The rise of $\Delta_\tau^{(2)}(X, L)$ is higher than that of $\Delta_\tau^{(1)}(X, L)$.

$\Delta_\tau^{(1)}(X, L)$ is the difference between $S_\tau(X, L)$ and the value $S_{\tau,c}(X, L)$ calculated with the $B_c(X, L)$ of (11) in which the kinetic energy is proportional to A as in the Fermi gas model. If we denote the kinetic energy part of $S_\tau(X, L)$ and $S_{\tau,c}(X, L)$ as $K_\tau(X, L)$ and $K_{\tau,c}(X, L)$,

$$K_\tau(X, L) = T(X, L) - T(X-1, L-1),$$

$$K_{\tau,c}(X, L) = T_c(X, L) - T_c(X-1, L-1) \doteq \text{constant}$$

where $T(X, L)$ are the kinetic energy of the nucleus of $A=2L-X$ and $|Z-N|=X$. In the single particle model, the energy level distance $D_\tau(X, L)$ between $(L-1)$ th and L th τ -nucleons in the nucleus (X, L) are large at

$$L=9, 21, \quad (A)$$

and small at

$$L=10, 11, 12, 13, 14; 16; 18, 19, 20, \quad (B)$$

when the spin-orbit coupling exists. Since in extreme $j-j$ coupling scheme, $L=8$ and 20

*) A. E. S. Green and D. F. Edwards⁹⁾ analysed in detail the pairing effects and the shell effects of nuclei referring to their mass surface and emphasized that $36/A^{3/4}$ or $140/A$ exaggerate the average magnitude of the pairing effect for light nuclides.

are magic numbers and three groups of nucleons, $L=9, 10, 11, 12, 13, 14$, $L=15, 16$ and $L=17, 18, 19, 20$ are in the same configuration $d_{5/2}$, $s_{1/2}$ and $p_{3/2}$ respectively.⁴⁾ Hence we can expect that $K_\tau(X, L) - K_{\tau,c}(X, L)$ are large at L in (A) and small at L in (B). Thus the falls at $L=9, 21$ and rises at $L=11, 13, (19)$ are considered to be the consequence of the variations of $D_\tau(X, L)$ such as (A) and (B). (shell effects) The above effect acts doubly to $\Delta_\tau^{(2)}(X, L)$.

The decrease of the falls of $\Delta_\tau^{(1)}$ and $\Delta_\tau^{(2)}$ at L in (A) with the decrease of X , and the decrease of the rises of $\Delta_\tau^{(1)}$ and $\Delta_\tau^{(2)}$ at $L=11, 13$ with the increase of X would be the consequence of that the $D_\tau(X, L)$ is the increasing function* of X . The shell effect which is caused by the largeness of $D_\tau(X, L)$ at L in (A) decreases with the decrease of X . The shell effect which is caused by the smallness of $D_\tau(X, L)$ at L in (B) increases with the decrease of X .

In Fig. 6, the connected lines of $\Delta_\tau^{(1)}(X, L)$ and $\Delta_\tau^{(2)}(X, L)$ coincide well on the abscissa except for negative X , and this fact seems to show that the even-odd variations of the binding energies are well explained by the term JL_1/A in (10) with the value (13). The large deviation of $\Delta_\tau^{(2)}(X, L)$ for negative X at $L=11, 13$ (and $19, 23$) are interpreted above by the kinetic energy part of the separation energy.

Green and Edwards⁹⁾ emphasized from the analysis of the discontinuities in nuclear mass surface that the expressions for the pairing effect involving shell quantum numbers which were derived by Mayer using the shell model are not in consistent with the available data. At $L=11, 13, 21$, and 23 , the $\Delta_\tau^{(2)}(X, L)$ is certainly larger than $\Delta_\tau^{(1)}(X, L)$ when X is negative. It may be possible that the large difference between $J_\tau^{(2)}(X, L)$ and $J_\tau^{(1)}(X, L)$ comes from the pairing effect involving j of the last two nucleons, because at $L=11, 13, 21, 23$, the last two nucleons, for which $J_\tau^{(2)}(X, L)$ are considered, have high j values in comparison with others. They are $j=5/2, j=5/2, j=7/2$ and $j=7/2$ respectively. But then the fact that these large differences ($J_\tau^{(2)} - J_\tau^{(1)}$) disappear when X is positive is not well understood. In view of the importance of the pairing effect in the shell model theory, the even-odd variations of separation energies must be studied in more detail and in more wide mass number region.

§ 4. Some theoretical considerations

The potential energy V_s given by (6) with the value of L_1 given by (12) has given the excellent explanation about the symmetry effects on the separation energies. If the total potential energy is given by V_s , the kinetic energy per nucleon should have been $\langle \sum T_i \rangle_{0AZ} = (14.03 - 3/8 L_1) A + \dots = 2.5A + \dots$ (Mev), and is too small.⁵⁾ There is another reason which will be mentioned soon later, that the potential energy of the above mentioned two-body interaction is smaller (about a half) than the total potential energy of the nucleus. This fact suggests the existence of other kinds of forces which is insensitive to the symmetry effects. We assume in this paper the existence of the potential energy which has the above

* The change of $D_\tau(X, L)$ with X seems to be larger than what is expected from the fact that the $D_\tau(X, L)$ is a decreasing function of A in the case of square well potential.

nature and we denote it as V_A .

The main parts of the fluctuations of J_z in Fig. 6 are explained by the variations of the kinetic energies of nuclei which are expected from the I. P. M. We now consider the separation energy by the I. P. M. which assumes that the wave function Φ_{0AZ} in (4) has the form

$$\Phi_{0AZ} = \mathcal{O} \phi_1(x_1) \phi_2(x_2) \cdots \phi_A(x_A), \quad (15)$$

where \mathcal{O} represents the antisymmetrizing operator. The fruitful success of the nuclear shell model has shown that this approximation is better than it has been expected. (In the shell model theory, Φ_{0AZ} in (15) is the zeroth order eigenfunction and the wave function is an appropriate linear sum of (15), which is determined by the interactions between outer equivalent particles). The I. P. M. is not inconsistent with the consideration by the uniform model in the previous section, when the I. P. M. wave function (15) gives the larger probability of finding two nucleons in the nuclear force range for the symmetric pairs than for the antisymmetric pairs. We write the wave function (15) as

$$\Phi_{0AZ} = \frac{1}{\sqrt{A}} \left(1 - \sum_{i=1}^{A-1} P_i \right) \phi_{0AZ}^n(x_A) \Psi_{0AZ}^n(x_1 \cdots x_{A-1}), \quad (16)$$

where ϕ_{0AZ}^n is the wave function of the last nucleon which is assumed here to be a neutron and Ψ_{0AZ}^n is the wave function of the residual nucleus in the nucleus (A, Z) , and P_i represents the exchange between x_A and x_i .

The Hamiltonian $H(A, Z)$ of the nucleus of mass number A can be written as

$$H(A, Z) = H(A-1, Z) + T_n + M_n + V_n(A, Z), \quad (17)$$

where T_n and M_n are the kinetic energy and mass of the last neutron and $V_n(A, Z)$ is the interaction between the last neutron and the residual nucleus $(A-1, Z)$. The variation principle gives the following equations for Φ_{0AZ} of the form (16) :

$$\begin{aligned} & [T_n + \{ d\tau_{A-1} \Psi_{0AZ}^{n*} V_n(A, Z) (1 - \sum P_i) \Psi_{0AZ}^n \\ & + \{ \Psi_{0AZ}^{n*} H(A-1, Z) \Psi_{0AZ}^n d\tau_{A-1} + M_n - M(A, Z) \} \phi_{0AZ}^n = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} & [H(A-1, Z) + \{ dx_A \phi_{0AZ}^{n*} V_n(A, Z) (1 - \sum P_i) \phi_{0AZ}^n \\ & + \{ \phi_{0AZ}^{n*} T_n \phi_{0AZ}^n dx_A + M_n - M(A, Z) \} \Psi_{0AZ}^n = 0. \end{aligned} \quad (19)$$

These equations express the interaction between the last nucleon and the residual nucleus in the Hartree approximation. The difference between Φ_{0AZ} and $\Psi_{0(A-1)Z}$ caused by the removal of the last nucleon is considered here as caused by the difference of radii of nuclei (A, Z) and $(A-1, Z)$, since the effect by the change of radius is very large and the constancy of the nuclear densities is known from the analysis of the nuclear coulomb energies.⁵⁾¹⁰⁾ Then the difference

$$\{ \Psi_{0AZ}^{n*} H(A-1, Z) \Psi_{0AZ}^n d\tau_{A-1} - \{ \Phi_{0(A-1)Z}^* H(A-1, Z) \Phi_{0(A-1)Z} d\tau_{A-1} \equiv R_{(AZ)}^n \quad (20)$$

is estimated as follows. We consider the kinetic part $T(A-1, Z)$ and potential part $V(A-1, Z)$ separately as

$$H(A-1, Z) = T(A-1, Z) + V(A-1, Z) + ZM_p + (A-1-Z)M_n. \quad (21)$$

We assume that the kinetic energy of i -th nucleon in the nucleus A of radius R_A is given by $a_i/(R_A)^{3\eta}$, where a_i and η are constants.

$$\begin{aligned} & \int \Psi_{0AZ}^{n*} T(A-1, Z) \Psi_{0AZ}^n d\tau_{A-1} - \int \Phi_{v(A-1)Z}^{n*} T(A-1, Z) \Phi_{v(A-1)Z}^n d\tau_{A-1} \\ &= \sum_{i=1}^{A-1} (a_i/R_A^{3\eta} - a_i/R_{A-1}^{3\eta}) = \sum_{i=1}^{A-1} a_i' (A^{-\eta} - (A-1)^{-\eta}) \\ &= -\eta \sum_{i=1}^{A-1} a_i' (A-1)^{-\eta-1} = -\eta \frac{1}{A-1} \sum_{i=1}^{A-1} a_i/R_A^{3\eta} = -\eta \bar{T}. \end{aligned} \quad (22)^*$$

\bar{T} is the mean kinetic energy per nucleon. Next, we calculate the potential part of (20) assuming that there is only V_s energy in (6) which is inversely proportional to the volume of the nucleus v_A .⁵⁾

$$\begin{aligned} & \int \Psi_{0AZ}^{n*} V(A-1, Z) \Psi_{0AZ}^n d\tau_{A-1} - \int \Phi_{v(A-1)Z}^{n*} V(A-1, Z) \Phi_{v(A-1)Z}^n d\tau_{A-1} \\ &= -2L_1 \frac{n_+(A-1, Z)}{v_A} + 2L_1 \frac{n_+(A-1, Z)}{v_{A-1}} = -2L_1 n_+(A-1, Z) \{A^{-1} - (A-1)^{-1}\} \\ &= \frac{1}{(A-1)} \frac{2n_+(A-1, Z)}{A-1} = -\bar{V}_s. \end{aligned} \quad (23)$$

\bar{V}_s is the mean V_s potential energy per nucleon. Then we have

$$R_{AZ}^n = -\eta \bar{T} - \bar{V}_s. \quad (24)$$

The Schrödinger equation for the last neutron (18) can be written as

$$[T_n + V_{n,AZ}(x_A) + S_n(A, Z) + R_{AZ}^n] \phi_{0AZ}^n = 0, \quad (25)$$

where $V_{n,AZ}(x_A)$ means the average potential energy for last neutron x_A , averaged over all co-ordinates of other nucleons, and when the exchanges P_i are neglected it is:

$$V_{n,AZ}(x_A) = \int \Psi_{0AZ}^{n*} V_n(A, Z) \Psi_{0AZ}^n d\tau_{A-1}. \quad (26)$$

The equation (25) means that the last neutron moves in the average potential $V_{n,AZ}$ with the energy

$$E_{n,0AZ} = -S_n(A, Z) - R_{AZ}^n. \quad (27)$$

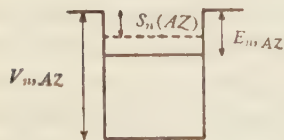


Fig. 7.

Thus, as is shown in Fig. 7,** the level of the last nucleon in I. P. M. is somewhat lower than $-S_n(A, Z)$ by the amount R_{AZ}^n . $R_{AZ}^n = -(\bar{V}_s + \eta \bar{T})$ has nearly constant value for all nuclei.

R_{AZ}^n is the energy required when the nucleus expands

*) K. Woeste¹¹⁾ noticed that these kinetic energy changes are appreciably large.

**) Fig. 1 in the Woeste's paper¹¹⁾ and Fig. 7 have the different meanings. In Fig. 7, V means the potential which determines the wave function of the last nucleon and is not equal to the mean potential energy per nucleon.

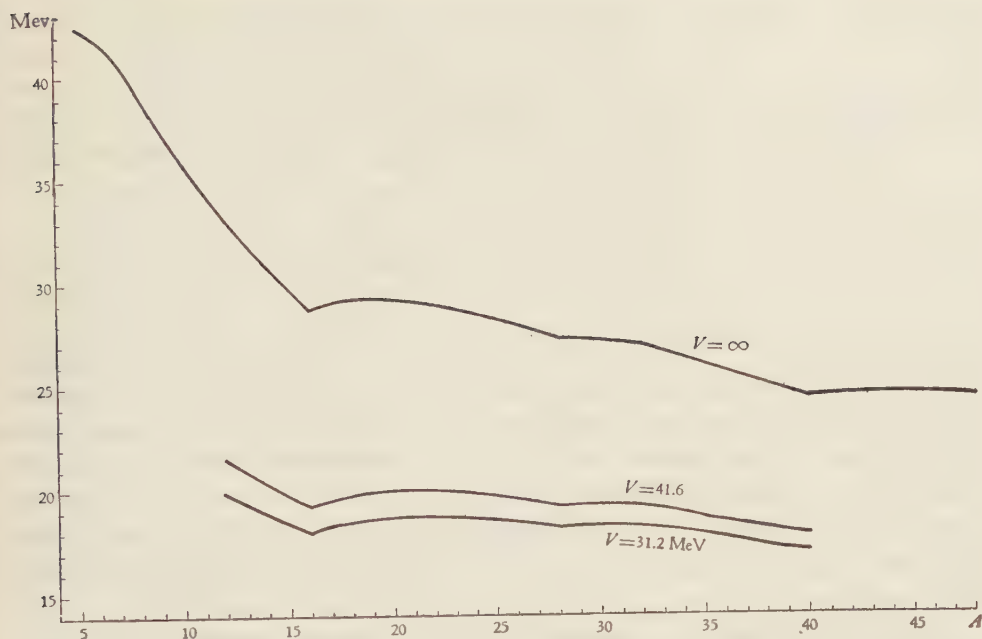


Fig. 8. Mean energy per nucleon (measured from the bottom of the potential) of the nucleus which consists of A nucleons moving independently in a square well potential of the depth V and of the radius $1.45 \times 10^{-13} A^{1/3}$ cm. Here the effect on the wave function by the spin-orbit coupling is neglected. We assumed that after $1s$ and $1p$ levels are filled, 12 nucleons fill $(1d)$, and 4 nucleons fill $(2s)$ and then 4 nucleons fill $(1d)$ successively, where $(1d)$, $(2s)$ are levels without the spin-orbit coupling.

from the equilibrium volume to the volume which the nucleus with one more nucleon has in the equilibrium. Since the nucleus has minimum energy at the equilibrium, R_{AZ}^n is positive and we call in this paper $\epsilon(A, Z)$ given by (28) as the coefficient of "expansibility" at the equilibrium.

$$\epsilon(A, Z) = R_{AZ}^n / (4/3) \pi r_0^3 > 0. \quad (28)$$

\bar{V}_s is estimated from (10) and (12) as

$$\bar{V}_s \sim -3/8 L_1 \sim -16.5 \text{ Mev}. \quad (29)$$

From (24), (28)

$$\bar{T} < 24.7 \text{ Mev}. \quad (30)$$

In (29), $\eta = 2/3$ was assumed. The energy shift R_{AZ}^n , of the last nucleon from S_n seems to be not large but rather small, then the mean kinetic energy is nearly equal in absolute magnitude to the mean V_s potential energy.

$$\bar{T} \lesssim -2/3 \bar{V}_s, \quad \bar{T} \sim 24 \text{ Mev}. \quad (31)$$

\bar{T} in (31) is considerably larger than the mean kinetic energy in the Fermi gas model*

*) The kinetic energy increases if the position correlations are taken into account.⁵⁾

(13~14 Mev). But the mean kinetic energy in the single particle model with infinite square well potential is 24~29 Mev in the mass number region $A=20\sim40$. For comparisons, the mean energies per nucleon $E_V(A)$ in the single particle model with square well potential of depth $V=\infty$, 30 and 40 Mev were calculated graphically and $E_V(A) - V$ are plotted in Fig. 8. Since the total binding energy is $14.03A + \dots$, the total potential energy per nucleon \bar{V} is estimated from \bar{T} in (31).

$$\bar{V} - \bar{T} \sim -14 \text{ Mev}, \quad \bar{V} \sim -38 \text{ Mev},$$

Then we have

$$\bar{V} - \bar{V}_s \sim -21 \text{ Mev}. \quad (32)$$

Here again we have the fact that the potential energy V_s given by (6) in the nucleus is only about a half of the total potential energy of the nucleus. We presumed besides V_s , the existence of the potential energy V_A which is insensitive to the symmetry effect of the nucleus. Here, we presume further that V_A is also insensitive to the "expansibility" of the nucleus. We want to discuss the possible interactions of such a kind in a separate paper. Of course, the estimation of $(V - V_s)$ is very crude and the surface energy was neglected. And we assumed for the wave function the form (16).

Next, we will estimate the magnitude of $V_{n,AZ}(x_A)$ in (25) assuming the square well shape.

The contribution to $V_{n,AZ}$ from V_s is denoted as $V_{n,AZ}^{(s)}$ and is from (6), (7) and (9)

$$\begin{aligned} V_{n,AZ}^{(s)} &= (\int \Psi_{0AZ}^* V(A-1, Z) \Psi_{0AZ} d\tau_{A-1})_s \\ &= -2(L_1/A) (n_+(A, Z) - n_+(A-1, Z)) \\ &= -2(L_1/A) ((3/8)A + (3/4) - (1/2)\xi(A, Z) + (1/2)\xi(A-1, Z)) \\ &= -L_1((3/4) + (3/2)A^{-1} - 1.75(X-1)A^{-1} \pm (JA^{-1}) \pm (J_0 A^{-1}, 0)), \end{aligned} \quad (33)^*$$

where $X=N-Z$. Since $\bar{V}_s = -2L_1 n_+(A, Z)/A^2 \sim -3/8 L_1$, (33) is nearly equal to $2\bar{V}_s$. The V_s potential energy difference between two nuclei (A, Z) and $(A-1, Z)$ is $\sim V_s$. When a neutron is removed from the nucleus, about a half of the potential energy $V_{n,AZ}^{(s)}$ is consumed by the decrease of the potential energy of the residual nucleus according to (23). Using the value (12)

$$V_{n,AZ}^{(s)} = -33 + 66A^{-1} - 77(X-1)A^{-1} \mp L_1 JA^{-1} \mp (L_1 J_0 A^{-1}, 0). \quad (34)$$

(in Mev)

The contributions from V_A to $V_{n,AZ}$ is denoted by $V_{n,AZ}^{(A)}$. \bar{V}_A is estimated from (32)

$$\bar{V}_A \sim -21 \text{ Mev}, \quad (35)$$

Since V_A was assumed insensitive to the change of the volume, $V_{n,AZ}^{(A)}$ is estimated to be

*) The signs and values of the last two terms in (33) and (36) must be chosen properly referring to the values in (9).

equal to the mean potential energy of V_A . Then estimated value for the depth of the square well potential $V_{n,AZ}$ is

$$V_{n,AZ} \sim -54 + 143A^{-1} - 77XA^{-1} \mp L_1 A^{-1} \mp (L_1 A_0 A^{-1}, 0). \text{ (in Mev)} \quad (36)^*$$

$V_{p,AZ}$ is the same with (34) except for the coulomb energy.

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*) Feshbach, Porter and Weisskopf's analysis¹²⁾ of the data of the low energy neutron scattering from nuclei by their model with the potential $V_0 + iW_0$ shows that $V_0 = 42$ Mev gives better fits to the experiment than $V_0 = 19$ Mev. $V_{n,AZ}$ in (36) is not inconsistent with their V_0 .

Appendix

The most of the separation energies are based on the Q_m values in the Ajzenberg and Lauritsen's paper¹⁵⁾ and the mass defects in the Endt and Kluyver's paper.¹⁶⁾ Ajzenberg and Lauritsen adopted the values of mass defects in the papers of Li, Whaling, Fowler, Lauritsen¹³⁾ and Li,¹¹⁾ which were derived from the reaction data. Endt and Kluyver adopted the values in Li's table¹¹⁾ and the weighted means of mass spectrographic data by Collins, Nier and Johnson¹⁷⁾ and several other authors. In the regions which are not included in the papers of Ajzenberg and Lauritsen and Endt and Kluyver, the values of mass defects in the tables of Li¹¹⁾ and Collins, Nier and Johnson⁶⁾ are adopted.* Dr. M. Sato kindly informed me about the excellent summary of Feather¹⁶⁾ about the separation energies, and from Feather's table 9 Sp 's and 16 Sn 's are added to Tables I and II. Other Sp 's and Sn 's in Tables I, II were compared with Feather's values, and most of them coincide within the limit of errors. 16 Sp 's and 21 Sn 's in Tables I and II are not found in Feather's Table. For the discussions in this paper, the general distribution of separation energies is important rather than the very precise value of each separation energy, so we use the values above mentioned without further scrutiny.

*) Q_m values of (p, γ) , (n, γ) etc., in the Ajzenberg and Lauritsen's paper coincide with Sp 's and Sn 's calculated from the mass defects of the Li's table except for six values. Exceptions arise from the different values of the mass defects of O¹⁸⁾ and F¹⁸⁾ in two references. (~ 0.2 Mev) We adopted Ajzenberg and Lauritsen's values. The mass defects derived from reaction data in the Endt and Kluyver's paper are the same with the mass defects in Li's table except M_n ²⁷⁾ for which we adopted the Endt and Kluyver's value. Some of the mass defects in Collins, Nier and Johnson differ more than 0.1 Mev from the values in Endt and Kluyver's paper which are weighted means of several mass spectrographic data. We adopted the latter values, but the mass defects of Collins, Nier and Johnson are adopted without corrections which are not found in Endt and Kluyver.

Table I

A	Z	S_p	S_p from the mass formula (1)						dev.	ref.
			vol.	surf.	sym.	coul.	δ	total		
10	5	6.6	14.0	-4.1	2.1	-2.3	-6.0	3.9	2.7	(15)
10	6	3.8	"	-4.1	-5.8	-2.7	6.0	7.4	-3.6	(19)
11	5	11.2	"	-4.0	6.0	-2.2	-6.0	7.8	3.4	(15)
11	6	8.7	"	-4.0	-1.8	-2.7	6.0	11.6	-2.9	(15)
12	6	15.9	"	-3.8	1.8	-2.6	5.2	14.5	1.4	(15)
12	7	0.3	"	-3.8	-4.7	-3.0	-5.2	-2.7	3.0	(19)
13	6	17.5	"	-3.7	5.0	-2.6	5.2	17.9	-0.4	(15)
13	7	1.9	"	-3.7	-1.5	-3.0	-5.2	0.6	1.3	(15)
14	7	7.5	"	-3.7	1.5	-2.9	-4.7	4.3	3.2	(15)
14	8	4.7	"	-3.7	-4.0	-3.3	4.7	7.7	-3.0	(19)
15	7	10.2	"	-3.6	4.2	-2.9	-4.7	7.2	3.0	(15)
15	8	7.3	"	-3.6	-1.2	-3.3	4.7	10.6	-3.3	(15)
16	7	10.6	"	-3.5	6.8	-2.8	-4.2	10.3	0.3	(15)
16	8	12.1	"	-3.5	1.2	-3.2	4.2	12.7	-0.6	(15)
17	8	13.7	"	-3.4	3.7	-3.2	4.2	15.3	-1.6	(15)
17	9	0.6	"	-3.4	-1.1	-3.6	-4.2	1.7	-1.1	(15)
18	8	16.1	"	-3.4	5.9	-3.1	3.8	17.3	-1.2	(15)
18	9	5.6	"	-3.4	1.1	-3.5	-3.8	4.5	1.1	(15)

19	9	8.0	14.0	-3.2	3.3	-3.5	-3.8	6.8	1.2	(15)
19	10	6.4	"	-3.2	-1.0	-3.8	3.8	9.8	-3.4	(15)
20	9	10.3	"	-3.2	5.3	-3.4	-3.5	9.1	1.2	(15)
20	10	12.9	"	-3.2	1.0	-3.8	3.5	11.6	1.3	(15)
20	11	0.8	"	-3.2	-2.9	-4.1	-3.5	0.3	0.5	(14) (16)
21	10	13.0	"	-3.2	2.9	-3.7	3.5	13.6	-0.6	(15)
21	11	2.5	"	-3.2	-0.9	-4.1	-3.5	2.3	0.2	(16)
22	11	6.7	"	-3.1	0.9	-4.1	-3.3	4.5	2.2	(16)
23	11	8.8	"	-3.1	2.7	-4.0	-3.3	6.3	2.5	(16)
23	12	7.5	"	-3.1	-0.8	-4.3	3.3	9.1	-1.6	(16)
24	11	10.6	"	-3.0	4.4	-4.0	-3.1	8.3	2.3	(14)
24	12	11.7	"	-3.0	0.8	-4.3	3.1	10.7	1.0	(16)
24	13	1.8	"	-3.0	-2.4	-4.6	-3.1	1.0	0.8	(16)
25	12	12.1	"	-3.0	2.4	-4.2	3.1	12.3	-0.2	(16)
25	13	2.3	"	-3.0	-0.8	-4.6	-3.1	2.5	-0.2	(16)
26	12	14.0	"	-2.9	4.0	-4.2	2.9	13.8	0.2	(16)
26	13	6.6	"	-2.9	0.8	-4.6	-2.9	4.4	2.2	(16)
27	13	8.2	"	-2.9	2.3	-4.5	-2.9	5.9	2.4	(16)
27	14	7.2	"	-2.9	-0.7	-4.8	2.9	8.4	-1.2	(16)
28	13	9.5	"	-2.9	3.7	-4.5	-2.8	7.6	1.9	(16)
28	14	11.6	"	-2.9	0.7	-4.8	2.8	9.8	1.8	(16)
28	15	2.2	"	-2.9	-2.0	-5.1	-2.8	1.2	1.0	(16)
29	13	10.4	"	-2.8	5.1	-4.4	-2.8	9.1	1.3	(16)
29	14	12.3	"	-2.8	2.1	-4.7	2.8	11.3	1.0	(16)
29	15	2.7	"	-2.8	-0.7	-5.1	-2.8	2.7	0.0	(16)
30	14	13.6	"	-2.8	3.4	-4.7	2.6	12.7	0.9	(16)
30	15	5.5	"	-2.8	0.7	-5.0	-2.6	4.3	1.2	(16)
31	15	7.3	"	-2.8	2.0	-5.0	-2.6	5.6	1.7	(16)
31	16	6.5	"	-2.8	-0.6	-5.4	2.6	7.9	-1.4	(16)
32	15	8.6	"	-2.8	3.2	-4.9	-2.5	7.0	1.6	(16)
32	16	8.9	"	-2.8	0.6	-5.3	2.5	9.0	-0.1	(16)
32	17	0.8	"	-2.8	-1.3	-5.6	-2.5	1.4	-0.6	(16)
33	15	9.4	"	-2.7	4.4	-4.8	-2.5	8.3	1.1	(16)
33	16	9.6	"	-2.7	1.8	-5.2	2.5	10.4	-0.8	(16)
33	17	2.5	"	-2.7	-0.6	-5.5	-2.5	2.8	-0.3	(16)
34	16	10.9	"	-2.7	3.0	-5.2	2.4	11.5	-0.6	(16)
34	17	5.0	"	-2.7	0.6	-5.5	-2.4	4.1	0.9	(16)
35	16	11.3	"	-2.6	4.1	-5.1	2.4	12.8	-1.5	(16)
35	17	6.4	"	-2.6	1.7	-5.4	-2.4	5.3	1.1	(16)
35	18	6.6	"	-2.6	-0.6	-5.8	2.4	7.4	-0.8	(16)
36	17	8.0	"	-2.6	2.8	-5.4	-2.3	6.5	1.5	(16)
36	18	8.5	"	-2.6	0.6	-5.7	2.3	8.5	0.0	(16)
37	17	9.2	"	-2.6	3.9	-5.4	-2.3	7.7	1.5	(16)
37	18	8.8	"	-2.6	1.6	-5.6	2.3	9.7	-0.9	(16)
37	19	2.5	"	-2.6	-0.5	-6.0	-2.3	2.7	-0.2	(16)
38	17	9.6	"	-2.6	5.0	-5.3	-2.2	8.9	0.7	(16)
38	18	10.2	"	-2.6	2.7	-5.7	2.2	10.6	-0.4	(16)
38	19	5.1	"	-2.6	0.5	-5.9	-2.2	3.9	1.2	(16)
39	18	10.8	"	-2.5	3.7	-5.6	2.2	11.7	-0.9	(16)
39	19	6.5	"	-2.5	1.5	-5.9	-2.2	5.0	1.5	(16)
39	20	5.7	"	-2.5	-0.5	-6.2	2.2	7.0	-1.3	(16)
40	18	12.1	"	-2.5	4.7	-5.5	2.1	12.8	-0.7	(16)
40	19	7.7	"	-2.5	2.5	-5.8	-2.1	6.1	1.6	(16)
40	20	8.4	"	-2.5	0.5	-6.1	2.1	8.0	0.4	(16)
41	19	7.8	"	-2.5	3.5	-5.8	-2.1	7.1	0.7	(16)
41	20	8.8	"	-2.5	1.5	-6.1	2.1	9.1	-0.3	(16)
41	21	1.7	"	-2.5	-0.5	-6.3	-2.1	2.6	-0.9	(17)
42	19	9.1	"	-2.5	4.4	-5.8	-2.0	8.2	0.9	(16)
42	20	10.2	"	-2.5	2.4	-6.1	2.0	9.9	0.3	(16)
43	20	10.8	"	-2.5	3.3	-6.0	2.0	10.9	-0.1	(16)
44	20	11.4	"	-2.5	4.2	-6.0	2.0	11.7	-0.3	(16)
45	21	6.8	"	-2.5	3.2	-6.2	-2.0	6.5	0.3	(17)

46	21	8.4	14.0	-2.4	4.0	-6.1	-1.9	7.6	0.8	(19)
46	22	10.5	"	-2.4	2.2	-6.5	1.9	9.2	1.3	(17)
47	22	10.3	"	-2.4	3.0	-6.5	1.9	10.0	0.3	(19)
48	21	10.2	"	-2.4	5.7	-6.1	-1.8	9.4	0.8	(16) (17)
48	22	10.9	"	-2.4	3.8	-6.4	1.8	11.0	-0.1	(19)
48	23	6.8	"	-2.4	2.1	-6.8	-1.8	5.2	1.6	(17)
49	22	10.1	"	-2.4	4.6	-6.4	1.8	11.8	-1.7	(17)
49	23	7.4	"	-2.4	2.9	-6.7	-1.8	6.0	1.4	(17)
49	24	7.6	"	-2.4	1.2	-6.8	1.8	7.9	-0.3	(17)
50	22	11.6	"	-2.4	5.4	-6.3	1.8	12.6	-1.0	(19)
50	23	7.7	"	-2.4	3.7	-6.6	-1.8	6.9	0.8	(17)
50	24	8.4	"	-2.4	2.0	-6.8	1.8	8.6	-0.2	(19)
51	23	7.8	"	-2.3	4.4	-6.6	-1.8	7.8	0.0	(17)
51	24	9.6	"	-2.3	2.8	-6.8	1.8	9.5	0.1	(17)

Table II

A	N	S_n	S_n from the mass formula (1)						dev.	ref.
			vol.	surf.	sym.	coul.	δ	total		
10	5	8.4	14.0	-4.1	2.0	0.3	-6.0	6.2	2.2	(15)
10	6	6.8	"	-4.1	-5.6	0.2	6.0	10.5	-3.7	(15)
11	5	13.4	"	-4.0	6.0	0.3	-6.0	10.4	3.0	(19)
11	6	11.5	"	-4.0	-1.8	0.2	6.0	14.5	-3.0	(15)
12	6	18.7	"	-3.8	1.8	0.3	5.2	17.4	1.3	(15)
12	7	3.4	"	-3.8	-4.7	0.2	-5.2	0.5	2.9	(15)
13	6	20.5	"	-3.7	5.0	0.3	5.2	20.8	-0.3	(19)
13	7	5.0	"	-3.7	-1.5	0.2	-5.2	3.9	1.1	(15)
14	7	10.6	"	-3.7	1.5	0.3	-4.7	7.5	3.1	(15)
14	8	8.2	"	-3.7	-4.0	0.2	4.7	11.2	-0.3	(15)
15	7	13.3	"	-3.6	4.2	0.4	-4.7	10.4	2.9	(19)
15	8	10.8	"	-3.6	-1.3	0.3	4.7	14.1	-3.3	(15)
15	9	2.2	"	-3.6	-6.1	0.2	-4.7	0.1	2.1	(15)
16	8	15.6	"	-3.5	1.3	0.3	4.2	16.3	-0.7	(15)
16	9	2.6	"	-3.5	-3.5	0.3	-4.2	3.1	-0.5	(15)
17	9	4.1	"	-3.4	-1.1	0.3	-4.2	5.6	-1.5	(15)
17	10	5.7	"	-3.4	-5.4	0.2	4.2	9.7	-4.0	(19)
18	9	9.2	"	-3.4	1.1	0.4	-3.8	8.3	0.9	(15)
18	10	8.1	"	-3.4	-3.2	0.3	3.8	11.6	-3.5	(15)
19	10	10.4	"	-3.2	-1.0	0.3	3.8	13.9	-3.5	(15)
19	11	4.2	"	-3.2	-4.9	0.3	-3.8	2.4	1.8	(15)
20	10	16.9	"	-3.2	1.0	0.4	3.5	15.7	1.2	(15)
20	11	6.6	"	-3.2	-2.8	0.3	-3.5	4.7	1.9	(15)
21	10	18.6	"	-3.2	2.9	0.4	3.5	17.7	0.9	(16)
21	11	6.8	"	-3.2	-0.9	0.3	-3.5	6.7	0.1	(15)
22	11	11.0	"	-3.1	0.9	0.4	-3.3	8.9	2.1	(16)
22	12	10.4	"	-3.1	-2.6	0.3	3.3	11.9	-1.5	(16)
23	12	12.4	"	-3.1	-0.8	0.4	3.3	13.8	-1.4	(16)
23	13	5.2	"	-3.1	-4.0	0.3	-3.3	3.9	1.3	(16)
24	12	16.6	"	-3.0	0.8	0.4	3.1	15.4	1.2	(16)
24	13	7.0	"	-3.0	-2.4	0.4	-3.1	5.9	1.1	(16)
25	12	17.0	"	-3.0	2.4	0.5	3.1	17.0	0.0	(16)
25	13	7.3	"	-3.0	-0.8	0.4	-3.1	7.6	-0.3	(16)
25	14	9.1	"	-3.0	-3.7	0.4	3.1	10.7	-1.6	(16)
26	13	11.6	"	-2.9	0.8	0.4	-2.9	9.4	2.2	(16)
26	14	11.1	"	-2.9	-2.2	0.4	2.9	12.2	-1.1	(16)
27	14	12.8	"	-2.9	-0.7	0.4	2.9	13.7	-0.9	(16)
27	15	6.4	"	-2.9	-3.5	0.3	-2.9	5.0	1.4	(16)
28	14	17.2	"	-2.9	0.7	0.5	2.8	15.1	2.1	(16)

28	15	7.7	14.0	-2.9	-2.0	0.4	-2.8	6.7	1.0	(16)
28	16	8.6	"	-2.9	-4.6	0.3	2.8	9.6	-1.0	(16)
29	14	17.7	"	-2.8	2.1	0.5	2.8	16.6	1.1	(16)
29	15	8.5	"	-2.8	-0.7	0.4	-2.8	8.2	0.3	(16)
29	16	9.4	"	-2.8	-3.2	0.4	2.8	11.1	-1.7	(16)
30	15	11.3	"	-2.8	0.7	0.5	-2.6	9.8	1.5	(16)
30	16	10.6	"	-2.8	-1.9	0.4	2.6	12.4	-1.8	(16)
31	16	12.4	"	-2.8	-0.6	0.5	2.6	13.7	-1.3	(16)
31	17	6.6	"	-2.8	-3.0	0.4	-2.6	6.0	0.6	(16)
32	16	14.8	"	-2.8	0.6	0.5	2.5	14.9	-0.1	(16)
32	17	7.9	"	-2.8	-1.8	0.4	-2.5	7.4	0.5	(16)
32	18	9.3	"	-2.8	-4.0	0.4	2.5	10.1	-0.8	(16)
33	16	16.5	"	-2.7	1.8	0.6	2.5	16.2	0.3	(16)
33	17	8.6	"	-2.7	-0.6	0.5	-2.5	8.8	-0.2	(16)
33	18	10.1	"	-2.7	-2.9	0.4	2.5	11.4	-1.3	(16)
34	17	11.2	"	-2.7	0.6	0.5	-2.4	10.1	1.1	(16)
34	18	11.4	"	-2.7	-1.7	0.4	2.4	12.5	-1.1	(16)
34	19	6.6	"	-2.7	-3.8	0.4	-2.4	5.5	1.1	(16)
35	18	12.8	"	-2.6	-0.6	0.5	2.4	13.7	-0.9	(16)
35	19	7.0	"	-2.6	-2.7	0.4	-2.4	6.8	0.2	(16)
36	18	14.7	"	-2.6	0.6	0.6	2.3	14.7	0.0	(16)
36	19	8.6	"	-2.6	-1.6	0.4	-2.3	7.9	0.7	(16)
36	20	9.2	"	-2.6	-3.6	0.5	2.3	10.5	-1.3	(16)
37	19	8.9	"	-2.6	-0.5	0.5	-2.3	9.1	-0.2	(16)
37	20	10.4	"	-2.6	-2.6	0.5	2.3	11.7	-1.3	(16)
37	21	5.7	"	-2.6	-4.5	0.4	-2.3	5.0	0.7	(16)
38	19	11.5	"	-2.6	0.5	0.6	-2.2	10.4	1.1	(16)
38	20	11.8	"	-2.6	-1.5	0.5	2.2	12.6	-0.8	(16)
38	21	6.1	"	-2.6	-3.4	0.5	-2.2	6.2	-0.1	(16)
39	20	13.2	"	-2.5	-0.5	0.5	2.2	13.7	-0.5	(16)
39	21	6.7	"	-2.5	-2.4	0.5	-2.2	7.3	-0.6	(16)
39	22	8.6	"	-2.5	-4.3	0.4	2.2	9.9	-1.3	(16)
40	20	15.9	"	-2.5	0.5	0.5	2.1	14.6	1.3	(16)
40	21	7.9	"	-2.5	-1.4	0.5	-2.1	8.5	-0.6	(16)
40	22	9.9	"	-2.5	-3.3	0.5	2.1	10.9	-1.0	(16)
41	21	8.3	"	-2.5	-0.5	0.6	-2.1	9.6	-1.3	(16)
41	22	10.0	"	-2.5	-2.3	0.5	2.1	11.8	-1.8	(16)
41	23	6.1	"	-2.5	-4.1	0.4	-2.1	5.8	0.3	(16)
42	22	11.4	"	-2.5	-1.4	0.5	2.0	12.7	-1.3	(16)
42	23	7.4	"	-2.5	-3.1	0.5	-2.0	6.8	0.6	(16)
43	23	8.0	"	-2.5	-2.2	0.5	-2.0	7.9	0.1	(16)
43	24	10.8	"	-2.5	-3.9	0.5	2.0	10.3	0.5	(16)
44	24	11.4	"	-2.5	-3.0	0.5	2.0	11.0	0.4	(16)
45	25	7.4	"	-2.5	-3.7	0.5	-2.0	6.4	1.0	(16)
46	24	13.3	"	-2.4	-1.3	0.7	1.9	12.9	0.4	(19)
46	25	8.9	"	-2.4	-2.9	0.6	-1.9	7.4	1.5	(19)
47	25	8.6	"	-2.4	-2.0	0.5	-1.9	8.2	0.4	(17)
47	26	10.5	"	-2.4	-3.6	0.5	1.9	10.4	0.1	(19)
48	26	11.7	"	-2.4	-2.7	0.6	1.8	11.4	0.3	(17)
48	27	7.8	"	-2.4	-4.2	0.4	-1.8	6.0	1.8	(19)
48	28	9.8	"	-2.4	-5.6	0.5	1.8	8.3	1.5	(17)
49	26	12.2	"	-2.4	-1.9	0.7	1.8	12.3	-0.1	(19)
49	27	8.0	"	-2.4	-3.4	0.5	-1.8	6.8	1.2	(17)
49	28	10.1	"	-2.4	-4.8	0.5	1.8	9.1	1.0	(19)
49	29	5.1	"	-2.4	-6.2	0.4	-1.8	4.1	1.0	(17)
50	26	13.3	"	-2.4	-1.2	0.7	1.8	13.0	0.4	(17)
50	27	8.0	"	-2.4	-2.6	0.5	-1.8	7.8	0.2	(19)
50	28	11.0	"	-2.4	-4.1	0.5	1.8	9.9	1.1	(17)
51	27	9.1	"	-2.3	-1.9	0.6	-1.8	8.7	0.4	(17)
51	28	11.1	"	-2.3	-3.3	0.6	1.8	10.8	0.3	(17)

The $\text{Cu}(n; 2n)$ Excitation Function and the Knock-out Process in Nuclear Reactions

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The $(n; 2n)$ excitation cross section in copper is calculated on the basis of the compound nucleus formalism. If the probability for the formation of the compound nucleus by the incident neutron is assumed to be equal to unity, the calculated cross section becomes much larger than the experimental one. When the knock-out process is taken into account and a smaller probability is adopted, the agreement with the experiment can be made better to some extent. The estimate of the cross section of the knock-out process is obtained by the use of a very simplified model.

§ 1. Introduction

In the previous papers^{1,2)}, cited as I and II, we were able to show that the compound nucleus theory is of good validity in accounting for the overall behaviour of gamma ray yields and the shape of a gamma ray spectrum, induced by medium energy neutrons. Although these results may be considered to support the statistical description of nuclear reactions, it has been pointed out by several authors³⁻⁵⁾ that the compound nucleus theory might be insufficient to explain details of experimental results. Discrepancies between the theory and the experiments seem to be due to contributions from such reaction processes that are neglected in the theoretical calculation. Among various processes to be considered, the knock-out process may be of primary importance, which has first been recognized by Bernardini et al. through their Monte Carlo calculation.⁶⁾ The direct process occurring at the nuclear surface was investigated quantum-mechanically by Austern, Butler and Mc Manus.⁷⁾ The recent work⁸⁾ by Hayakawa, Kawai and Kikuchi, cited as III, has shown that the knock-out process can not be neglected in describing the nucleon spectrum which, if interpreted in terms of the simple evaporation theory, results in a queer tendency of the nuclear temperature, as pointed out by Cohen⁹⁾, and gives the main contribution in the case of inelastic scattering of high energy protons. Their calculation based on the knock-out process is in fair agreement with the experimental energy and angular dependences of protons induced by 31 Mev protons, while small difference in proton yields between the forward and the backward directions in the 18 Mev proton inelastic scattering¹¹⁾ indicates that the proton yield due to the evaporation is actually predominant over that due to the knock-out process in this energy range. In other words, the probability for the formation of a compound nucleus by an incident particle seems to become smaller than unity in the energy range above, say, 20 Mev. It may, therefore, be worth while to examine this point on a

little different example, namely on the analysis of the excitation cross section of $\text{Cu}(n; 2n)$ reaction¹²⁾ with incident energies ranging from the threshold ($=11.1$ Mev) to 30 Mev. Since the kinetic energy of a directly knocked out nucleon is usually much larger than that due to the evaporation, the residual nucleus left after the direct neutron emission can hardly evaporate the second neutron in the above energy region. Then if the knock-out process takes place, it diminishes the magnitude of the $(n; 2n)$ cross section in comparison with that expected from the evaporation theory.

In § 2 we shall proceed to calculate the cross section on the basis of compound nucleus formalism assuming the usual empirical formula for the nuclear level density. The result obtained is that the $\text{Cu}(n; 2n)$ cross section would become much larger than the experimental one at high incident energies, unless the probability for the formation of a compound nucleus is assumed to be smaller than unity. This smallness of the probability allows us to estimate the magnitude of the cross section for the knock-out process, since the inelastic total cross section is thought to be approximately equal to the geometrical cross section. In § 3, is given a crude estimate of the cross section for this process by the use of a very simple model. In treating the actual knock-out reaction, the following two effects must not be overlooked. One is the attenuation of both impinging and scattered particles in the nuclear matter, which can not be neglected for the reason that the mean free path is much smaller than the nuclear radius in the energy range concerned here. The other is the transmission and refraction of nucleon beams at the nuclear boundary, which is more important for scattered particles than for incident ones. In our semi-classical treatment these effects are taken into account based upon a very simplified model. The methods we used in § 2 and § 3 are essentially the same as those in II and III respectively.

§ 2. Outline of calculation by the evaporation theory

Our methods of calculations and notations are the same as those in the previous paper II, except that here we take the three steps of evaporations into account, since we should be concerned with all decay processes energetically possible. For this purpose we define the branching probability $P_{ijk}(E)$, like as P_{ij} in II, which expresses the probability of emitting particle k from the intermediate nucleus left after the successive emissions of particles i and then j , by

$$P_{ijk}(E) = F_k(E - S_{ijk}) / \sum_m F_m(E - S_{ijm}) \quad (1)$$

where E is the excitation energy of the intermediate nucleus concerned and S_{ijk} is the separation energy of particle k from it. The quantity F_i is proportional to the partial width for disintegration with emission of i and is defined by (I; 5, 6). If a particle i emerges from the compound nucleus with its kinetic energy ϵ_i , the probability of emission of j and then k from the residual nucleus is proportional to quantity $I_{jk}^{(i)}$, which is given, if j is any particle, by

$$I_{jk}^{(i)}(E_{ex} - S_i - S_{ij} - S_{ijk} - \epsilon_i)$$

$$= \frac{2M_j}{b^2} \int_0^{E_{ex} - S_i - S_{ij} - S_{ijk} - \epsilon_i} \epsilon_j \sigma_c^{(j)}(\epsilon_j) w_{ji}^{(i)}(E_{ex} - S_i - S_{ij} - \epsilon_i - \epsilon_j) P_{ijk}(E_{ex} - S_i - S_{ij} - \epsilon_i - \epsilon_j) d\epsilon_j, \quad (2)$$

and, if j is a gamma ray, by

$$I_{ijk}^{(i)}(E_{ex} - S_i - S_{ik} - \epsilon_i) \approx \frac{3}{4} \cdot \frac{e^2}{\hbar c} \cdot \left(\frac{R}{\hbar c}\right)^2 \cdot \frac{1}{D_0} \int_0^{E_{ex} - S_i - S_{ik} - \epsilon_i} \epsilon_j^3 w_{ji}^{(i)}(E_{ex} - S_i - \epsilon_i - \epsilon_j) P_{ijk}(E_{ex} - S_i - \epsilon_i - \epsilon_j) d\epsilon_j, \quad (3)$$

where E_{ex} is the excitation energy of the compound nucleus. In (3) we take only the electric dipole radiation into account, because of its major contribution. Other notations adopted here are the same as those in I and II and of usual meaning. Then the probability $p_{jk}^{(i)}$ for emission of j and k from the intermediate residual nucleus left after the emission of particle i is expressed, as a function of E_{ex} and ϵ_i , as

$$p_{jk}^{(i)}(E_{ex} - S_i - \epsilon_i) = \frac{\sum_l I_{ijk}^{(i)}(E_{ex} - S_i - S_{ij} - S_{ijk} - \epsilon_i)}{\sum_{l,m} I_{lm}^{(i)}(E_{ex} - S_i - S_{il} - S_{ilm} - \epsilon_i)}. \quad (4)$$

When the compound nucleus can evaporate three particles in succession, the probability for having emission i , j and k is

$$K_{ijk} / \sum_{l,m,n} K_{lmn} \quad (5)$$

where the summation is extended over all possible channels. The quantity K_{ijk} is written as

$$K_{ijk} = \frac{2M_i}{b^2} \int_0^{E_{ex} - S_i - S_{ij} - S_{ijk}} \epsilon_i \sigma_c^{(i)}(\epsilon_i) w_{il}^{(i)}(E_{ex} - S_i - \epsilon_i) p_{jk}^{(i)}(E_{ex} - S_i - \epsilon_i) d\epsilon_i. \quad (6)$$

Thus we can calculate the cross section $\sigma_c(l; i, j, k)$ of the $(l; i, j, k)$ reaction by the usual formula

$$\sigma_c(l; i, j, k) = \sigma_c^{(i)}(E_l) [K_{ijk} / \sum_{m,n,p} K_{mnp}]. \quad (7)$$

$\sigma_c^{(i)}(E_l)$ is the cross section for the formation of a compound nucleus due to the particle l with incident energy E_l . On account of $\sum_{j,k} p_{jk}^{(i)} = 1$ and the definition of F_i , we have

$$\sum_{n,p} K_{mnp} = F_m. \quad (8)$$

In order to carry out our calculation, we have to consider the following three quantities: the separation energies, the cross sections for the formation of compound nuclei and the level densities. Since our calculation refers to the case when incident neutrons bombard ^{63}Cu nuclei, the most abundant isotope of copper (71%), the compound nucleus is ^{64}Cu with its excitation energy $E_{in} + S_n$ Mev. The emission of α , d and t , etc. may be of negligible probability, so that we may take account only of that of n , p and γ .

As our knowledge about the separation energies is very scanty, we are obliged to employ those obtained by the Weizsäcker's mass formula. The values we used are shown in Table 1.

Table 1.	$S_n = 7.8$ Mev	$S_{nnn} = 8.9$	$S_{nnp} = 10.7$
	$S_p = 6.3$	$S_{nnp} = 4.9$	$S_{ppp} = 8.2$
	$S_{nn} = 11.3$	$S_{nnp} = 19.6$	
	$S_{np} = 5.6$	$S_{pnn} = 10.7$	
	$S_{pn} = 7.1$	$S_{npp} = 9.6$	
	$S_{pp} = 10.3$	$S_{ppn} = 6.5$	

For the level density we have employed the following semi-empirical formula based upon the Fermi gas model :

$$w(E) = c \exp (2\sqrt{aE}), \quad (9)$$

$$\text{with } c_{\text{even-even}} = 1/2 \quad c_{\text{even-odd}} = 1/4 \quad c_{\text{odd-odd}}$$

where a and c are two adjustable parameters. The cross sections for different emission processes are sensitively dependent upon the magnitude of a , so that we shall make the calculations for the two limiting cases of a values. Blatt and Weisskopf⁽¹³⁾ give a value of a for the nucleus concerned here

$$a = 2 \text{ Mev}^{-1}, \quad (i)$$

on the other hand Graves and Rosen⁽¹⁴⁾ obtained

$$a = 23.8 \text{ Mev}^{-1} \quad (ii)$$

by observing the neutron spectrum from copper excited by 14 Mev neutrons.

Cross sections for the formation of a compound nucleus appearing in integrals F , I and K are calculated by the same way as in II, i.e. those for neutrons are by the formula given by Feshbach and Weisskopf⁽¹⁵⁾ and for protons are interpolated from the table listed by Blatt and Weisskopf⁽¹³⁾. These values of σ_c we used are shown in Fig. 1. The cross section for the formation of the compound nucleus ^{64}Cu by the incident neutron may be thought smaller than the reaction cross section given by Feshbach and Weisskopf as the

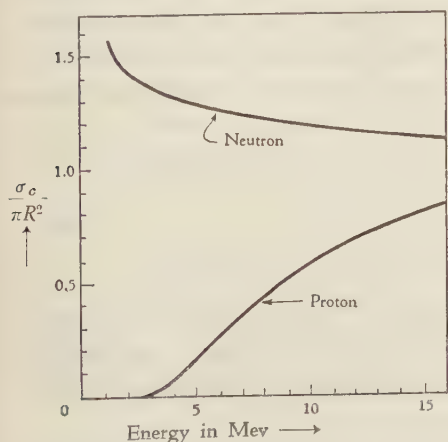


Fig. 1. Cross sections for the formation of a compound nucleus by neutrons and protons.

incident energy becomes higher. Leaving the discussions on the energy dependence of $\sigma_c^{(n)}(E)$ to the following section § 4, here we assume that all reaction processes are due to the evaporation from the compound nucleus, that is, the probability for the formation of the compound nucleus equals to unity. Calculated results and the comparison with the experiment are shown in Fig 2.

It is remarkable that with choice (ii) of a , $\sigma(n; 2n)$ is much smaller than the experimental cross section. This is because the radiation width increases more rapidly than other particle widths with increasing value of a .

These results show that the choice (ii) of a may not be suitable and the magnitude of a must be much smaller as pointed out by Tomasini.¹⁶⁾

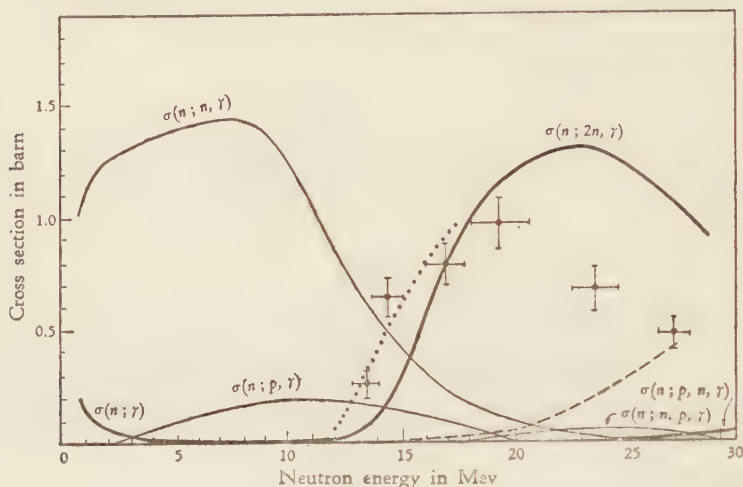


Fig. 2. Cross sections for various processes calculated on the assumption that the probability of the formation of the compound nucleus by incident neutrons is equal to unity. Solid curves refer to results calculated for $a=2 \text{ Mev}^{-1}$ (case (i)) and the dashed curve indicates $(n; 2n)$ cross section calculated for $a=23.8 \text{ Mev}^{-1}$ (case (ii)). The dotted curve indicates $(n; 2n)$ cross section obtained without competition.

§ 3. The estimate of the knock-out process

The calculated value of $\sigma(n; 2n)$ much larger than the experimental cross section supports the occurrence of the knock-out process with increasing incident energy. The nucleon-nucleon collision in the nuclear matter has been studied by several authors¹⁷⁻¹⁹⁾ on the basis of the Fermi gas model and by the use of the experimental cross section of free nucleon-nucleon collision, taking the effect of the Pauli principle into account. The realistic form of the cross section for the free nucleon-nucleon collision is too complicated to allow the analytical expression. Here in accordance with Goldberger¹⁷⁾, we make the simplest assumption of the energy independent and isotropic cross section for the cross section of the free nucleon-nucleon collision. In this case the analytical calculation is readily carried out with the result

$$\bar{\sigma} = \begin{cases} \sigma_f \left(1 - \frac{7}{5} \cdot \frac{E_f}{E_1}\right) & \text{for } E_1 \geq 2E_f, \\ \sigma_f \left\{1 - \frac{7}{5} \cdot \frac{E_f}{E_1} + \frac{2}{5} \cdot \frac{E_f}{E_1} \left(2 - \frac{E_1}{E_f}\right)^{5/2}\right\} & \text{for } E_1 \leq 2E_f \end{cases} \quad (10)$$

where σ_f is the cross section of the free nucleon-nucleon collision and $\bar{\sigma}$ is that of the nucleon-nucleon collision in the nucleus. E_1 and E_f are the incident energy and the Fermi energy ($=21.5 \text{ Mev}$) respectively, both measured from the bottom of the Fermi well. Thus the total collision cross section σ_t is given by

$$\sigma_t = Z\bar{\sigma}_{np} + N\bar{\sigma}_{nn}, \quad (11)$$

Z and N being the proton and the neutron numbers in the target nucleus. $\bar{\sigma}_{np}$ and $\bar{\sigma}_{nn}$ are the cross section of neutron-proton collisions and that of neutron-neutron collisions, calculated by the above formula (10). One should, however, notice that the total cross section given by (11) fails to reproduce the observed facts for the reason that the effects of the attenuation and the transmission are not yet taken into account. In actual nuclear reactions we have to deal with the attenuation within a nucleus and with the transmission and refraction at the nuclear surface, for both the incident and the scattered beams. These effects can in principle be handled with the Monte Carlo method, which, however, seems so tedious that it will be impracticable to cover a wide energy range. Accordingly we prefer to treat

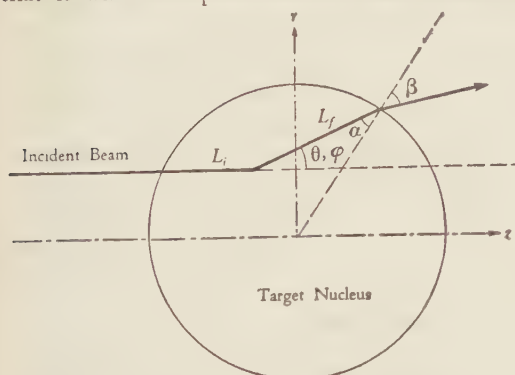


Fig. 3. Schematic representation for the knock-out process.

the knock-out process on a very simple model in this section. The main feature of this process is represented schematically in Fig 3, in which z -axis is along the direction of the incident beam and r is perpendicular to it. L_i and L_f are the lengths of paths within nucleus before and after the collision respectively. The total cross section σ_d of the knock-out process is expressed as

$$\sigma_d = \frac{1}{\pi R^2} \int_0^R 2\pi r dr T(E_i) \int \frac{L_i}{\lambda_i} \cdot \frac{\partial \sigma}{\partial \Omega} \cdot T(E_f) \exp\left(-\frac{L_i}{\lambda_i} - \frac{L_f}{\lambda_f}\right) d\Omega dL_i, \quad (12)$$

where T 's are the transmission coefficients and E_i is the incident energy and E_f is the mean value of the final energy, which is given by the formula (3.19) in III. The mean free path λ is expressed in terms of the cross section σ and the density of the nuclear matter ρ as

$$\lambda = 1/\bar{\sigma}\rho. \quad (13)$$

In the integration of (12), since T 's are slowly decreasing with increasing r , we may drop the integration over r and adopt the assumption that the incident neutrons impinge with zero impact parameter and then do not suffer the diffraction on entering into the nucleus. Putting L_i equal to λ_i (12) is rewritten as

$$\sigma_d = \frac{\langle T_i \rangle \langle T_f \rangle}{e} \int \frac{\partial \sigma}{\partial \Omega} \cdot \exp\left(-\frac{L_f}{\lambda_f}\right) d\Omega \quad (14)$$

where T 's are put to the average value $\langle T \rangle$ over angles. The angular distribution of scattered particles $\partial \sigma / \partial \Omega$ has a very complicated form in spite of the simple model we employed. The general feature of the distribution, shown in III, suggests us that the differential cross section may be approximated by a cosine function as follows:

$$\frac{\partial \sigma}{\partial \Omega} = \begin{cases} (\sigma_i/2\pi) \cos \theta & \text{for } 0 \leq \theta \leq \pi/2, \\ 0 & \text{for } \pi/2 \leq \theta. \end{cases} \quad (15)$$

With this approximation the integration is reduced to

$$\sigma_d = \sigma_i \frac{\langle T_i \rangle \langle T_f \rangle}{e} \cdot \int_0^{\theta_{max}} \cos \theta \sin \theta \exp\left(-\frac{L_f}{\lambda_f}\right) d\theta. \quad (16)$$

θ_{max} is the upper limit of θ , which is determined from the fact that the scattered beam suffers refraction on emerging from the nuclear surface. If we express the wave numbers of an emerging nucleon inside and outside of the nucleus k_{int} and k_{out} respectively, the analogy to geometrical optics leads to the Snellius' relation

$$k_{out} \sin \beta = k_{int} \sin \alpha. \quad (17)$$

Since $\sin \beta \leq 1$, the equation

$$\sin \alpha = k_{out}/k_{int} \quad (18)$$

determines the maximum value of the angle α , which gives

$$\theta_{max} = \sin^{-1}\left(\frac{R}{a} \cdot \frac{k_{out}}{k_{int}}\right) \quad (19)$$

where $a = R - \lambda_i$.

The integration (16) is calculated into the following form

$$\sigma_d = \frac{\sigma_i}{2a^2} \cdot \frac{\langle T_i \rangle \langle T_f \rangle}{e} \Psi(\lambda_f). \quad (20)$$

The function $\Psi(\lambda_f)$ is given by

$$\Psi(\lambda_f) = \lambda_f^2 (1+x) e^{-x} \Big|_{x=x_0}^{x_1} + \frac{b^4}{\lambda_f^2} \int_{x_0}^{x_1} \frac{e^{-x}}{x^3} dx \quad (21)$$

with

$$b^2 = R^2 - a^2,$$

$$x_0 = (a+R)/\lambda_f,$$

$$x_1 = (1/\lambda_f) \{a \cos \theta_{max} + \sqrt{a^2 \cos^2 \theta_{max} + b^2}\}.$$

Calculated value of σ_d is shown in Fig 4.

§ 4. Discussions and comparison with experiment

The evaluation of the cross section for the knock-out process allows us to make a careful comparison between the theoretical calculations and the experimental data. Although the cross section $\sigma_c^{(n)}(E_n)$ for the formation of the compound nucleus by the incident neutron is assumed to be equal to the reaction cross section σ_r in the treatment of § 2, the estimate of the knock-out process in § 3 facilitate us to make a more exact investigation on this problem. The occurrence of this process has the effect of diminishing the magnitude of $\sigma_c^{(n)}$, for the sum of $\sigma_c^{(n)}$ and σ_d is approximately equal to the reaction cross section

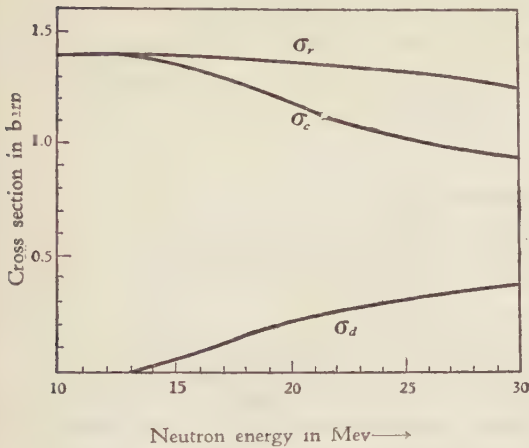


Fig. 4. Calculated cross section σ_d for the knock-out process. The cross section σ_c for the formation of a compound nucleus is obtained by the subtraction of σ_d from the reaction cross section σ_r .

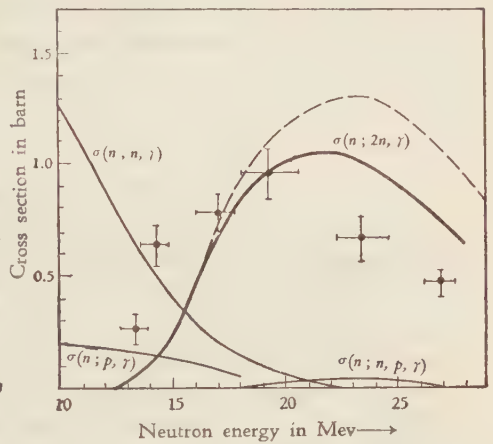


Fig. 5. Cross sections for various processes. The fact that the occurrence of the knock-out process diminishes the probability of the formation of the compound nucleus by incident neutrons is taken into account.

Φ 's indicate experimental $(n; 2n)$ cross section and the dashed curve is $(n; 2n)$ cross section shown in Fig. 2.

σ_r . Thus the formula (7) is revised as

$$\sigma_c(l; i, j, k) = (\sigma_r - \sigma_d) K_{ijk} / \sum_{m,n,p} K_{mnp} \quad (22)$$

The reaction cross section σ_r is given by Feshbach and Weisskopf and is approximately equal to the geometrical cross section πR^2 for high energies. The cross sections for various processes calculated by (22) is of smaller magnitude than those shown in Fig 2. There are some discrepancies between the theoretical result and the experimental one, as seen from Fig 5. The discrepancy in the low energy portion comes from the fact that the formula of F_γ gives the larger value than the actual radiation width. When the available energy for the neutron emission is large enough, the large magnitude of F_γ does not matter because it is much smaller than the particle width. However, as the available energy decreases and the particle width becomes smaller, the magnitude of F_γ tends to decrease the particle emission probability. This prospect is supported by the fact that the calculated $\sigma(n; 2n)$ is in fair agreement with the experiment, if we make the simplifying assumption that the residual nucleus always emits a neutron if its excitation energy is sufficient to do so.

The figure also indicates some discrepancies in the high energy region, which might have come from the neglect of the direct process occurring at the nuclear surface, from the ambiguity in the choice of the values of a , or from the disregard of α , d and t emission. Each effect mentioned above seems to make the $(n; 2n)$ cross section smaller, if it is taken into account. Though there remain some ambiguities in these respects, it is clear that the knock-out process can not be ignored in the nuclear reactions with the incident energy above 20 Mev.

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On the Quenching of Molecular Rotation of Ortho-Hydrogen in Solid State

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By using the data measured by Hill and Ricketson, we can express the anomalous specific heat C_V of solid hydrogen at sufficiently high temperatures approximately as

$$C_V T^2/R = \alpha c + \beta c^2,$$

$$\alpha = 1.1, \quad \beta = 15.7,$$

where c is the concentration of ortho-hydrogen molecules, T the absolute temperature, and R the gas constant. It is an aim of this paper to inquire into the origin of these two terms. Through calculations by the method of moment expansion, the magnitude of β can be explained from the quadrupolar interactions among ortho-molecules, while the anisotropic nature of the valence and van der Waals forces is shown to be effective only as corrective terms. On the other hand, the appearance of α shows that the molecular rotation of an ortho-molecule surrounded by para-molecules should also be quenched. A possible mechanism of this phenomenon may be the rotation-vibration coupling, for which the valence and van der Waals forces should be exclusively responsible. However, we are not yet able to get a complete understanding of this quenching process. Deviation from the above formula, i. e. the linear dependence on c of the quantity $C_V T^2/R$ at temperatures not high enough is also derived and compared with experiment. Further, a more exact treatment than the moment expansion method is given for the case of low ortho-concentrations and is applied to the case of 7% ortho.

§ 1. Introduction

The anomalous specific heat of solid hydrogen was originally observed by Simon *et al.*¹⁾ Schaefer then attempted to explain them.²⁾

The rotational degeneracy temperature of hydrogen molecule is 86°K, and is sufficiently higher than the temperatures of solid hydrogen. Hence, the rotational state of ortho-molecule is almost in $J=1$, while that of para-molecule is in $J=0$. Here J denotes the rotational quantum number. Thus, each para-molecule has a non-degenerate state corresponding to its rotation of spherical symmetry, while each ortho-molecule has a three-fold degeneracy in its state of free rotation. If this degeneracy is lifted, the free rotation of an orthomolecule will be quenched. The entropy removal arising from such a quenching process will be responsible for the anomalous specific heat. Schaefer thought that the hindering potential for ortho-molecule proportional to $P_2(\cos \theta)$ was sufficient to quench the rotation of ortho-molecule, where $P_2(\cos \theta)$ is the Legendre polynomial and θ the polar angle of the molecular axis. But he did not inquire into the origin of such a term. Now, Nagamiya and Kisi³⁾ pointed out the fact that the coefficient of $P_2(\cos \theta)$ originating from

the valence force of Margenau⁴⁾ vanishes identically in the case of the hexagonal close-packed structure, which corresponds to the crystal structure of solid hydrogen.⁵⁾ We may say therefore that Schaefer's theory does not elucidate the quenching of molecular rotation of ortho-hydrogen.

However, recent development of proton magnetic resonance gave us valuable informations about the molecular rotation in solids. Thus, Hatton and Rollin studied the proton resonance in solid hydrogen⁶⁾ and Reif and Purcell measured the line shape in normal hydrogen with a higher accuracy⁷⁾. They confirmed the quenching of molecular rotation of the ortho-hydrogen. As a result of a detailed analysis of these resonance data, Tomita concluded that it was impossible to explain the abrupt change in the resonance width at about 1.5°K if one disregards the cooperative character of quenching of molecular rotation.⁸⁾

In quite recent years, Hill and Ricketson measured the anomalous specific heat of solid hydrogen of different concentrations of ortho-molecules.⁹⁾ They observed a λ -shaped transition in the cases of 74% and 66% ortho-concentrations, whose transition points are 1.6° and 1.35°K respectively. This fact suggests that the quenching of molecular rotation proceeds cooperatively, in accord with Tomita's conclusion from resonance data. According to further experiments by Hill and Ricketson,¹⁰⁾ the transition temperature is proportional to ortho-concentration at higher concentrations than 60% ortho-molecules and the λ -shaped transition disappears at 60% ortho-molecules and below. Hence we may conclude that the interactions between ortho-molecules play an essential role in the λ -transition.

What types of intermolecular forces are responsible for this phenomenon? In order to seek them, let us study the tails of the anomalous specific heat curves at moderate high temperatures.*) If the anomalous specific heat per mole, C_V , in this region is proportional to the inverse square of temperature, $C_V T^2/R$ will approach to a constant value with increasing temperatures, where T is the absolute temperature and R the gas constant. We plot $C_V T^2/R$ against T for different concentrations, as shown in Fig. 1. The saturation seems to be realized, though approximately. These saturated values should be a function of ortho-concentration, which we denote by c . Considering the cooperative behaviour of quenching at high concentrations of ortho-molecules, we may suppose that the contributions to $C_V T^2/R$ will predominantly be from c^2 . So we plot the saturated values of $(C_V T^2/R)^{1/2}$ against c . The curve thus obtained shows a straight line at higher concentrations than $c=0.25$, as can be seen in Fig. 2. But the linear part with respect to c is not negligibly small, because that straight line cuts the ordinate at about 0.14. Thus, we can put

$$C_V T^2/R = \alpha c + \beta c^2, \quad (1)$$

where $C_V T^2/R$ denotes its saturated value. Then we have

$$(C_V T^2/R)^{1/2} \simeq \sqrt{\beta} c + \alpha/2\sqrt{\beta} \quad (2)$$

* The anomalous parts C_V were approximated by the difference of C_p at 0.5% ortho- from C_p at $x\%$ ortho-concentrations.

in the regions of high concentrations. The numerical values obtained from Fig. 2 are given by

$$\sqrt{\beta} = 3.96, \quad \alpha/2\sqrt{\beta} = 0.14, \quad (3)$$

whence we get

$$\alpha = 1.1, \quad \beta = 15.7. \quad (4)$$

β is about fifteen times larger than α .

Whatever the origin of β may be, it should be attributed to the interactions among ortho-molecules. On the other hand, the small part proportional to c shows that the molecular rotation of an ortho-molecule is quenched by the interactions common to ortho-ortho and ortho-para pairs. We have now to look for the origin of these two kinds of quenching process.

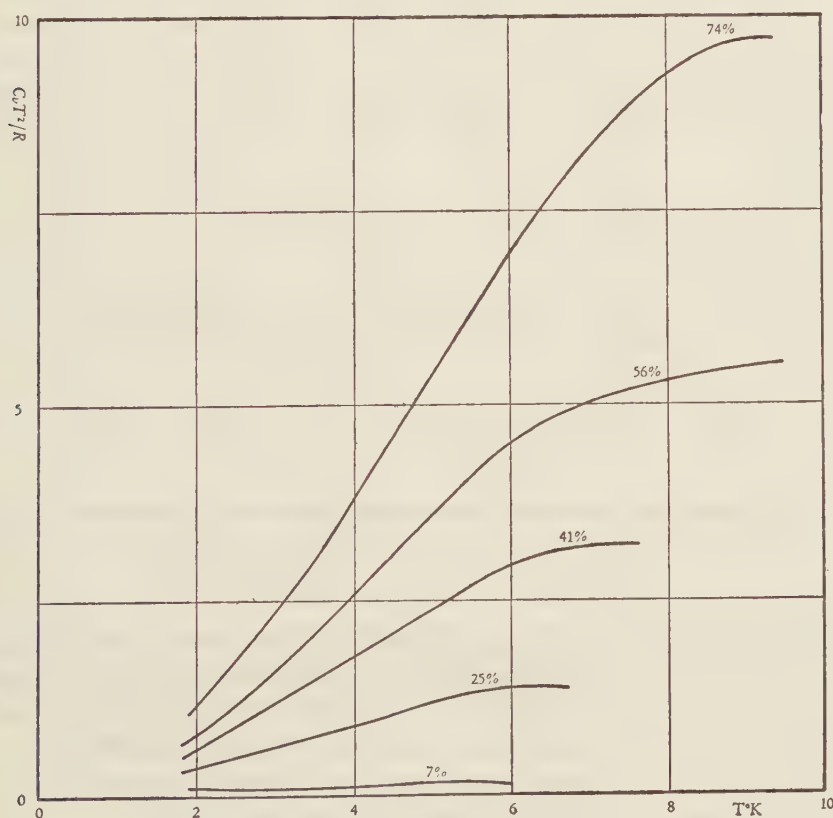


Fig. 1. Curves for $C_V T^2 / R$ versus T .

Our program is the following:

- 1). We consider the anisotropic part of interactions between two hydrogen molecules (§ 2), which is expressible in terms of the orientation angles of two molecules, (θ_1, φ_1) and (θ_2, φ_2) .

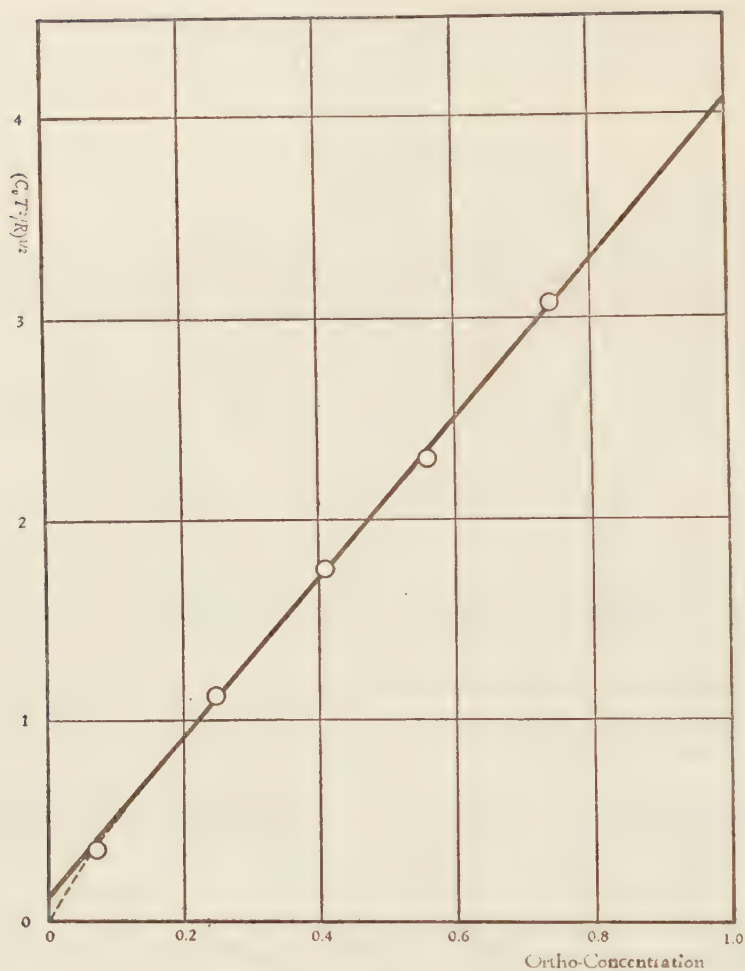


Fig. 2 Curve for saturated $(C_v T^2 / R)^{1/2}$ versus c .

2). We transform the representation thus obtained to the representation in which the magnitude of \mathbf{J} and J_z are diagonal, where \mathbf{J} denotes the rotational angular momentum operator and J_z its z -component. Our interest is of course in the parts of $J=1$ and $J=0$. In such a way, we get the interactions for ortho-ortho and ortho-para pairs (§ 3).

3). It can be shown that interactions common to ortho-ortho and ortho-para pairs vanish identically in the case of the hexagonal close-packed structure with

$$c/a = 2\sqrt{2/3}. \quad (5)$$

Thus, we have only to consider the interactions which are characteristic of the ortho-ortho pair. A calculation of the second moment for the total Hamiltonian leads to $\beta=20.5$ (§ 4).

4). As a greater part of theoretical β proves to arise from the quadrupolar interactions,

we confine our Hamiltonian to the quadrupolar parts to calculate the third moment for the total Hamiltonian. The result obtained is in a better agreement with the experimental β (§ 5, Fig. 5).

5). In order to treat the temperature dependence of C_V at very low concentrations, the eigen-values of a pair Hamiltonian are obtained, where our attention is again confined to the quadrupolar part alone. Supposing ortho-ortho pairs to be independent of each other, we calculate C_V without help of the $1/T$ -expansion method (§ 6).

6). The splitting of rotational levels, JE , in the absence of the interactions characteristic of ortho-ortho pair, may be obtained from α of (4). Assuming that the rotational level with $J_z'=0$ is lower than that with $J_z'=\pm 1$,*) we take ΔE to be 3.1×10^{-16} erg. If we take into account C_V arising from this splitting JE , the experimental values of C_V at 7% ortho-concentration can well be reproduced (§ 6, Fig. 6).

7). The origin of JE may arise from the interactions which are common to ortho-ortho and ortho-para pairs. These interactions, which vanish identically in the rigid lattice, may produce the quenching through the vibrational motion of molecule. But we have not yet succeeded to explain the magnitude of JE by such a mechanism, so we shall not present any of our results in this paper.

§ 2. Anisotropic interactions between two hydrogen molecules

There are three important types of interactions between two hydrogen molecules, that is, the valence (repulsive), van der Waals, and quadrupolar interactions. We shall give a review which is necessary for our calculations.¹¹⁾

In the first place, the valence force E_{val} was obtained by Margenau and de Boer. Especially, de Boer's expression is written in a convenient form¹²⁾

$$E_{\text{val}} = V(e^{-r_{ao}/\rho} + e^{-r_{bo}/\rho} + e^{-r_{ad}/\rho} + e^{-r_{bd}/\rho}), \quad (6)$$

where (a, b) and (c, d) denote proton positions of a hydrogen molecule respectively and r_{ac} etc. can be seen in Fig. 3. The numerical values of V and ρ are given by

$$V = 2.78 e^2/a_0, \quad \rho = a_0/1.87, \quad (7)$$

where e is the usual charge quantum and a_0 the Bohr radius.

Let us denote the distance between two protons within a single molecule by r_1 , and the distance between the centres of gravity of two molecules by r (Fig. 3). Assuming $r \gg r_1$, we may write (6) as³⁾

$$E_{\text{val}} = 4Ve^{-r/\rho} \cosh(\gamma \cos \theta_1) \cosh(\gamma \cos \theta_2), \quad (8)$$

$$\cosh(\gamma \cos \theta) = \lambda_0 P_0(\cos \theta) + 5\lambda_2 P_2(\cos \theta) + \dots, \quad (9)$$

$$\lambda_0 = \sinh \gamma / \gamma, \quad (9a)$$

$$\lambda_2 = (1/\gamma + 3/\gamma^3) \sinh \gamma - (3/\gamma^2) \cosh \gamma, \quad (9b)$$

where

* By J_z' we mean the eigen-value of the z -component of rotational angular momentum operator.

Then, the average van der Waals force \bar{E}_{dis} becomes

$$\bar{E}_{\text{dis}} = -(2P/3)/r^6, \quad (17)$$

which was evaluated by Margenau in the following form:

$$\bar{E}_{\text{dis}} = -10.9 e^2/a_0 \cdot (a_0/r)^6. \quad (18)$$

With the help of the numerical values of α_{\parallel} and α_{\perp} , (18) is sufficient to determine the constant K in (14). By using α_{\parallel} and α_{\perp} evaluated by Ishiguro *et al.*,¹³⁾ we have the anisotropic part of E_{dis}

$$\tilde{E}_{\text{dis}} = -\tilde{c}(\theta_1, \varphi_1; \theta_2, \varphi_2)/r^6, \quad (19)$$

$$\tilde{c}(\theta_1, \varphi_1; \theta_2, \varphi_2) = c(\theta_1, \varphi_1; \theta_2, \varphi_2) - 2P/3. \quad (19 \text{ a})$$

Some numerical values at $r=r_0$ ($\equiv 3.75 \text{ \AA}$) can be seen in Table 2.

Thirdly, the quadrupolar interaction E_Q is written

$$E_Q = (3e^2\mu_Q^2/4r^5)f(\theta_1, \varphi_1; \theta_2, \varphi_2), \quad (20)$$

$$\begin{aligned} f(\theta_1, \varphi_1; \theta_2, \varphi_2) = & 1 - 5(\cos^2\theta_1 + \cos^2\theta_2) - 15\cos^2\theta_1\cos^2\theta_2 \\ & + 2\{-4\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)\}^2. \end{aligned} \quad (21)$$

We may rewrite (21) as

$$\begin{aligned} f(\theta_1, \varphi_1; \theta_2, \varphi_2) = & 8P_2(\cos\theta_1)P_2(\cos\theta_2) \\ & - 16\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2\cos(\varphi_1 - \varphi_2) \\ & + \sin^2\theta_1\sin^2\theta_2\cos 2(\varphi_1 - \varphi_2). \end{aligned} \quad (22)$$

Table 2.
Some numerical values for the van der Waals
and quadrupolar forces.

$\bar{E}_{\text{dis}} = -(2P/3)/r_0^6$ (Margenau)	$37.50 \times 10^{-16} \text{ erg}$
Q/P	0.0483
R/P	0.0138
Q/r_0^6	$2.72 \times 10^{-16} \text{ erg}$
R/r_0^6	$0.77 \times 10^{-16} \text{ erg}$
μ_Q	$0.110 \times 10^{-16} \text{ cm}^2$
$3e^2\mu_Q^2/4r_0^5$	$2.82 \times 10^{-16} \text{ erg}$

As can easily be seen from (22), the value of E_Q averaged over the orientation of a single molecule vanishes identically, and hence the quadrupolar interactions consist totally of fluctuating parts. The electrical quadrupole moment, μ_Q , can be determined from the rotational magnetic moment of hydrogen molecule. Thus Harrick and Ramsey gave $\mu_Q = 0.110 \times 10^{-16} \text{ cm}^2$.¹⁴⁾ The numerical value of $3e^2\mu_Q^2/4r_0^5$ with this μ_Q can be seen also in Table 2.*

§ 3. Expressions for the ortho-ortho and ortho-para interactions

In the preceding section, we gave a survey of the anisotropic parts of intermolecular forces. Let us consider how these parts are modified by the rotational state of the molecule.

* It may be noted that Hirschfelder, Curtiss and Bird defined the quadrupole moment as twice our μ_Q .

Now, the following operator equivalence is written down within the subspace of a single J :

$$\begin{aligned} P_2(\cos \theta) &\cong p_J \left\{ \frac{3}{2} J_z^2 - \frac{1}{2} J(J+1) \right\}, \\ \cos \theta \sin \theta e^{\pm i\varphi} &\cong p_J \cdot \frac{1}{2} (J_z J_{\pm} + J_{\pm} J_z), \\ \sin^2 \theta e^{\pm 2i\varphi} &\sim p_J J_{\pm} J_{\pm}, \end{aligned} \quad (23)$$

where J_x , J_y and J_z are the angular momentum operator and

$$J_{\pm} = J_x \pm iJ_y. \quad (24)$$

The constant p_J can be determined from the comparison of both sides of (23) with respect to one matrix element, for example $(J, J_z' | P_2(\cos \theta) | J, J_z')$. Thus we have

$$p_J = -2/5, \quad J=1. \quad (25)$$

Using (23) and (25), we get the operator equivalence of E_{\parallel} , in which both molecules are in the state $J=1$. That is, (21) is now written

$$\begin{aligned} f &\cong 8/25 \cdot \{ (3J_{z1}^2 - 2)(3J_{z2}^2 - 2) - (J_{z1}J_{+1} + J_{+1}J_{z1})(J_{z2}J_{-2} + J_{-2}J_{z2}) \\ &\quad - (J_{z1}J_{-1} + J_{-1}J_{z1})(J_{z2}J_{+2} + J_{+2}J_{z2}) \\ &\quad + 1/4 \cdot (J_{+1}J_{+1}J_{-2}J_{-2} + J_{-1}J_{-1}J_{+2}J_{+2}) \}. \end{aligned} \quad (26)$$

In the same way, $\tilde{c}(\theta_1, \varphi_1; \theta_2, \varphi_2)$ of an ortho-ortho pair is transformed into

$$\begin{aligned} \tilde{c}(\theta_1, \varphi_1; \theta_2, \varphi_2) &\cong -1/5 \cdot (2Q - 4R/3) \{ (3J_{z1}^2 - 2) + (3J_{z2}^2 - 2) \} \\ &\quad + 2R/25 \cdot \{ (3J_{z1}^2 - 2)(3J_{z2}^2 - 2) - (J_{z1}J_{+1} + J_{+1}J_{z1})(J_{z2}J_{-2} + J_{-2}J_{z2}) \\ &\quad - (J_{z1}J_{-1} + J_{-1}J_{z1})(J_{z2}J_{+2} + J_{+2}J_{z2}) \\ &\quad + 1/2 \cdot (J_{+1}J_{+1}J_{-2}J_{-2} + J_{-1}J_{-1}J_{+2}J_{+2}) \}, \end{aligned} \quad (27)$$

where \tilde{c} is defined by (15) and (19a).

As the operator form of \tilde{E}_{val} defined by (11) can easily be obtained by use of (23), we may omit, for the sake of brevity, to write down its explicit form.

Next, let us consider the ortho-para interaction. It may simply be obtained from averaging (27) and (26) over the orientation of the second molecule, which we assume to be a para-molecule. Then, (26) vanishes identically, while (27) reduces to

$$\tilde{c}(\theta_1, \varphi_1; \theta_2, \varphi_2) \cong -1/5 \cdot (2Q - 4R/3) (3J_{z1}^2 - 2). \quad (28)$$

In the same way, the ortho-para representation of \tilde{E}_{val} may easily be obtained.

It should be noted that we took the representation in which J_z is diagonal, where our z -axis is along the line connecting two molecules. Hence, our quantization axes are different with each other for different pairs of molecules.

§ 4. The rotational part of Hamiltonian and the specific heat arising from ortho-ortho interactions

The rotational part of Hamiltonian consists of kinetic and potential terms. The kinetic part is diagonal in our representation. The kinetic energy of ortho-molecule is $2b^2/2I$, and that of paramolecule is zero, where I is the moment of inertia of hydrogen molecule and b the Planck constant divided by 2π . Since this part is almost independent of temperature, we may put it out of consideration.

Thus, we have only to take into account \tilde{E}_{val} , \tilde{E}_{dis} and E_Q . Let us confine our consideration to the interactions between two nearest neighbouring molecules. Then, the coupling constants of three kinds of interactions are given in Tables 1 and 2. Including \tilde{E}_{val} (cf. Eqs. (11), (23) and (25)), \tilde{E}_{dis} (cf. Eqs. (19), (27) and (28)), and E_Q (cf. Eqs. (20) and (26)), we get the rotational Hamiltonian H :

$$H = \sum_i H_i + \sum_{i>k} H_{ik}, \quad (29)$$

$$H_i = - (4\lambda_0\lambda_2 V e^{-r_0/\rho} - 2Q/5r_0^6 + 4R/15r_0^6) \\ \times \sum_k (3J_{zi}^2 - 2)_k, \quad (30)$$

$$H_{ik} = (8/25 \cdot (3e^2\mu_Q^2/4r_0^5) + 4\lambda_2^2 V e^{-r_0/\rho} - 2R/25r_0^6) \\ \times (3J_{zi}^2 - 2)(3J_{zk}^2 - 2) \\ - (8/25 \cdot (3e^2\mu_Q^2/4r_0^5) - 1/25 \cdot (2R/r_0^6)) \{ (J_{zi}J_{+i} + J_{+i}J_{zi}) (J_{zi}J_{-k} + J_{-k}J_{zi}) \\ + (J_{zi}J_{-i} + J_{-i}J_{zi}) (J_{zi}J_{+k} + J_{+k}J_{zi}) \} \\ + 2/25 \cdot (3e^2\mu_Q^2/4r_0^5 - R/2r_0^6) (J_{+i}J_{+i}J_{-k}J_{-k} + J_{-i}J_{-i}J_{+k}J_{+k}), \quad (31)$$

where the indices i and k in (29) run over all pairs of ortho-molecules neighbouring each other, but the index k in (30) runs over all nearest neighbours irrespectively of para- or ortho-molecule. And $(3J_{zi}^2 - 2)_k$ means that the quantization axis is along the line connecting i and k molecules.

If we take the representation in which the component of \mathbf{J} along the c -axis is diagonal, H_i is transformed to

$$H_i = - (4\lambda_0\lambda_2 V e^{-r_0/\rho} - 2Q/5r_0^6 + 4R/15r_0^6) \\ \times \sum_k \{ P_2(\cos \theta_k) (3J_{zi}^2 - 2) \\ + 1/2 \cdot P_2^1(\cos \theta_k) [e^{-i\Phi_k} (J_{zi}J_{+i} + J_{+i}J_{zi}) + e^{i\Phi_k} (J_{zi}J_{-i} + J_{-i}J_{zi})] \\ + 1/4 \cdot P_2^2(\cos \theta_k) (e^{-2i\Phi_k} J_{+i}J_{+i} + e^{2i\Phi_k} J_{-i}J_{-i}) \}. \quad (32)$$

with the help of (40), where (θ_k, Φ_k) denotes the direction from the i -th molecule to the k -th one in the coordinate system whose polar axis is the c -axis. However, (32) vanishes identically after summation over k , because of the hexagonal close-packed structure with (5). Hence, we shall neglect it hereafter. This neglect is valid only in the case of rigid lattice.

Accordingly, our Hamiltonian has a simple form

$$H = \sum_{i>k} H_{ik}. \quad (33)$$

The Slater sum Z is then written

$$Z = \text{Tr} \exp(-H/kT), \quad (34)$$

whence we obtain the specific heats, C_V , due to ortho-ortho interactions by the moment expansion method of Waller-Van Vleck.^{15), 16)} The result is

$$C_V = (cN/2kT^2) \sum_k \langle H_{ik}^2 \rangle_{AV}, \quad (35)$$

where we may note that $\langle H_{ik} \rangle_{AV} = 0$ and $\langle H_{ik} H_{jk} \rangle_{AV} = 0$. Assuming the random distribution of ortho-molecule on the lattice points we may write (35) as

$$C_V T^2 / R = \beta c^2, \quad (36)$$

$$\beta = (z/2k^2) \cdot \langle H_{ik}^2 \rangle_{AV}, \quad (37)$$

in which z is the number of nearest neighbours.

Now, we have to evaluate $\langle H_{ik}^2 \rangle_{AV}$. It is sufficient for this purpose to note the following relations:

$$\begin{aligned} \langle (3J_z^2 - 2)^2 \rangle_{AV} &= 2, \\ \langle (J_z J_+ + J_+ J_z) (J_z J_- + J_- J_z) \rangle_{AV} &= \\ &= \langle (J_z J_- + J_- J_z) (J_z J_+ + J_+ J_z) \rangle_{AV} = 4/3, \\ \langle J_+ J_+ J_- J_- \rangle_{AV} &= \langle J_- J_- J_+ J_+ \rangle_{AV} = 4/3. \end{aligned} \quad (38)$$

The average values of the other cross products prove to vanish. Thus, (31) leads to

$$\begin{aligned} \langle H_{ik}^2 \rangle_{AV} &= 4 \left(\frac{8}{25} \cdot \frac{3e^2 \mu_Q^2}{4r_0^5} + 4\lambda_2^2 V e^{-r_0/\rho} - \frac{2}{25} \cdot \frac{R}{r_0^6} \right)^2 \\ &+ \frac{32}{9} \left(\frac{8}{25} \cdot \frac{3e^2 \mu_Q^2}{4r_0^5} - \frac{2}{25} \cdot \frac{R}{r_0^6} \right)^2 \\ &+ \frac{32}{9} \left(\frac{2}{25} \right)^2 \left(\frac{3e^2 \mu_Q^2}{4r_0^5} - \frac{R}{2r_0^6} \right)^2. \end{aligned} \quad (39)$$

The numerical values of it can be evaluated with the help of Tables 1 and 2, and we have $\langle H_{ik}^2 \rangle_{AV}/k^2 = 3.41$. Substituting this value into (37) with $z=12$, we obtain $\beta=20.5$, which should be compared with its experimental value $\beta=15.7$ (cf. Eq. (4)). The agreement is fairly good.

It should be noted that the main contributions to $\langle H_{ik}^2 \rangle_{AV}$ come from the quadrupolar interactions. Both valence and van der Waals forces play their role only as a correction. In actuality, the numerical value obtained by neglecting $4\lambda_2^2 V e^{-r_0/\rho}$ and R/r_0^6 is given by $\beta=19.9$. Although this situation is due to the fact that the contributions of $4\lambda_2^2 V e^{-r_0/\rho}$ and R/r_0^6 are in opposite directions to each other, the absolute values of them are not so appreciable in comparison with the magnitude of $3e^2 \mu_Q^2 / 4r_0^5$.

Therefore, we suppose that the λ -transition in solid hydrogen corresponds to the ordering process arising mainly from the quadrupolar interactions, and hence the ordered array of the orientations of ortho-molecule may be determined by the lowest energy of quadrupolar interactions.

§ 5. Specific heat at high ortho-concentrations

The numerical value of β evaluated in § 4 is larger than that of experimental β by about 20%. This disagreement is due to the situation that our temperature region is not high enough to make $C_V T^2/R$ constant. Let us calculate the $(1/T)^3$ -term of C_V in order to see this situation.

The calculation is somewhat complicated. For, the $(1/T)^3$ -term comes from the third moment of H , which is related to $\langle H_{ij} H_{jk} H_{ki} \rangle_{AV}$. Hence, it is necessary to use the quantized axis which is common to three molecules.

The transformation formula from (x, y, z) - to (x', y', z') -system is well known, where the relation between two systems can be seen in Fig. 4. Then, the operator equivalence leads to

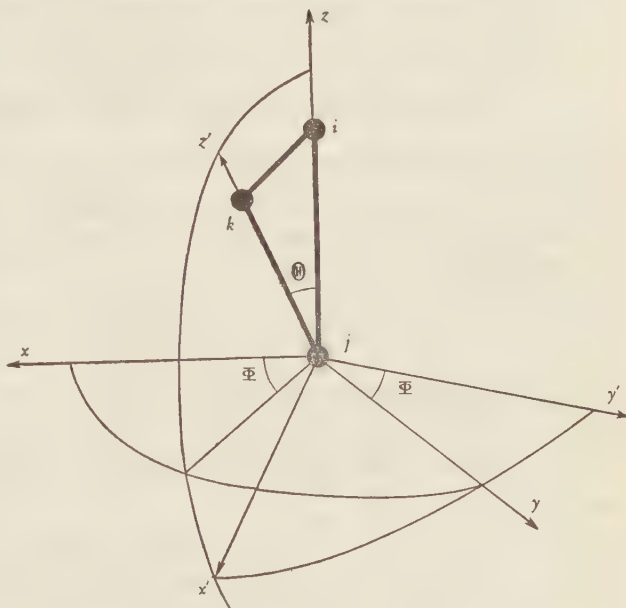


Fig. 4. The transformation of quantized axis.

$$\begin{aligned} J_+' &= 1/2 \cdot (1 + \cos \theta) e^{-i\Phi} J_+ + 1/2 \cdot (-1 + \cos \theta) e^{i\Phi} J_- - \sin \theta \cdot J_z, \\ J_-' &= 1/2 \cdot (-1 + \cos \theta) e^{-i\Phi} J_+ + 1/2 \cdot (1 + \cos \theta) e^{i\Phi} J_- - \sin \theta \cdot J_z, \\ J_z' &= 1/2 \cdot \sin \theta (e^{-i\Phi} J_+ + e^{i\Phi} J_-) + \cos \theta \cdot J_z. \end{aligned} \quad (40)$$

For the sake of simplicity, we shall neglect the parts of valence and van der Waals forces. According to the discussion made at the end of § 4, such a neglect brings no serious error.

Our object is an equilateral triangle ijk which gives the most important contribution to the third moment of H . Let us denote f in (26) by f_{12} . And let us take the z -axis so that it coincides with \bar{ij} (see Fig. 4). The polar coordinates of the directions \bar{jk} and \bar{ik} may then be written $(\pi/3, 0)$ and $(2\pi/3, 0)$ respectively.

Now, the transformation formulas (40) give the expressions for f_{jk} and f_{ik} in the representation which diagonalizes J_z . We shall define a_s by the following relations:

$$25/8 \cdot f_{ik} = \sum_s a_s A_s; \quad (41)$$

$$\begin{aligned} A_1 &= (3J_{zi}^2 - 2)(3J_{zk}^2 - 2), \\ A_2 &= (J_{zi}J_{+i} + J_{+i}J_{zi})(J_{zk}J_{-k} + J_{-k}J_{zk}) + (i \rightleftharpoons k), \\ A_3 &= J_{+i}J_{+i}J_{-k}J_{-k} + J_{-i}J_{-i}J_{+k}J_{+k}, \\ A_4 &= (3J_{zi}^2 - 2)\{(J_{zk}J_{-k} + J_{-k}J_{zk}) + (J_{zk}J_{+k} + J_{+k}J_{zk})\} + (i \rightleftharpoons k), \\ A_5 &= J_{-i}J_{-i}(J_{zk}J_{+k} + J_{+k}J_{zk}) + J_{+i}J_{+i}(J_{zk}J_{-k} + J_{-k}J_{zk}) + (i \rightleftharpoons k), \\ A_6 &= J_{+i}J_{+i}(J_{zk}J_{+k} + J_{+k}J_{zk}) + J_{-i}J_{-i}(J_{zk}J_{-k} + J_{-k}J_{zk}) + (i \rightleftharpoons k), \\ A_7 &= (3J_{zi}^2 - 2)(J_{+k}J_{+k} + J_{-k}J_{-k}) + (i \rightleftharpoons k), \\ A_8 &= (J_{zi}J_{+i} + J_{+i}J_{zi})(J_{zk}J_{+k} + J_{+k}J_{zk}) + (J_{zi}J_{-i} + J_{-i}J_{zi})(J_{zk}J_{-k} + J_{-k}J_{zk}), \\ A_9 &= J_{+i}J_{+i}J_{+k}J_{+k} + J_{-i}J_{-i}J_{-k}J_{-k}, \end{aligned} \quad (42)$$

where $(i \rightleftharpoons k)$ means the terms obtained by permuting i and k in the preceding terms. The numerical values of a_s are given in the Table 3. f_{ij} has the same form as (26).

By using these results and (38), we get

$$\langle f_{ij} f_{jk} f_{ki} \rangle_{AV} = \left(\frac{8}{25}\right)^2 \cdot \left(\frac{1}{128}\right)^2 \cdot \frac{99040}{3}. \quad (43)$$

The $1/T$ -expansion of the Slater sum (34) gives

$$Z = 3^{eN} \left\{ 1 + \frac{1}{2!} \left(\frac{1}{kT}\right)^2 \langle H^2 \rangle_{AV} - \frac{1}{3!} \left(\frac{1}{kT}\right)^3 \langle H^3 \rangle_{AV} + \dots \right\}. \quad (44)$$

And

$$\langle H^2 \rangle_{AV} = (N\zeta c^2/2) \cdot \langle H_{ik}^2 \rangle_{AV}, \quad (45)$$

$$\langle H^3 \rangle_{AV} = 4N\zeta c^3 \langle H_{ij} H_{jk} H_{ki} \rangle_{AV}, \quad (46)$$

because the numbers of the least equilateral triangles in hexagonal close-packed lattice are $2\zeta N/3$. (Note that there are 2ζ triangles which have a lattice point as their vertices.)

The internal energy due to H is then given by

$$E = -k\partial(\log Z)/\partial(1/T) \quad (47)$$

$$= - \left\{ \left(\frac{1}{k^2 T}\right) \langle H^2 \rangle_{AV} - \frac{1}{2} \left(\frac{1}{k^3 T^2}\right) \langle H^3 \rangle_{AV} + \dots \right\},$$

which leads to

$$\begin{aligned} C_V &= \frac{dE}{dT} \\ &= k \{ \langle (H/kT)^2 \rangle_{AV} - \langle (H/kT)^3 \rangle_{AV} \}. \end{aligned} \quad (48)$$

Substituting (45) and (46) into (48), and taking (39) and (43) into account, we have

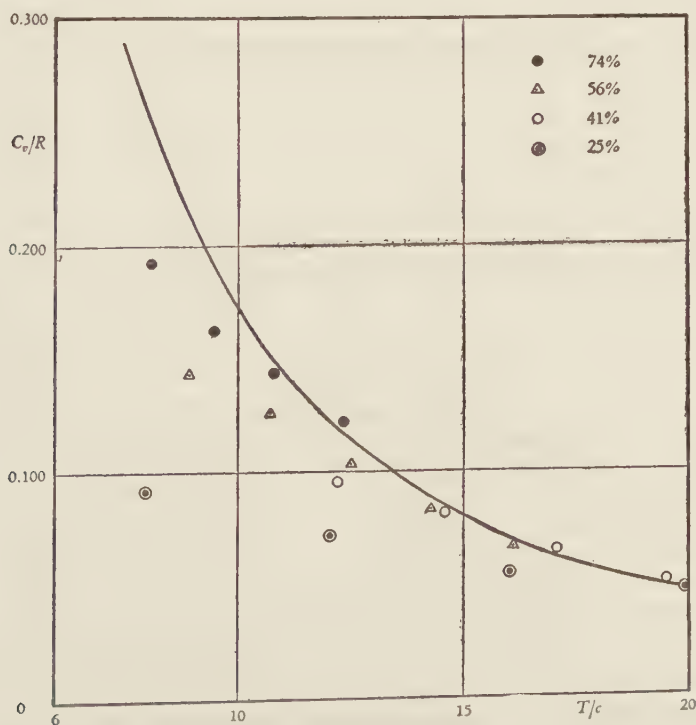
$$\frac{C_V}{R} = \frac{140}{3} \left(\frac{8c}{25kT} \cdot \frac{3c^2 \mu_Q^2}{4r_0^5} \right)^2 - 48 \cdot \frac{3095}{1536} \left(\frac{8c}{25kT} \cdot \frac{3c^2 \mu_Q^2}{4r_0^5} \right)^3. \quad (49)$$

Now we plot C_V/R against T/c in Fig. 5. The experimental C_V/R seems to approach to the theoretical curve with increasing temperature. Since the experimental C_V/R contains α in (4), the deviation of its value from the theoretical one should still remain to exist, especially for low ortho-concentrations. Actually, $C_V T^2/R$ estimated from (49) is about 17.7 at 9°K for 74% ortho-concentration, which should be compared with $\beta=15.7$.

Table 3. The numerical values of a_s defined by (41) and (42).

	a_1	a_2	a_3	a_4	a_5
f_{jk}	$-37/128$	$37/128$	$-1/4 \cdot 37/128$	$-\sqrt{3} \cdot 25/128$	$\sqrt{3} \cdot 2 \cdot 25/128$
f_{ik}	$-37/128$	$37/128$	$-1/4 \cdot 37/128$	$\sqrt{3} \cdot 25/128$	$-\sqrt{3} \cdot 2 \cdot 25/128$

	a_6	a_7	a_8	a_9
f_{jk}	$3\sqrt{3} \cdot 2 \cdot 35/128$	$1/2 \cdot 45/128$	$45/128$	$9/4 \cdot 35/128$
f_{ik}	$-3\sqrt{3} \cdot 2 \cdot 35/128$	$1/2 \cdot 45/128$	$45/128$	$9/4 \cdot 35/128$

Fig. 5. The curve for C_V/R versus T/c at high ortho-concentrations.

§ 6. Specific heats at very low ortho-concentrations

As can be seen from Fig. 1, the value of $(C_V T^2/R)^{1/2}$ in the case of 7% ortho-concentration deviates considerably from a linear law. Of course, it will be due to the appreciable contribution arising from α in (4). However, since the anomalous specific heat of 7% ortho-concentration does not appear appreciably until we go down to about 5 K,^{*} we are not allowed to use the $1/T$ -expansion method. The analysis of the situation for this very low ortho-concentration will, moreover, serve to obtain more detailed informations about the state of ortho-molecules.

Let us first evaluate the Slater sum of a pair Hamiltonian

$$Z_{\text{pair}} = \text{Tr} \exp(-H_{ik}/kT). \quad (50)$$

For this purpose, let us find the eigen-values of H_{ik} . Of course, we have only to find the eigen-values of $(25/8)f_{ik} = g_{ik}$, in which f_{ik} is given by (26). As is easily shown, g_{ik} proves to commute with the z -component of the total angular momentum: $J_z^{\text{tot}} = J_{z,i} + J_{z,k}$. Hence, our eigen-value problem is factorized into five parts specified by

$$J_z^{\text{tot}} = \pm 2, \pm 1, 0,$$

where J_z^{tot} denotes an eigen-value of J_z^{tot} . And let ϵ be an eigen-value of g_{ik} . Then,

$$(1) \quad J_z^{\text{tot}} = \pm 2 : \text{ we get } \epsilon = 1 \text{ easily.} \quad (51)$$

$$(2) \quad J_z^{\text{tot}} = \pm 1 : \text{ we have the matrix representation of } g_{ik}$$

$$\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix},$$

whence

$$\epsilon = -4, 0. \quad (52)$$

$$(3) \quad J_z^{\text{tot}} = 0 : \text{ the matrix representation of } g_{ik} \text{ is given by}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

whence

$$\epsilon = 0, 0, 6. \quad (53)$$

The Slater sum (50) is then written

$$Z_{\text{pair}} = 2e^{1x} + 4 + 2e^{-x} + e^{-6x}, \quad (54)$$

$$x = 8/25 \cdot (1/kT) (3e^2 \mu_Q^2 / 4r_0^6). \quad (55)$$

* We could not accurately read the numerical values above 5°K from the curve given by Hill and Ricketson.

Assuming that pairs of ortho-molecules are all independent of one another, and taking a procedure similar to (47) and (48), we get

$$\frac{C_V}{R} = \frac{1}{2} \kappa c^2 \left(\frac{x}{2e^{4x} + 4 + 2e^{-x} + e^{-6x}} \right)^2 \{ (32e^{4x} + 2e^{-x} + 36e^{-6x}) \times (2e^{4x} + 4 + 2e^{-x} + e^{-6x}) - (8e^{4x} - 2e^{-x} - 6e^{-6x})^2 \}. \quad (56)$$

We plotted C_V/c^2R evaluated from (56) against T . The result is the curve *a* in Fig. 6, where the experimental C_V/c^2R is also plotted for the case of 7% ortho-concentration. By comparison between them, we see that the experimental values are about twice larger than the theoretical ones. The linear term with respect to c is therefore important.

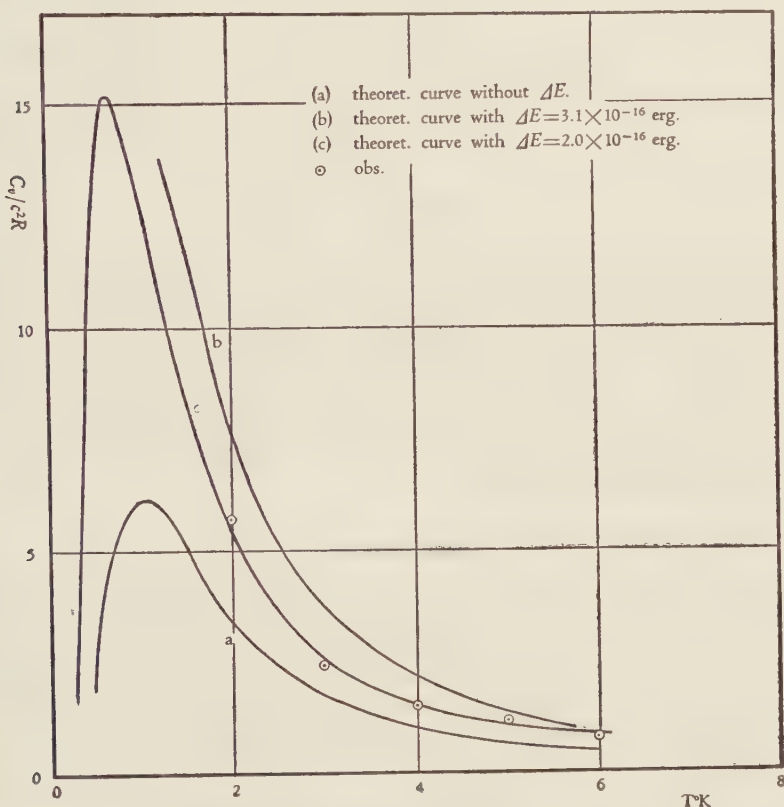


Fig. 6. The theoretical and experimental curves for C_V/c^2R versus T at very low concentration ($c=7\%$).

However, we have already obtained this term from the informations about the concentration dependency of $C_V T^2/R$. As can be seen from (2), the estimation of α , which is given by (4), should be regarded as based on the data at high ortho-concentrations.

Let us consider that the threefold degenerate level of an ortho-molecule is split into two different ones in the absence of the interactions characteristic of ortho-ortho pair, where the

upper level corresponds to $J_z' = \pm 1$, while the lower one to $J_z' = 0$. Here the quantized axis may be taken as the c -axis. Let ΔE be the splitting of these two levels. Then, the contribution from this splitting to $C_V T^2/R$ can easily be obtained as

$$C_V' T^2/R = 2/9 \cdot (\Delta E/k)^2 c. \quad (57)$$

The right-hand side of it is nothing but αc . Hence, we get the magnitude of ΔE with the help of α given by (4). That is,

$$\Delta E = 3.1 \times 10^{-16} \text{ erg}. \quad (58)$$

The specific heat at low temperatures is then estimated according to

$$C_V'/R = 2cu^2 \cdot e^{-u} / (1 + 2e^{-u})^2, \quad (59)$$

$$u = \Delta E/kT. \quad (60)$$

Superposing this contribution on (56), the experimental $C_V/c^2 R$ can be reproduced better. If we assume $\Delta E = 2 \times 10^{-16}$ erg, the reproduction proves to be most satisfactory (see Fig. 6). Thus we may conclude that the magnitude of ΔE given by (58) is fairly reliable.

Our final step will be to inquire into the elementary process causing ΔE -splitting. If we remember that the specific heat arising from ΔE is linear with respect to c , the origin of ΔE should be from the interactions which are common to ortho-ortho and ortho-para pairs. But, the rotational Hamiltonian coming from those interactions, (32), should vanish due to the crystalline symmetry in the case of rigid lattice. If we consider the vibrational motion of molecule, however, Hamiltonian (32) will cause the quenching of molecular rotation. We hope that we shall be able to discuss such a mechanism in a near future.

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On the Renormalization of Heisenberg Treatment

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Dyson's renormalization method has been applied to the simultaneous integral equations of the vacuum expectation values of Heisenberg operators. The renormalized integral equations will always be free from the well-known divergence difficulties, not only in the perturbation expansions but for any other approximations.

The same method is also applicable to the bound state equations. Further, for the real nucleons which are surrounded by the cloud of virtual mesons, expressions are given for the S-matrix, the expectation values and the normalized wave functions.

§ 1. Introduction

Since a few years ago the treatment of Heisenberg operators in the Green function formalism has been advanced by Schwinger and other authors¹⁾. In connection with these investigations the simultaneous integral equations satisfied by the vacuum expectation values of Heisenberg operators have been proposed^{2,3,4)}, the solution of which, if carried out practically, may offer the expectation values and S-matrix elements. Before reaching the concrete results directly applicable to the discussions of experiments, we must first give answer to some important problems. For these problems many considerations have been proposed.

The most important problem concerns the well-known divergence difficulties of the self-energy, the vacuum polarization, etc. It should be possible to adopt some cut-off method so as to avoid these difficulties, but it is necessary to clarify the mechanism of the well-known divergencies so that one may be sure of the appropriateness of the cut-off method.

It has been known that the treatments in the Heisenberg representation are convenient, by their covariant form, for the study of the correspondence to Dyson's perturbation method. Also in the divergence problems, we shall assume, the whole results obtained from Dyson's renormalization method⁴⁾ in the perturbation theory can be applied directly to the Heisenberg treatment and any new divergence will not appear. The renormalization of Heisenberg treatment on this line has been carried out by a few authors⁵⁾ in the approximation neglecting the Fermion closed loops and restricting the meson number. Our attempt is only a generalization of their tasks. The same idea is applied to the original integral equations themselves. And in consequence of no restriction on the meson number, Dyson's

* Hereafter we refer to this paper as I.

renormalization method is applied straight-forwardly. The essential parts of Dyson's renormalization method are formulated with reference to some inner network taken out freely from some S-matrix diagram. So, also for the above integral equations, we may apply the same method to each network containing the new integrals in it. The results will have more general meaning than the renormalization in the perturbation method, that is, if we can solve the renormalized integral equations by any suitable method, the solutions will certainly be always free from the well-known divergence difficulties.

The further application of Gell-Mann-Low's method⁶⁾ becomes sufficient for obtaining the renormalized bound state equations. In conclusion, Dyson's renormalization method can be applied to the general integral equations for the S-matrix and the bound states without using the perturbation expansions. Of course the above conclusions hold only for the systems which are renormalizable in the perturbation theory, namely the systems containing only the so-called first kind interactions.⁷⁾ In this paper just as in I, the practical calculations are carried out only for the system of Fermions interacting with neutral scalar mesons. And the divergences which are originated in the terms of meson emission, absorption and scattering by a meson are not considered. The renormalization of these terms may be carried out quite similarly to the others. But its procedure and resultant equations are so complicated that we should hardly be able to obtain any profit from them.

Another problem lies in the situation that the real particles are surrounded by the clouds of virtual quanta of interacting fields even in their free motions. From this situation it follows that the solution of the S-matrix obtained by the method in I will not give the values immediately corresponding to the experimental scattering cross-sections. The same situation will be true for the case of the expectation values. As to this problem a detailed analysis has been carried out by Nishijima.⁸⁾ We shall show more simple and concrete expressions for the approximation neglecting Fermion closed loops. Namely we shall give the expressions of the expectation values or S-matrix elements connecting one bound state to another bound state, and a definition of the covariant orthonormality for the bound state wave functions in that approximation.

As these procedures may easily be generalized to the solutions for two particle bound states, we should be able to make clearer the physical meanings of Salpeter-Bethe wave functions.⁹⁾

§ 2. Renormalization process of the integral equations

Hereafter in the course of the renormalization for the integral equations of Heisenberg operators, the results obtained by Dyson's⁴⁾ and Salam's¹⁰⁾ analyses of the divergences in the perturbation expansions will be utilized. It is said that the perturbation expansion is the asymptotic expansion for it is not analytical at the origin. But the great successes of the renormalization method in the electrodynamics make us believe that the ideas in the renormalization process in the perturbation theory may be applied more generally over the limits of its approximation. We shall use as before the ideas of the primitive

divergence, the overlapping divergence, etc., and their natures in the analysis of the perturbation expansions.¹⁰⁾¹¹⁾ The practical calculations are carried out for the Fermion (nucleon) and neutral scalar meson system.

(a) *Isolation of the divergent terms*

Adding the renormalization terms to the ordinary interaction Lagrangian density the original unrenormalized integral equations of the vacuum expectation values of Heisenberg operators (Feynman amplitudes) (37), (38) and (39) of I are given by

$$\begin{aligned} \langle \psi(k), \bar{\psi}(l) \rangle &= \delta(k-l) S(k) + i g S(k) \int dp \langle \psi(p), \phi(k-p), \bar{\psi}(l) \rangle \\ &+ i \partial \kappa_0 S(k) \langle \psi(k), \bar{\psi}(l) \rangle, \end{aligned} \quad (1)$$

$$\begin{aligned} \langle \phi(h_1), \phi(h_2) \rangle &= \delta(h_1+h_2) \Delta(h_1) + i g \Delta(h_1) \int dp \langle \bar{\psi}(-h_1+p), \psi(p), \phi(h_2) \rangle \\ &+ i \partial \mu^2 \Delta(h_1) \langle \phi(h_1), \phi(h_2) \rangle, \end{aligned} \quad (2)$$

$$\begin{aligned} \langle \psi(k_1), \dots, \psi(k_m), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) \rangle \\ = i g S(k_1) \int dp \langle \psi(p) \phi(k-p), \dots \rangle + i \partial \kappa_0 S(k_1) \langle \psi(k_1), \dots \rangle, \\ \langle \psi(k_1), \dots, \psi(k_m), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) \rangle \\ = i g \Delta(h_1) \int dp \langle \dots, \bar{\psi}(-h_1+p) \psi(p), \dots \rangle + i \partial \mu^2 \Delta(h_1) \langle \dots, \phi(h_1), \dots \rangle, \end{aligned} \quad (3)$$

where $S(k)$ and $\Delta(h)$ are the momentum expressions of the ordinary Dyson's Green functions

$$S(k) = i(k\gamma - \kappa_0) / (k^2 + \kappa_0^2 - i\epsilon), \quad (4)$$

$$\Delta(h) = -i / (h^2 + \mu^2 - i\epsilon). \quad (5)$$

$\langle \psi(k_1), \dots, \phi(h_1), \dots, \bar{\psi}(l_1), \dots \rangle$'s are four dimensional Fourier components of the Feynman amplitudes $\langle \mathcal{EP}(\psi(x_1), \dots, \phi(y_1), \dots, \bar{\psi}(z_1), \dots) \rangle_0$'s.

We have rewritten here $g/(2\pi)^2$ of I as g . There are equations on $\bar{\psi}(l)$ conjugate to (1) and the first equations of (3) on $\psi(k)$.³⁾ As their treatments are quite symmetric to (1) and (3), we don't express them especially.

According to Dyson we take the following expressions:

$$\langle \psi(k), \bar{\psi}(l) \rangle = \delta(k-l) S'(k), \quad (6)$$

$$\langle \phi(h_1), \phi(h_2) \rangle = \delta(h_1+h_2) \Delta'(h_1), \quad (7)$$

$$\langle \psi(k), \phi(h), \bar{\psi}(l) \rangle = \delta(k+h-l) S'(k) \Gamma(k, l) S'(l) \Delta'(h), \quad (8)$$

where Γ contains the coupling constant.

(1) and (2) are written as follows:

$$\begin{aligned} S'(k) &= S(k) + i g S(k) \int dp S'(p) \Delta'(k-p) \Gamma(p, k) S'(k) \\ &+ i \partial \kappa_0 S(k) S'(k), \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{A}'(h) = & \mathcal{A}(h) - ig\mathcal{A}(h) \int dp S'(-h+p) S'(p) \Gamma(p, -h+p) \mathcal{A}'(h) \\ & + i\delta\mu^2 \mathcal{A}(h) \mathcal{A}'(h). \end{aligned} \quad (10)$$

There are five types of the primitive divergences for the Fermion and scalar meson system, namely, i) Fermion self-energy, ii) meson self energy, iii) vertex (V part), iv) meson emission or absorption by meson and v) meson-meson scattering. By the reasons stated in the introduction we shall not consider here the divergences of the type iv) and v). Therefore we may regard that there are only three primitive divergences i), ii) and iii), and apply Dyson's renormalization method to them.

It is well known that the divergence difficulties arise from the point interaction of the field operators in the interaction Lagrangian density. Correspondingly, new divergences always arise in the integral equations, due to the integrals of the point interaction

$$\int dp \psi(p) \phi(k-p) \quad \text{or} \quad \int dp \bar{\psi}(-h+p) \psi(p).$$

Therefore, if we isolate the terms which are expected to diverge by the integration and renormalize them, we shall obtain the divergence-free integral equations, that is, their solutions will not diverge in any approximation.

To isolate the terms expected to diverge by the integration we utilize Salam's idea of the "class"^{10,11)}. We call the irreducible V part as the V part of class I. All the classes of V part may be constructed by the process of inserting the irreducible V part in the end point of the irreducible V part (class 2), and again inserting the irreducible V part in the end point of the V part of class 2 (class 3), and so on. The general V part is the sum of all the classes.

Using the kernel K which is obtained opening the end point of the irreducible V part, the above relation is written as

$$\Gamma = ig \left\{ 1 + \int dp \sum_{n=1}^{\infty} K^n \right\}. \quad (11)$$

The exact expression of (11) will be given later. In the abbreviations as (11), we omit hereafter S' or \mathcal{A}' connecting the kernels. K is the total sum of the graphs which cannot be split into two parts which are joined by a Fermion line and a meson line.

Now we consider the amplitude $\langle \psi(k), \phi(h_1), \phi(h_2), \bar{\psi}(l) \rangle$. It consists of the two parts of the Dyson diagrams, which are the graphs $\Gamma S' \Gamma'$ joined by a Fermion line and the graphs $\sum_{n=1}^{\infty} K^n$ joined by a Fermion line and a meson line. That is

$$\langle \psi \phi \phi \bar{\psi} \rangle = \Gamma S' \Gamma + \sum_{n=1}^{\infty} K^n. \quad (12)$$

If we put

$$\langle \psi \phi \phi \bar{\psi} \rangle' = \sum_{n=1}^{\infty} K^n = \mathfrak{R}, \quad (13)$$

as the relation between the kernels K and \mathfrak{K} , we get the following integral equation :

$$\mathfrak{K} = K + K\mathfrak{K}. \quad (14)$$

Exact expression of (14) is

$$\begin{aligned} \mathfrak{K}(k, h_1; h_2) &= K(k, h_1; h_2) \\ &+ \int dh K(k, h_1; h) S'(k+h_1+h) A'(h) \mathfrak{K}(k+h_1+h, -h; h_2). \end{aligned} \quad (14')$$

(14') is obtained by the following procedure : From the first integral equation of $m=1, n=1$ of (3) we have

$$\begin{aligned} \langle \psi(k), \phi(h), \bar{\psi}(l) \rangle &= igS(k) \left\{ \int dp \langle \psi(p), \phi(k-p), \phi, \bar{\psi} \rangle \right. \\ &\quad \left. + A'(h) \langle \psi(k+h), \bar{\psi} \rangle \right\} + i\partial\kappa_0 S(k) \langle \psi(k), \phi, \bar{\psi} \rangle \\ &= igS \left\{ \int dp S' A' \Gamma \langle \psi, \phi, \bar{\psi} \rangle + \int dp S' A' \mathfrak{K} S' A' \delta(k+h-l) \right. \\ &\quad \left. + A' S' \delta(k+h-l) \right\} + i\partial\kappa_0 S \langle \varphi, \phi, \bar{\psi} \rangle, \end{aligned} \quad (15)$$

which is reduced by (8) and (9) as follows

$$\begin{aligned} \{1 - igS \int dp S' A' I' - i\partial\kappa_0 S\} \langle \psi, \phi, \bar{\psi} \rangle &= S A' \Gamma S' \delta(k+h-l) \\ &= Sig \left\{ 1 + \int dp S' A' \mathfrak{K} \right\} S' A' \delta(k+h-l). \end{aligned} \quad (16)$$

From (16) Γ is shown to be of the following form :

$$\Gamma(k, l) = ig \left\{ 1 + \int dp S'(p) A'(k-p) \mathfrak{K}(p, k-p; l-k) \right\}. \quad (11')$$

In the case of more general type $\{dp \langle \psi(p), \phi(k-p), \dots \rangle$ the divergences arise from the three parts with $\Gamma \langle \psi(p), \dots \rangle$, $\mathfrak{K} \langle \rangle \langle \rangle$ and $\mathfrak{K} \langle \rangle''$. The remaining part $\langle \rangle''$ consists only of the graphs which cannot be split into parts which are joined by a Fermion line and a meson line. We put the summation of the third graphs $\mathfrak{K} \langle \rangle''$ and the graphs $\langle \rangle''$ as

$$\langle \rangle' = \langle \rangle'' + \mathfrak{K} \langle \rangle''. \quad (17)$$

The exact relations among these functions become

$$\begin{aligned} \langle \psi(p), \phi(k-p), \dots \rangle &= S'(p) A'(k-p) \Gamma(p, k) \langle \psi(k), \dots \rangle \\ &+ \sum \mathcal{E} S'(p) A'(k-p) \int dp_1 \mathfrak{K}(p, k-p; p_1-k) \langle \psi(p_1), \dots \rangle \langle \phi(k-p_1), \dots \rangle \\ &+ \langle \psi(p), \phi(k-p), \dots \rangle'. \end{aligned} \quad (18)$$

Here $\sum \mathcal{E}$ represents the sum of all the possible terms which are separated into two Feynman amplitudes, multiplied by ± 1 according to the permutation of Fermion operators.

Hereafter we omit ε in the case of similar summations.

And (17) is written as

$$\begin{aligned} \langle \psi(p), \phi(k-p), \dots \rangle' &= \langle \psi(p), \phi(k-p), \dots \rangle'' \\ &+ S'(p) A'(k-p) \int dp_1 \mathfrak{R}(p, k-p; p_1-k) \langle \psi(p_1), \phi(k-p_1), \dots \rangle''. \end{aligned} \quad (17')$$

Substituting (18) into the general integral equations (3) we have

$$\begin{aligned} \{1 - igS \int dp S' A' \Gamma - i\partial \kappa_0 S\} \langle \psi, \dots \rangle \\ = igS \int dp \{ \langle \psi, \phi, \dots \rangle' + \sum \langle \psi, \dots \rangle \langle \phi, \dots \rangle \\ + \sum S' A' \int dp_1 \mathfrak{R} \langle \psi, \dots \rangle \langle \phi, \dots \rangle \}, \end{aligned}$$

where the second term $\sum \langle \psi, \dots \rangle \langle \phi, \dots \rangle$ appears by (19) of I.

By the consideration of (9) and (17') similarly in the case of reduction (11') and the use of (11') itself, we get the following equations:

$$\begin{aligned} \langle \psi(k), \dots \rangle &= igS'(k) \int dp \{1 + \int dp_1 S'(p_1) A'(k-p_1) \mathfrak{R}(p_1, k-p_1; p-k)\} \\ &\times \{ \langle \psi(p), \phi(k-p), \dots \rangle'' + \sum \langle \psi(p), \dots \rangle \langle \phi(k-p), \dots \rangle \} \\ &= S'(k) \int dp \Gamma(k, p) \{ \langle \psi(p), \phi(k-p), \dots \rangle'' + \sum \langle \psi(p), \dots \rangle \langle \phi(k-p), \dots \rangle \}, \end{aligned} \quad (19)$$

where $\langle \rangle''$ are defined by (18) and (17').

We may apply the same treatment to the integrals $\int dp \langle \bar{\psi}(-h+p), \psi(p), \dots \rangle$ in the second equations of (3), and will get the similar equations. By the kernel H which is obtained opening the top point of the irreducible V part, we may write Γ as follows:

$$\Gamma = ig \{1 + \int dp \sum_{n=1}^{\infty} H^n\}. \quad (20)$$

Corresponding to (12), (13) and (14), we have

$$\langle \bar{\psi} \psi \phi \bar{\psi} \rangle = \Gamma A' \Gamma + \sum_{n=1}^{\infty} H^n, \quad (21)$$

$$\langle \bar{\psi} \psi \phi \bar{\psi} \rangle' = \sum_{n=1}^{\infty} H^n = \mathfrak{S}, \quad (22)$$

$$\mathfrak{S} = H + H \mathfrak{S}. \quad (23)$$

From the second equation of $m=1, n=1$ of (3) we obtain the relation (20) by the calculation similar to (15) and (16), that is exactly

$$\Gamma(k, l) = ig \{1 + \int dp S'(-h+p) S'(p) \mathfrak{S}(-h+p, p; k)\}, \quad (20')$$

where $h=k-l$.

The terms which will be divergent in the integration $\int dp \langle \bar{\psi}(-h+p), \psi(p), \dots \rangle$ are represented by the graphs as $\Gamma \langle \phi(h), \dots \rangle$, $\mathfrak{S} \langle \rangle \langle \rangle$ and $\mathfrak{S} \langle \rangle''$. Similarly to (18) and (17) we may define the following functions:

$$\begin{aligned} \langle \bar{\psi}, \psi, \dots \rangle &= -S'S'\Gamma \langle \phi(h), \dots \rangle \\ &+ \sum S'S' \int dp_1 \mathfrak{S} \langle \bar{\psi}(-h+p_1), \psi(p_1), \dots \rangle + \langle \bar{\psi}, \psi, \dots \rangle', \end{aligned} \quad (24)$$

$$\langle \bar{\psi}, \psi, \dots \rangle' = \langle \bar{\psi}, \psi, \dots \rangle'' + S'S' \int dp_1 \mathfrak{S} \langle \bar{\psi}(-h+p_1), \psi(p_1), \dots \rangle''. \quad (25)$$

By a quite similar way we have the equations corresponding to (19),

$$\begin{aligned} \langle \phi(h), \dots \rangle &= \mathcal{A}'(h) \int dp \Gamma(-h+p, p) \\ &\times \{ \langle \bar{\psi}(-h+p), \psi(p), \dots \rangle'' + \sum \langle \dots, \bar{\psi}(-h+p) \rangle \langle \psi(p), \dots \rangle \}. \end{aligned} \quad (26)$$

Thus the equations (19) and (26), which are convenient for applying the renormalization method, have been derived from the original equations (3).

We wish to add different expressions for the relations between $\langle \rangle'$ and $\langle \rangle''$. If the kernel K is given explicitly from (14) as

$$K = \mathfrak{R}(1 + \mathfrak{R})^{-1}, \quad (27)$$

we may rewrite $\langle \rangle''$ from (17) also explicitly

$$\langle \rangle'' = \langle \rangle' - \mathfrak{R}(1 + \mathfrak{R})^{-1} \langle \rangle' = \langle \rangle' - K \langle \rangle'. \quad (28)$$

Similarly in place of (25) we get

$$\langle \rangle'' = \langle \rangle' - H \langle \rangle'. \quad (29)$$

(b) Application of Dyson's renormalization

According to Dyson's consideration, if we write the coupling constant

$$g = z_1 z_2^{-1} z_3^{-1/2} g_1, \quad (30)$$

and Heisenberg operators

$$\psi = z_2^{1/2} \phi_1, \quad \bar{\psi} = z_2^{1/2} \bar{\phi}_1, \quad \phi = z_3^{1/2} \phi_1, \quad (31)$$

with the infinite constants z_1 , z_2 and z_3 , we may regard the constant g_1 and the operators ϕ_1 , $\bar{\phi}_1$ and ϕ_1 as a finite constant and finite operators. Corresponding to these facts we may define in the same way as treated by Matthews and Salam and other authors,^{(13),(14)} as follows:

$$\begin{aligned} &\langle \psi(k_1) \dots, \psi(k_m), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) \rangle \\ &= z_2^m z_3^{n/2} \langle \psi(k_1), \dots, \phi(h_1), \dots, \bar{\phi}(l_1), \dots \rangle_1, \end{aligned} \quad (32)$$

and regard the function $\langle \rangle_1$ as finite.

Specially we may write

$$S' = z_2 S_1', \quad A' = z_3 A_1', \quad \Gamma' = z_2^{-1} z_3^{-1/2} \Gamma_1, \quad (33)$$

and

$$\mathfrak{R} = z_2^{-1} z_3^{-1} \mathfrak{R}_1, \quad \mathfrak{S} = z_2^{-2} \mathfrak{S}_1. \quad (34)$$

According to Dyson's renormalization procedure we obtain from (9), (10) and (11') or (20') the integral equations containing only the finite functions :

$$S_1'(k) = S(k) + S(k) \sum_c(k) S_1'(k), \quad (35)$$

$$A_1'(h) = A(h) + A(h) H_c(h) A_1'(h), \quad (36)$$

$$\Gamma_1(k, l) = ig_1 \{1 + A_c(k, l)\}. \quad (37)$$

In (18), (17'), (24) and (25) substituting the functions S' , A' , etc. to S_1' , A_1' , etc., we may obtain from $\langle \rangle_1$ finite relations for the functions $\langle \rangle_1'$ and $\langle \rangle_1''$.

Consequently substituting these functions in to (19) and (26) and dividing both sides of these equations by the just equal infinite constant $z_2^{-n} z_3^{-n/2}$, we obtain the integral equations containing only the finite functions as follows :

$$\begin{aligned} & \langle \psi(k_1), \dots, \phi(h_1), \dots, \bar{\psi}(l_1), \dots \rangle_1 \\ &= S_1'(k_1) \int dp \Gamma_1(k_1, p) \\ & \times \{ \langle \psi(p), \phi(k_1-p), \dots \rangle_1'' + \sum \langle \psi(p), \dots \rangle_1 \langle \phi(k_1-p), \dots \rangle_1 \} \\ &= A_1'(h_1) \int dp \Gamma_1(-h_1+p, p) \\ & \times \{ \langle \dots, \bar{\psi}(-h_1+p), \psi(p), \dots \rangle_1'' + \sum \langle \dots, \bar{\psi}(-h_1+p) \rangle_1 \langle \psi(p), \dots \rangle_1 \}. \quad (38) \end{aligned}$$

Solving these simultaneous equations (35), (36), (37) and (38) and using the finite relations corresponding to (18), (17'), (24) and (25), we shall obtain the finite solutions in any approximation.

We have not started from its proper stand point of the Heisenberg representation, but rather utilized the consequences in the perturbation expansions. For instance we have eliminated the terms which are expected to diverge in the perturbation series. However as referred at the beginning of this section, we may say that there is no divergence in all arbitrary processes, until there should actually happen any new difficulty.

§ 3. Renormalization of the bound state equations

We may apply the renormalization method to the bound state equations in the same way. By the use of Gell-Mann-Low's treatment, as we showed in § 5 of I, the bound states of one nucleon system surrounded by the meson cloud and nucleon pair cloud are represented by

$$\langle \psi(k_1), \dots, \psi(k_{m+1}), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) \rangle_K, \quad (39)$$

where $m, n=0, 1, 2, \dots$. And the bound states of two nucleon system with the meson and pair cloud are represented by

$$\langle \psi(k_1), \dots, \psi(k_{m+2}), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) | \kappa \rangle. \quad (40)$$

K is the four momentum of these states. Generally s nucleon bound states are represented in the same way. We call a set of functions such as (39) or (40), the wave functions of those bound state similarly to the Schroedinger wave functions.²⁾ We have the bound state equations from (1), (2) and (3)

$$\begin{aligned} \langle \psi(k), \dots | \kappa = i g S(k) \int dp \{ \langle \psi(p), \phi(k-p), \dots | \kappa \\ + \sum \langle \phi(k-p), \dots \rangle \langle \psi(p), \dots | \kappa + \sum \langle \psi(p), \dots \rangle \langle \phi(k-p), \dots | \kappa \} \\ + i \delta k_0 S(k) \langle \psi(k), \dots | \kappa, \text{ etc.} \end{aligned} \quad (41)$$

By the isolation of the divergent terms in § 2 and the replacement corresponding to (32)

$$\begin{aligned} \langle \psi(k_1), \dots, \psi(k_{m+s}), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) | \kappa \\ = z_2^{m+(s/2)} z_3^{n/2} \langle \psi(k_1), \dots, \bar{\psi}(l_m) |_{1K}, \end{aligned} \quad (42)$$

we obtain the renormalized equations as

$$\begin{aligned} \langle \psi(k), \dots |_{1K} = S_1'(k) \int dp \Gamma_1(k, p) \{ \langle \psi(p), \phi(k-p), \dots |_{1\kappa''} \\ + \sum \langle \phi(k-p), \dots \rangle_1 \langle \psi(p), \dots |_{1\kappa} + \sum \langle \psi(p), \dots \rangle_1 \langle \phi(k-p), \dots |_{1\kappa} \}, \text{ etc.} \end{aligned} \quad (43)$$

The relations for $\langle \dots |_{1\kappa''}$ are similar to the relations for $\langle \dots \rangle_1''$.

We may obtain the other equations from the second equations of (3). These equations will be obtained also with Gell-Mann-Low's averaging process from the renormalized amplitude equations of (38).

Therefore it may be regarded that the renormalization and Gell-Mann-Low's process are independent of each other and Dyson's renormalization may be applied to the bound state equations similarly to the equations for Feynman amplitudes.

Especially in the one nucleon system from the renormalized equation (35) and (33) we have

$$\langle \psi(k), \bar{\psi}(l) \rangle_1 = S(k) \delta(k-l) + S(k) \sum_c(k) \langle \psi(k), \bar{\psi}(l) \rangle_1. \quad (35')$$

Generally we may regard the experimental nucleon mass K_0 to be different from the bare nucleon mass κ_0 , and then we cannot give Dyson's condition to decide the renormalization constant,

$$\sum_c(k^2 = -\kappa_0^2) = 0. \quad (44)$$

If we use the averaging process, s term vanishes and we obtain the equation corresponding to $n=0$ in (41) as follows;

$$\langle \psi(k) |_{1K} = S(k) \sum_c(k) \langle \psi(k) |_{1K}, \quad (45)$$

or

$$S^{-1}(k) \langle \psi(k) |_{1K} = i(ik\gamma + \kappa_0) \langle \psi(k) |_{1K} = \sum_c(k) \langle \psi(k) |_{1K}. \quad (45')$$

As we can write

$$\langle \psi(k) |_{1K} = \chi_K \delta(k - K), \quad (46)$$

if we consider that there are four states with the same momentum K , (45') becomes

$$\sum_c(K) = i(\kappa_0 - K_0). \quad (47)$$

We may regard that (47) is the equation from which one can get the bound state momentum K , that corresponds to the Salpeter-Bethe equation of two nucleon system, or the condition for the renormalization constant corresponding to (44).

We may rewrite (35) as

$$S_1' = \{1 - S \sum_c\}^{-1} S. \quad (35'')$$

Then the bound state momentum K is a singular point of the function $S_1'(k)$, and the bare state momentum $k^2 = -\kappa_0^2$ is not.¹⁴⁾ If we take the condition (44) according to Dyson, it corresponds to $\kappa_0 = K_0$ in (47) and the singular point of S_1' agrees with the bare nucleon pole*.

If we know all the Feynman amplitudes and can get the eigenvalue K satisfying (47), we can obtain immediately the one nucleon wave functions. Namely, as we can put generally

$$\begin{aligned} & \langle \psi(k_1), \dots, \psi(k_m), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) \rangle_1 \\ &= S_1'(k_1) \cdots A_1'(h_1) \cdots S(k_1, \dots, k_m; h_1, \dots, h_{n-1}; l_1, \dots, l_m) \\ & \times \cdots S_1'(l_m) \delta(k_1 + \cdots + h_1 + \cdots - l_1 - \cdots), \end{aligned} \quad (48)$$

the wave functions (39) become

$$\begin{aligned} & \langle \psi(k_1), \dots, \psi(k_{m+1}), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l_1), \dots, \bar{\psi}(l_m) |_{1K} \\ &= S_1'(k_1) \cdots A_1'(h_1) \cdots S(k_1, \dots, k_m; h_1, \dots, h_n; l_1, \dots, l_m, K) \chi_K \\ & \times \cdots S_1'(l_m) \delta(k_1 + \cdots + h_1 + \cdots - l_1 - \cdots). \end{aligned} \quad (49)$$

We should be able to get the bound state wave function (49) by means of the renormalized bound state equations (43), but before that we must know the explicit forms of S_1' , A_1' , I_1 , \mathfrak{R}_1 , etc., which may be obtained only from the amplitude equations (38). Therefore it seems inevitable to solve the amplitude equations so as to obtain the bound state solutions.

Lastly we point out that if we know the wave functions for all bound states of the systems, we could obtain the amplitudes by the following relations;

* Here we do not consider the imaginary poles of (35'') which are pointed out by Feldman.¹⁵⁾

$$\langle \psi(k_1), \dots, \bar{\psi}(l_m) \rangle_1 = \sum_K \langle \psi(k_1), \dots |_{1K} {}_K \bar{\psi}(l_m) \rangle_1. \quad (50)$$

When we try to solve (38) or (43), we must deal with the non-linear simultaneous equations even in the approximation neglecting the nucleon closed loops and they are more complicated because of the existence of the additional equations for $\langle \rangle''$.

§ 4. Expectation values and S-matrix for the real nucleons

We have shown that the expectation values and S-matrix for the free particles can be obtained from the amplitudes in § 4 of I. In this paper we shall give their expressions for the "real" nucleons, which will be the bound states surrounded with the meson cloud. In this section we restrict our calculations to the approximation neglecting the graphs including the nucleon closed loops. Therefore we can neglect the nucleon pair cloud attached to the real nucleons and then take into account only the components of $m=0$ in (39) and (40) as the wave functions. Also the free mesons may be regarded as the real mesons. It is unnecessary to use the second equations of (3) or (38), so the renormalization of meson self-energy $\partial \mu^2$ and the modification of meson lines Δ_1' of (36) may be disregarded. In this approximation we know that, if we get the one nucleon amplitudes, we can immediately obtain the two nucleon amplitudes. Accordingly it should be most necessary to get the one nucleon amplitudes, the relation of which for the bound state wave functions

$$\langle \psi(k), \phi(h_1), \dots, \phi(h_n) |_{1K} = \chi_K(h_1, \dots, h_n) \delta(k + h_1 + \dots + h_n - K) \quad (39')$$

have been given by (49) of § 3.

From these functions we may obtain the expectation values, isobar energies and the nucleon-meson S-matrix with the emission and absorption of mesons by the real nucleons. Next we proceed to the formation of two nucleon amplitudes by the one nucleon amplitudes, and then we may obtain the nucleon-nucleon S-matrix with the emission and absorption of mesons. As we may obtain the kernel \mathfrak{H} in § 2 from them, we could proceed to the approximation containing a few nucleon closed loops. In the same way as we may construct the kernel of the Salpeter-Bethe equation we can proceed to the treatment of two nucleon bound states.

In the following we shall show the way to derive the expectation values and S-matrix from the one nucleon amplitudes or one nucleon bound state wave functions, and lastly add the covariant normalization.

Now disregarding the renormalization process by the one nucleon amplitudes which satisfies the first equations of (3)

$$\begin{aligned} & \langle \psi(k), \phi(h_1), \dots, \phi(h_n), \bar{\psi}(l) \rangle \\ &= igS(k) \left\{ \int dp \langle \psi(p), \phi(k-p), \phi(h_1), \dots, \bar{\psi}(l) \rangle \right. \\ & \quad \left. + \sum_{s=1}^n \Delta(h_s) \langle \psi(k+h_s), \dots, \phi(h_{s-1}), \phi(h_{s+1}), \dots, \bar{\psi}(l) \rangle \right\}, \end{aligned} \quad (51)$$

we shall put as

$$\begin{aligned}
 & \langle \psi(k_1), \psi(k_2), \phi(h_1), \dots, \phi(h_m), \bar{\psi}(l_1), \bar{\psi}(l_2) \rangle \\
 &= \sum_{n=1}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta^{-1}(h_1') \dots \Delta^{-1}(h_n') \\
 & \times \sum_{n^m} \{ \langle \psi(k_1), \phi(h_1), \dots, \phi(h_s), \phi(h_1'), \dots, \phi(h_n'), \bar{\psi}(l_2) \rangle \\
 & \times \langle \psi(k_2), \phi(h_{s+1}), \dots, \phi(h_m), \phi(-h_1'), \dots, \phi(-h_n'), \bar{\psi}(l_1) \rangle \\
 & - (l_1, l_2 \text{ exchanged}) \}, \tag{52}
 \end{aligned}$$

where

$$\Delta^{-1}(h) = i(h^2 + \mu^2), \tag{53}$$

and \sum_{n^m} represents the sum of all possible terms allotting $\phi(h_1), \dots, \phi(h_m)$ to two parts.

The expressions of (52) satisfy the integral equations for two nucleon amplitudes of (3), and we can show that the first few terms of the perturbation expansions of the both sides of (52) coincide. Thus the relations (52) are confirmed in the approximation that one neglects the nucleon closed loops. And these relations do not vary with the use of the renormalization method.

(a) Expectation values

We shall consider the expectation values of the product of field operators $\phi(x_1)\phi(x_2)\dots\phi(x_m)$ or $\bar{\psi}(x)O\psi(x)\phi(x_1)\dots\phi(x_m)$ between the real nucleon states K and K' , where O is the arbitrary product of γ and τ matrix.

Using Gell-Mann-Low's treatment $\psi(k) \rightarrow +\infty$ and $\bar{\psi}(l) \rightarrow -\infty$ for the amplitudes as

$$\left. \begin{aligned} & \langle \psi(k), \phi(h_1), \dots, \phi(h_m), \bar{\psi}(l) \rangle, \\ & \int dp \langle \psi(k), \bar{\psi}(-h+p)O\psi(p), \phi(h_1), \dots, \phi(h_m), \bar{\psi}(l) \rangle, \end{aligned} \right\} \tag{54}$$

we have

$$\begin{aligned}
 & {}_K \langle \phi(h_1), \dots, \phi(h_m) |_{1K'}, \\
 &= {}_K \mathcal{S}(K; h_1, \dots, h_m; K') {}_{K'} \mathcal{A}(h_1) \dots \mathcal{A}(h_m) \delta(K + h_1 + \dots + h_m - K'), \tag{55}
 \end{aligned}$$

and from (52)

$$\begin{aligned}
 & \int dp {}_K \langle \bar{\psi}(-h+p)O\psi(p), \phi(h_1), \dots, \phi(h_m) |_{K'} \\
 &= \int dp \sum_{n^m} {}_K \langle \phi(h_1), \dots, \phi(h_s), \bar{\psi}(-h+p) \rangle O \langle \psi(p), \phi(h_{s+1}), \dots, \phi(h_m) |_{K'} \\
 &+ \int dp \sum_{n=1}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta^{-1}(h_1') \dots \Delta^{-1}(h_n') \\
 & \times \sum_{n^m} \{ {}_K \langle \phi(h_1), \dots, \phi(h_s), \phi(h_1'), \dots, \phi(h_n'), \bar{\psi}(-h+p) \rangle
 \end{aligned}$$

$$\begin{aligned}
& \times O\langle \psi(p), \phi(h_{s+1}), \dots, \phi(h_m), \phi(-h_1'), \dots, \phi(-h_n') |_{K'} \\
& +_{\pi} | \phi(h_1), \dots, \phi(h_s), \phi(h_1'), \dots, \phi(h_n') |_{K'} \\
& \times \langle \bar{\psi}(-h+p) O\psi(p), \phi(h_{s+1}), \dots, \phi(h_m), \phi(-h_1'), \dots, \phi(-h_n') \rangle \}.
\end{aligned}$$

As the last terms may be neglected in the approximation of this section, we have

$$\begin{aligned}
& = \int dp \sum_{n=0}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta^{-1}(h_1') \dots \Delta^{-1}(h_n') \\
& \times \sum_{2^m} | \phi(h_1), \dots, \phi(h_s), \phi(h_1'), \dots, \phi(h_n'), \bar{\psi}(-h+p) \rangle \\
& \times O\langle \psi(p), \phi(h_{s+1}), \dots, \phi(h_m), \phi(-h_1'), \dots, \phi(-h_n') |_{K'} .
\end{aligned} \tag{56}$$

The expressions (55) and (56) are constructed by the wave functions of (39') or S -functions of (49).

(b) S -matrix

The S -matrix for the transition $K' \rightarrow K$ of one nucleon with the emission and absorption of m mesons corresponding to (53) of I are represented by

$$S(K; \mathbf{h}_1, \dots, \mathbf{h}_m; K') \tag{57}$$

of (48), where

$$\mathbf{h}_s = (\vec{h}_s, \sqrt{\vec{h}_s^2 + \mu^2}).$$

The S -matrix for the transition $K_1', K_2' \rightarrow K_1, K_2$ of two nucleons with the emission and absorption of m mesons are represented by the treatment $\psi(k_1), \psi(k_2) \rightarrow +\infty$ and $\bar{\psi}(l_1), \bar{\psi}(l_2) \rightarrow -\infty$ for (52), namely

$$\begin{aligned}
& S(K_1, K_2; \mathbf{h}_1, \dots, \mathbf{h}_m; K_1', K_2') \\
& = \sum_{n=1}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta(h_1') \dots \Delta(h_n') \\
& \times \sum_{2^m} \{ S(K_1; \mathbf{h}_1, \dots, \mathbf{h}_s, h_1', \dots, h_n'; K_2') S(K_2; \mathbf{h}_{s+1}, \dots, \mathbf{h}_m, -h_1', \dots, -h_n'; K_1') \\
& \times \delta(K_1 + \mathbf{h}_1 + \dots + \mathbf{h}_s + h_1' + \dots + h_n' - K_2') - (K_1', K_2' \text{ exchanged}) \}.
\end{aligned} \tag{58}$$

(c) Covariant orthonormality

Of course it is able to define the orthogonality and normalization of the bound state wave functions of (39) by returning to the Schrodinger wave functions as (73) and (74) of I. But we can give the different definition of the covariant orthogonality and normalization from the method of Nishijima.⁸⁾

Namely we use the operator

$$N = i \int \bar{\psi}(x) \gamma_{\mu} \psi(x) d\sigma_{\mu}, \tag{59}$$

which represents the nucleon number and commutes with the total energy momentum of the system. By the momentum representation of (59)

$$N = \int \frac{dh}{2\pi} \delta(\vec{h}) dp \bar{\psi}(-h+p) \gamma_4 \psi(p), \quad (60)$$

or its abbreviation

$$= \int \frac{dh_0}{2\pi} dp \bar{\psi}(-h_0+p) \gamma_4 \psi(p), \quad (60')$$

and (56), the definition of the orthonormality of one nucleon systems is written as

$$\begin{aligned} {}_{\kappa} | N |_{\kappa'} &= \int \frac{dh_0}{2\pi} dp \sum_{n=0}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta^{-1}(h_1') \dots \Delta^{-1}(h_n') \\ &\times {}_{\kappa} | \phi(h_1'), \dots, \phi(h_n'), \bar{\psi}(-h_0+p) \rangle \gamma_4 \langle \psi(p), \phi(-h_1'), \dots, \phi(-h_n') |_{\kappa'} \\ &= \delta_{nn'} \delta(\vec{K} - \vec{K}'), \end{aligned} \quad (61)$$

where n and n' are the level numbers of the isobar states.

Similarly for the bound states of two nucleons we have

$$\begin{aligned} \frac{1}{4} {}_{\kappa} | N^2 |_{\kappa'} &= \frac{1}{4} \int \frac{dh_{10}}{2\pi} \frac{dh_{20}}{2\pi} dp_1 dp_2 \sum_{n=0}^{\infty} \frac{1}{n!} \int dh_1' \dots dh_n' \Delta^{-1}(h_1') \dots \Delta^{-1}(h_n') \\ &\times {}_{\kappa} | \phi(h_1'), \dots, \phi(h_n'), \bar{\psi}(-h_{10}+p_1) \rangle \gamma_4, \bar{\psi}(-h_{20}+p_2) \rangle \gamma_4 \\ &\times \langle \psi(p_1), \psi(p_2), \phi(-h_1'), \dots, \phi(-h_n') |_{\kappa'} \\ &= \delta_{nn'} \delta(\vec{K} - \vec{K}'). \end{aligned} \quad (62)$$

These definitions are similar to those of the Schroedinger wave functions, in the circumstance that they are divergent in their original forms. By the renormalization process as in § 2, for the one nucleon systems (56) becomes

$${}_{\kappa} | N |_{\kappa'} = \int \frac{dh_0}{2\pi} dp {}_{\kappa} | \bar{\psi}(-h_0+p) \rangle \Gamma_4(-h_0+p, p) \langle \psi(p) |_{\kappa'} \quad (63)$$

where

$$\Gamma_{\mu}(-h+p, p) = \gamma_{\mu} \left\{ 1 + \int dp_1 S'(-h+p_1) S'(p_1) \mathfrak{S}(-h+p_1, p_1; p) \right\}. \quad (64)$$

Corresponding to Ward's identity of the charge renormalization in the quantum electrodynamics we may put⁽⁸⁾

$$\Gamma_{\mu} = z_2^{-1} \Gamma_{\mu 1}. \quad (65)$$

From (42) and (65) and (63)

$$\begin{aligned} &\int \frac{dh_0}{2\pi} dp {}_{\kappa} | \bar{\psi}(-h_0+p) \rangle_1 \Gamma_{41}(-h_0+p, p) \langle \psi(p) |_{1\kappa'} \\ &= \frac{1}{2\pi} \bar{\chi}_{\kappa} \Gamma_{41}(K, K') \chi_{\kappa'} \delta(\vec{K} - \vec{K}') = \delta_{nn'} \delta(\vec{K} - \vec{K}'). \end{aligned} \quad (66)$$

(66) is the finite relation, and by the definition of z_2 so as to

$$\Gamma_{41}(K, K) = \gamma_4, \quad (67)$$

it coincides the ordinary free nucleon normalization.

For the two nucleon systems the renormalized orthonormality is more complicated even by the relation (67), therefore we can not justify with our definition the normalization of Goldstein,¹⁶⁾ which is related only to the Salpeter-Bethe wave function $\langle \psi(k_1), \psi'(k_2) |_{\kappa}$.

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Letters to the Editor

On the Wave Equation for Spin 1 in Hamiltonian Form

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As is well known the vector meson is governed by Proca's set of field equations

$$\partial_\mu U_\nu - \partial_\nu U_\mu - \kappa U_{\mu\nu} = 0, \quad (\text{A})$$

$$\partial^\mu U_{\mu\nu} - \kappa U_\nu = 0. \quad (\text{B})$$

In 1939 Kemmer wrote this in the matrix form¹⁾

$$(\partial^\rho \beta_\rho + \kappa) \psi = 0 \quad (1)$$

with the four matrices β_ρ 's defined by the commutation relations

$$\beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \delta_{\lambda\mu} \beta_\nu + \delta_{\mu\nu} \beta_\lambda. \quad (2)$$

From (A) and (B) we get the equations

$$\begin{aligned} \partial_\lambda U_{\mu\nu} + \partial_\mu U_{\nu\lambda} + \partial_\nu U_{\lambda\mu} &= 0, \\ \partial^\rho U_\rho &= 0 \end{aligned} \quad (\text{C})$$

wherein no mass terms appear. The corresponding ones in Kemmer's formulation are obtained in the following way. By operating ∂_μ to eq. (1) we get

$$\partial_\mu \partial^\rho \beta_\rho \psi + \kappa \partial_\mu \psi = 0 \quad (3)$$

and the left multiplication of eq. (1) by $-\partial^\rho \beta_\rho \beta_\mu$ gives according to eq. (2)

$$-\partial_\mu \partial^\rho \beta_\rho \psi - \kappa \partial^\rho \beta_\rho \beta_\mu \psi = 0. \quad (4)$$

The sum of these equations divided by κ

$$\partial_\mu \psi - \partial^\rho \beta_\rho \beta_\mu \psi = 0 \quad (5)$$

is the required one as will easily be confirmed with the aid of the projection operator technique proposed by the author.²⁾

The transition from the covariant form of field equations such as eq. (1), which is characterized by the mass term having the matrix factor unity, to the Hamiltonian form wherein the time derivative has the matrix factor unity is straightforward in the case of the Dirac equation since each γ_ρ has its

reciprocal. But the β_ρ has no reciprocal on account of the eigenvalue 0, so that the said transition is not so easy in the case of Kemmer's β -formalism. In his recent note³⁾ Schrödinger has proposed a direct method of setting up a matrix formulation of the vector meson in Hamiltonian form, which is stated as follows. In terms of the symbols E and H for $U_{\mu\nu}$ and A and V for U_ρ eqs. (A) and (B) are rewritten respectively as

$$\begin{aligned} (\text{A}) \quad \text{grad } V + \dot{A} &= -\kappa E, & (\text{B}) \quad \text{curl } H - \dot{E} &= -\kappa A, \end{aligned} \quad (6)$$

$$\text{curl } A = \kappa H, \quad \text{div } E = -\kappa V \quad (7)$$

and eqs. (C) transform into

$$\text{curl } E + \dot{H} = 0, \quad \text{div } A + \dot{V} = 0. \quad (8)$$

He has expressed the field variables E , H , A and V in the form of the 10-componental column matrix ψ and rewritten eqs. (6) and (8) containing time derivatives in matrix form, getting the Schrödinger equation

$$i\dot{\psi} = H\psi \quad (9)$$

with the Hamiltonian

$$H = -i\partial^k \xi_k + \kappa \xi_4, \quad (k=1, 2, 3). \quad (10)$$

He has found that the 10-dimensional matrices ξ_ρ 's are all Hermitian and moreover by actual construction that they satisfy Kemmer's commutation relations (2). Author's remark concerns with that the essential part of these results are found in Kemmer's original treatments¹⁾ of eq. (1) which is nothing but the matrix formulation of eqs. (6) and (7) containing mass terms.

Now that $\beta_4^2 = \beta_4$, β_4^2 and $(1 - \beta_4^2)$ form two mutually orthogonal idempotents and the operation of these projectors split up the wave function ψ into two parts. The multiplication of eq. (1) by β_4 gives according to the relationship $\beta_k \beta_4^2 + \beta_4^2 \beta_k = \beta_k$

$$\partial_4 \beta_4^2 \psi + \partial^k \beta_k \beta_4^2 (1 - \beta_4^2) \psi + \kappa \beta_4 \cdot \beta_4^2 \psi = 0. \quad (11)$$

Then the left multiplication of eq. (1) by $(1 - \beta_4^2)$ annihilates the term with β_4 yielding

$$\partial^k \beta_k (1 - \beta_4^2) \psi + \kappa (1 - \beta_4^2) \psi = 0. \quad (12)$$

Thus one is led to conclude with the correspondences

$$\beta_4^2 \psi \leftarrow \rightarrow (E, A) \text{ and } (1 - \beta_4^2) \psi \leftarrow \rightarrow (H, V), \quad (13)$$

that eqs. (6) and (7) are identical with eqs. (11) and (12) respectively. In order to pass into the Hamiltonian form, which is nothing but the coupled equations (6) and (8) written in matrix form, we need the matrix form of eq. (8). This is supplied by eq. (5) for $\mu=4$:

$$\partial_4(1 - \beta_4^2) \psi - \partial^k \beta_k \beta_4 \cdot \beta_4^2 \psi = 0, \quad (14)$$

and the sum of eqs. (11) and (14) furnishes the required equation

$$\partial_4 \psi + [\partial^k (\beta_4 \beta_k - \beta_k \beta_4) + \kappa \beta_4] \psi = 0, \quad (15)$$

which shows that the Hamiltonian is

$$\mathfrak{H} = -i \partial^k \cdot i (\beta_4 \beta_k - \beta_k \beta_4) + \kappa \beta_4. \quad (16)$$

Now on comparison with Schrödinger's result (10) we find that

$$\xi_k = i (\beta_4 \beta_k - \beta_k \beta_4) \text{ and } \xi_4 = \beta_4. \quad (17)$$

Since $\exp(i a \beta_4) = (1 - \beta_4^2) + i \beta_4 \sin a + \beta_4^2 \cos a$,

we see with the aid of the relationship $\beta_4 \beta_k \beta_4 = 0$ that

$$\begin{aligned} \exp(i a \beta_4) \beta_k \exp(-i a \beta_4) \\ = i (\beta_4 \beta_k - \beta_k \beta_4) \sin a + \beta_k \cos a. \end{aligned}$$

Accordingly we get for $a = \pi/2$.

$$\xi_\mu = \exp(i \pi \beta_4/2) \beta_\mu \exp(-i \pi \beta_4/2). \quad (18)$$

This shows that the commutation relations (2) are satisfied by ξ_ρ 's.

In the case of the Dirac equation the transition to the Hamiltonian form is performed by a single operation of multiplication by γ_4 . Then it is interesting to find out what the operator is that produces such a transition in the β -formalism. The answer is simple enough, for the examination of eqs. (3), (4) for $\mu=4$ and (11) tells us that eq. (1) multiplied by

$$\begin{aligned} \mathfrak{K} &= \partial_4 - \partial^\rho \beta_\rho \beta_4 + \kappa \beta_4 \\ &= \partial_4 (1 - \beta_4^2) + (\kappa \beta_4 - \partial^k \beta_k \beta_4) \end{aligned} \quad (19)$$

affords κ times the equation (15). But the reverse process of passing from the Hamiltonian form to the original covariant form (1) is impossible. For rewriting eq. (15) as

$$(a^\rho \xi_\rho + \partial_t) \psi = 0 \quad (20)$$

with $\partial_t = i \partial_4$, $a_k = \partial_k$ and $a_4 = i \kappa$ we see at once that the operator corresponding to the above \mathfrak{K} is

$(a_4 - a^\rho \xi_\rho \xi_4 + \partial_t \xi_4)$ which yields on operating on eq. (20)

$$\partial_4 [a_4 + a^k (\xi_4 \xi_k - \xi_k \xi_4) + \partial_t \xi_4] \psi = 0. \quad (21)$$

The relationship $\xi_4 \xi_k - \xi_k \xi_4 = i \beta_k$ shows that this is the time derivative of eq. (1). Thus corresponding to the fact that eq. (6) coupled with eq. (8) furnishes not eq. (7) itself but its time derivative, it is not the eq. (12) itself but its time derivative that follows from the Hamiltonian formulation. In Hamiltonian formalism the equivalent to the covariant equation (1) is therefore eq. (9) coupled with eq. (12) regarded now as the initial conditions. The vanishing of the quantity on the left hand side of eq. (12) is ensured, when once assumed, automatically by eq. (21) derived from eq. (9) or eq. (15). This may also be stated alternatively as follows, as has been pointed out by Schrödinger. The above quantity is rewritten in terms of \mathfrak{H} as $(\kappa - \mathfrak{H} \beta_4) \psi$, so that its time derivative vanishes always on account of eq. (9) and the matrix identity $\mathfrak{H} \beta_4 \mathfrak{H} = \kappa \mathfrak{H}$. These considerations are not confined to a special irreducible representation of the β -matrices, so that they are equally applicable for the scalar meson.

As a further remark we add that the quantities $\gamma_{kl} = i (\xi_k \xi_l - \xi_l \xi_k)$ for $k, l = 1, 2, 3$ which Schrödinger has regarded as spin matrices, turn out to be the usual spin matrices $i (\beta_k \beta_l - \beta_l \beta_k)$ themselves since they are expressible as

$$\gamma_{kl} = i \exp(i \pi \beta_4/2) (\beta_k \beta_l - \beta_l \beta_k) \exp(-i \pi \beta_4/2)$$

and β_4 commutes with $(\beta_k \beta_l - \beta_l \beta_k)$ on account of eqs. (2).

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Note on the Lehmann's Modified Propagators

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Recently Lehmann¹⁾ has shown that the modified propagators,

$$\begin{cases} S_{F'}(p) = - \int e^{-i p(x-y)} \langle T(\phi(x) \bar{\phi}(y)) \rangle_0 d(x-y), \\ A_{F'}(k) = \int e^{-i k(x-y)} \langle T(\phi(x) \phi(y)) \rangle_0 d(x-y), \end{cases} \quad (1)$$

are given by

$$\begin{cases} S_{F'}(p) = \frac{i}{i\gamma p + M - i\varepsilon} \left[1 - (i\gamma p + M) \cdot \right. \\ \quad \left. \cdot \int_0^\infty d\kappa^2 \frac{(i\gamma p - \kappa) \sigma_1(\kappa^2) + \rho_2(\kappa^2)}{p^2 + \kappa^2 - i\varepsilon} \right], \\ A_{F'}(k) = \frac{-i}{k^2 + \mu^2 - i\varepsilon} \left[1 + (k^2 + \mu^2) \cdot \right. \\ \quad \left. \cdot \int_0^\infty d\kappa^2 \frac{\sigma(\kappa^2)}{k^2 + \kappa^2 - i\varepsilon} \right], \end{cases} \quad (2)$$

where $\psi(x)$ and $\phi(x)$ stand for the nucleon and meson field operators in the Heisenberg representation respectively. The density functions σ_1 , ρ_2 and σ satisfy the conditions:

$$\begin{aligned} 2\kappa\sigma_1 \geq \rho_2 \geq 0 \text{ for } \kappa^2 \geq (M + \mu)^2, \\ \sigma \geq 0 \text{ for } \kappa^2 \geq (3\mu)^2 \end{aligned} \quad (3)$$

and, if there are no bound states,

$$\begin{aligned} \sigma_1 = \rho_2 \equiv 0 \text{ for } (M + \mu)^2 > \kappa^2 \geq 0, \\ \sigma \equiv 0 \text{ for } (3\mu)^2 > \kappa^2 \geq 0. \end{aligned} \quad (4)$$

The renormalization constants Z_2 , Z_3 , δM and $\delta\mu^2$ are given by

$$\begin{cases} Z_2^{-1} = 1 + \int_0^\infty \sigma_1 d\kappa^2, \quad Z_3^{-1} = 1 + \int_0^\infty \sigma d\kappa^2, \\ \quad 1 \geq (Z_2, Z_3) \geq 0, \\ \delta M = Z_2 \int_0^\infty [(M - \kappa) \sigma_1 + \rho_2] d\kappa^2 \\ \text{and } \delta\mu^2 = -Z_3 \int_0^\infty (\kappa^2 - \mu^2) \sigma d\kappa^2. \end{cases} \quad (5)$$

The arguments which were used in deriving eqs. (1), (2), (3), and (4) do not matter whether the renormalization has been performed or not. Nevertheless the results are exclusively true of the renormalized case.

Under the same assumptions as Lehmann has made, we can find the expressions for the unrenormalized modified propagators. For example, the unrenormalized modified propagator $S_{F'}(p)$ is given by

$$\begin{cases} S_{F'}(p) = S_F(p) + S_F(p) \Sigma(p) S_F(p), \\ S_F(p) = \frac{i}{i\gamma p + M_0 - i\varepsilon}, \end{cases} \quad (6)$$

$$\begin{aligned} \Sigma(p) &= \int \langle T(O(x) \bar{O}(y)) \rangle_0 e^{-i p(x-y)} d(x-y) \\ &\text{and} \\ O(x) &= i g \gamma_5 \tau_i \phi_i(x) \psi(x) \quad \text{for the symmetrical} \\ &\quad \text{ps(ps) theory,} \end{aligned} \quad (7)$$

where M_0 stands for the unrenormalized nucleon mass. Using the same method as Lehmann has done, we get from eqs. (6) and (7) the following expressions

$$\begin{aligned} \Sigma(p, g, M_0) \\ = i \int_0^\infty d\kappa^2 \frac{(i\gamma p - \kappa) \bar{A}_1(\kappa^2, g, M_0) + \bar{A}_2(\kappa^2, g, M_0)}{p^2 + \kappa^2 - i\varepsilon} d\kappa^2, \\ S_{F'uv'}(p, g, M_0) \end{aligned} \quad (8)$$

$$= S_F(p) - \frac{i}{(i\gamma p + M_0 - i\varepsilon)^2} \int_0^\infty \frac{(i\gamma p - \kappa) \bar{A}_1 + \bar{A}_2}{p^2 + \kappa^2 - i\varepsilon} d\kappa^2.$$

where arguments g and M_0 express that the charge and mass renormalization is not performed.

We now proceed with the mass and charge renormalization without recourse to the perturbation method. If we perform the mass renormalization only and not the charge one, we get, instead of eqs. (7) and (8),

$$\begin{aligned} \Sigma(p, g, M) &= \int \langle T(O(x) \bar{O}(y)) \rangle_0 e^{-i p(x-y)} d(x-y) \\ &\quad - i\delta M, \end{aligned}$$

$$\begin{aligned} S_{F'}(p) &= \frac{i}{i\gamma p + M - i\varepsilon}, \\ O(x) &= i g \gamma_5 \tau_i \phi_i(x) \psi(x) - \delta M \psi(x), \end{aligned} \quad (9)$$

and this can be expressed in the integral form:

$$\begin{aligned} \Sigma(p, g, M) \\ = i \int_0^\infty d\kappa^2 \frac{(i\gamma p - \kappa) A_1(\kappa^2, g, M) + A_2(\kappa^2, g, M)}{p^2 + \kappa^2 - i\varepsilon} \\ - i\delta M. \end{aligned} \quad (10)$$

The density functions \bar{A}_1 , \bar{A}_2 , A_1 and A_2 in eqs. (8) and (10) have the similar inequalities as σ_1 and ρ_2 do:

$$\begin{aligned} 2\kappa \bar{A}_1 \geq \bar{A}_2 \geq 0 \text{ for } \kappa^2 \geq (M_0 + \mu_0)^2, \\ 2\kappa A_1 \geq A_2 \geq 0 \text{ for } \kappa^2 \geq (M + \mu)^2, \end{aligned} \quad (11)$$

and if there are no bound states,

$$\begin{aligned} \bar{A}_1 = \bar{A}_2 \equiv 0 \text{ for } (M_0 + \mu_0)^2 > \kappa^2 \geq 0, \\ A_1 = A_2 \equiv 0 \text{ for } (M + \mu)^2 > \kappa^2 \geq 0. \end{aligned} \quad (12)$$

Next, we can perform the charge renormalization after the prescription by Dyson. From eq. (10)

there result, then,

$$\begin{cases} \delta M = \int_0^\infty d\kappa^2 [(M - \kappa) \Sigma_1 + \Sigma_2], \\ 1 = \int_0^\infty d\kappa^2 [\Sigma_1 + \beta \delta (\kappa^2 - M^2)] \end{cases} \quad (13)$$

and

$$S_{F' re}(p) = \frac{i\beta}{Z_2(i\gamma p + M - i\varepsilon)} \left[1 - \frac{(i\gamma p + M)}{\beta} \cdot \int_0^\infty d\kappa^2 \frac{(i\gamma p - \kappa) \Sigma_1 + \Sigma_2}{p^2 + \kappa^2 - i\varepsilon} \right], \quad (14)$$

where

$$\begin{cases} \Sigma_1(\kappa^2, g, M) = \frac{(\kappa + M)^2 A_1 - 2MA_2}{(\kappa^2 - M^2)^2}, \\ \Sigma_2(\kappa^2, g, M) = \frac{(\kappa - M)^2 A_2}{(\kappa^2 - M^2)^2}, \end{cases} \quad (15)$$

and β is an arbitrary constant factor. $S_{F' re}$ should satisfy the following conditions, according to its physical meaning of the renormalized modified propagator:

1. $S_{F' re}(p) = S_F(p)$ for $(i\gamma p + M) = 0$ and $(p^2 + M^2) = 0$.
2. $S_{F' re}(p)$ should be expressed by the renormalized quantities alone. Using these conditions for $S_{F' re}$, we get from eq. (14)

$$\begin{cases} \beta = Z_2, \\ \Sigma_1 = Z_2 \sigma_1(\kappa^2, g', M), \quad \Sigma_2 = Z_2 \rho_2(\kappa^2, g', M) \end{cases} \quad (16)$$

and from eqs. (11), (12), (15) and (16)

$$\begin{aligned} Z_2 [2\kappa\sigma_1 - \rho_2] &\geq 0 \text{ for } \kappa^2 \geq (M + \mu)^2, \\ &= 0 \text{ for } (M + \mu)^2 > \kappa^2 \geq 0. \end{aligned} \quad (17)$$

Physically Z_2 is interpreted as probability that a physical particle will be found in the bare state, and it would have non-zero value, i.e., $1 > Z_2 > 0$, though the perturbation calculations show this is not the case, i.e., $Z_2 = 0$. The same arguments can also be applied to the unrenormalized modified propagator $\Delta_{F' uv}$. The eqs. (13)–(17), therefore, are exactly equal to the eqs. (2)–(5). We can thus conclude that the eq. (8) expresses the unrenormalized modified propagator and, when renormalized (without recourse to the perturbation method), it results in the Lehmann's one.

The foregoing statements can be confirmed by the perturbation calculation. In g^2 approximation, the results obtained by using the eqs. (8) and (10) really agree with those obtained by the use of the usual perturbation method without renormalization.

The imaginary parts of them are finite and coincide with those of renormalized ones, and the real parts diverge.

Now we discuss why Lehmann has got solely the renormalized modified propagator but not the unrenormalized one, though his discussions were independent of renormalization. In deriving eq. (2) it is assumed that the κ^2 integration is commutable with other ones. Then, in order to express the modified propagator in a parametrical integral form, the former integration must be left to the end being untouched. This necessarily results that the density functions σ_1 , ρ_2 and σ depend on κ^2 alone but not on p^2 or k^2 . As will be seen from eq. (8), however, σ_1 , ρ_2 and σ in eqs. (2) must depend on p^2 or k^2 as well as κ^2 , when the renormalization is not performed. This is the reason why Lehmann has got the renormalized modified propagator but not the unrenormalized one.

Finally we will briefly discuss about the causality problem. In eqs. (2) and (8) we can find the infinitesimal imaginary factor $i\varepsilon$, which gives a relation between the real and imaginary parts of the modified propagators. This relation expresses just the causality condition, which was pointed out by Nambu²⁾. Therefore, we may conclude that the causality requirements are also satisfied either before or after the renormalization is performed. The details will be published later in this journal.

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Magnetic Exchange Moment to Order eg^4 in Adiabatic Approximation

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Recently considerable progress was made in the meson theory of nuclear forces. Nuclear forces derived from the symmetrical pseudoscalar meson theory have succeeded in explaining many experimental data of two nucleon system.¹⁾ The next interesting test of the meson theory will be the electromagnetic properties of the two nucleon system. Many authors therefore calculated the mesonic corrections to the magnetic and quadrupole moments of the deuteron using pseudoscalar meson theory.²⁾

In these calculations of deuteron moments there is no contribution of meson current. However the meson current will contribute to the transition effects, e.g., photodisintegration of the deuteron, or to the magnetic moment of the three nucleon system, so we investigate here the expression for the magnetic exchange moment up to order eg^4 in the adiabatic approximation. This approximation is consistent with that adopted in the recent studies of nuclear forces.

The calculation is made in the same way as Villars' one.³⁾ We start from the following Hamiltonian

$$H = H_M + H_{MN}, \quad (1)$$

where H_M is the Hamiltonian for the free meson field and H_{MN} is

$$H_{MN} = (g/\mu) \sum_{s=1}^2 \sum_{\alpha=1}^3 (\boldsymbol{\sigma} \cdot \boldsymbol{\tau})^{(s)} \tau_{\alpha}^{(s)} \phi_{\alpha}(\mathbf{x}_s). \quad (2)$$

The current density due to the fields ϕ_1 and ϕ_2 is

$$S = \phi_2 \boldsymbol{\nabla} \phi_1 - \phi_1 \boldsymbol{\nabla} \phi_2 - (g/\mu) \sum_{s=1}^2 \boldsymbol{\sigma}^{(s)} \delta(\mathbf{x} - \mathbf{x}_s) \cdot (\phi_1 \boldsymbol{\tau}_2^{(s)} - \phi_2 \boldsymbol{\tau}_1^{(s)}). \quad (3)$$

The magnetic moment is related to the current density by

$$\mathbf{M} = \frac{1}{2} e \int [\mathbf{x} \times \mathbf{S}] dV. \quad (4)$$

We transform (1) by successive canonical transformations and eliminate non-diagonal parts with respect to meson occupation numbers step by step, and then apply these transformations to \mathbf{M} . In this manner we calculate magnetic exchange moment up to order eg^4 . The resulting expression in order eg^2 is the same with that of Villars³⁾, that is

$$\begin{aligned} \mathbf{M}^{(2)} &= \mathbf{M}_I^{(2)} + \mathbf{M}_I + \mathbf{M}_I', \\ \mathbf{M}_I^{(2)} &= e/2 \cdot [\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 [\mathbf{x}_1 \times \mathbf{x}_2] V^{(2)}(\mathbf{x}), \\ \mathbf{M}_I &= 1/4 \cdot [\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 (\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}) F_I(\mathbf{x}), \\ \mathbf{M}_I' &= 1/4 \cdot [\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 \{ 3 [\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)})] \mathbf{r}/r^2 \\ &\quad - (\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}) \} F_I'(\mathbf{x}), \end{aligned} \quad (5)$$

where

$$V^{(2)}(\mathbf{x}) = \frac{g^2}{4\pi\mu} \left\{ \frac{1}{3} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) S_{12} \right\} \cdot e^{-x/x},$$

$$S_{12} = 3(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r})/r^2 - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}),$$

$$F_I(\mathbf{x}) = -\frac{g^2}{4\pi} \frac{2e}{3\mu} (-2 + 1/x) e^{-x}, \quad (6)$$

$$F_I'(\mathbf{x}) = -\frac{g^2}{4\pi} \frac{2e}{3\mu} (1 + 1/x) e^{-x},$$

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2, \quad r = |\mathbf{r}| \quad \text{and} \quad x = \mu r.$$

From eg^4 terms we pick up only terms corresponding to diagrams (a) and (b) in Fig. 1, because terms corresponding to (c) give only renormalization of coupling constant in the non-covariant sense

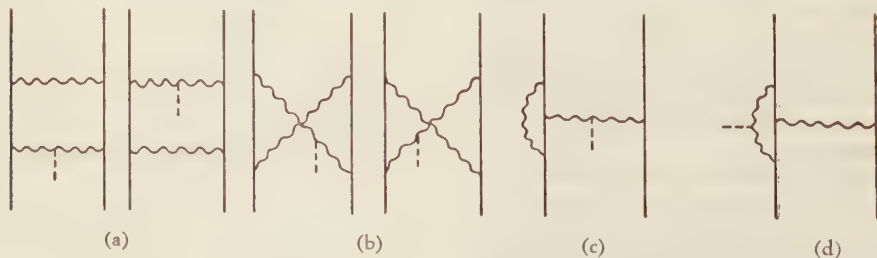


Fig. 1. ——— nucleon; ~~~~~ meson; - - - - - electromagnetic field

and those corresponding to (d) are connected with the difficulty of one nucleon magnetic moment problem and the strengths of these terms are probably small.

The expression thus obtained for the magnetic exchange moment in order eg^1 is

$$\begin{aligned} \mathbf{M}^{(4)} &= \mathbf{M}_I^{(4)} + \mathbf{M}_{II} + \mathbf{M}_{II}' + \mathbf{M}_{III} + \mathbf{M}_{III}', \\ \mathbf{M}_I^{(4)} &= e/2 \cdot [\mathbf{r}^{(1)} \times \mathbf{r}^{(2)}]_3 [\mathbf{x}_1 \times \mathbf{x}_2] V^{(4)}(x), \\ \mathbf{M}_{II} &= 1/4 \cdot (\tau_3^{(1)} - \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) F_{II}(x), \\ \mathbf{M}_{II}' &= 1/4 \cdot (\tau_3^{(1)} - \tau_3^{(2)}) \{3 [\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)})] \mathbf{r}/r^2 \\ &\quad - (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)})\} F_{II}'(x), \quad (7) \\ \mathbf{M}_{III} &= 1/4 \cdot (\tau_3^{(1)} + \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) F_{III}(x), \\ \mathbf{M}_{III}' &= 1/4 \cdot (\tau_3^{(1)} + \tau_3^{(2)}) \{3 [\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})] \mathbf{r}/r^2 \\ &\quad - (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\} F_{III}'(x) \end{aligned}$$

where

$$\begin{aligned} V^{(4)}(x) &= -(g^2/4\pi)^2 \frac{8\mu}{\pi} \left\{ \left(\frac{1}{x} + \frac{23}{4x^3} \right) K_0(2x) \right. \\ &\quad \left. + \left(\frac{3}{x^2} + \frac{23}{4x^4} \right) K_1(2x) \right\}, \\ F_{II}(x) &= -F_{III}(x) = -(g^2/4\pi)^2 \\ &\quad \times \frac{4e}{\pi\mu} \left(\frac{8}{3} + \frac{10}{3x^2} \right) K_1(2x), \quad (8) \\ F_{II}'(x) &= -F_{III}'(x) = -(g^2/4\pi)^2 (4e/\pi\mu) \\ &\quad \times \left(\frac{2}{x^3} K_0(2x) - \left(\frac{4}{3} - \frac{7}{3x^2} \right) K_1(2x) \right). \end{aligned}$$

The notation \mathbf{M} 's in (5) and (7) are analogous to Berger's.⁴⁾ Of course \mathbf{M} 's here obtained do not contribute to the magnetic moment of the deuteron. $V^{(2)}(x)$ and $V^{(4)}(x)$ are the second order nuclear potential and that part of the fourth order one⁵⁾ which is proportional to $\mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)}$ respectively, therefore $\mathbf{M}_I^{(2)}$ and $\mathbf{M}_I^{(4)}$ are the so-called Sachs exchange moment⁶⁾, the relation of which to the exchange potential was investigated by Dalitz⁷⁾.

For the deuteron the exchange moment operators \mathbf{M}_I , \mathbf{M}_{III} and \mathbf{M}_{III}' can give no contribution to the matrix elements for the photodisintegration. If we assume that the ground state of the deuteron is completely an S state, \mathbf{M}_I and \mathbf{M}_{II} give rise to transitions to a final singlet S state, while \mathbf{M}_I' and \mathbf{M}_{II}' give rise to transitions to final singlet S or singlet D states. As Berger⁴⁾ pointed out these \mathbf{M}_I , \mathbf{M}_I' , \mathbf{M}_{II} and \mathbf{M}_{II}' are the main terms of all the

possible exchange magnetic moment operators obtained in phenomenological method.

In phenomenological approach the radial dependences $F(x)$'s are left arbitrary, but our investigation determines these functions unambiguously. The radial dependences of \mathbf{M}_{II} and \mathbf{M}_{II}' have shorter ranges than \mathbf{M}_I and \mathbf{M}_I' , and may become significant at higher energies. It will be very interesting to calculate the exchange moment contributions to the photodisintegration of the deuteron or to the magnetic moment of the three nucleon system using operators here obtained and to investigate whether it is possible to explain these phenomena consistently with meson theory.

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On the Mass Reversal in the Quantized Field Theory

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Recently Tiomno¹⁾ has proposed to require the invariance of the fundamental equations under mass reversion. But in his discussions the physical meaning of the mass reversion operation is not so clear as charge-conjugation, etc., and we find some inadequacies there. So we examine this operation in the quantized field theory to know how many types of transformation we can take as mass reversal transformations. Next we try to restrict the types of interactions especially in taking account of the possibility that we can perform the mass reversion for some or all of the elementary particles concerned. We can show that there is an essential difference between Lorents- and mass reversal transformations.

We discuss this problem in the interaction representation in the same manner as time reversal is treated.²⁾ The field equations, which can be derived from Lagrangian, should be invariant under the mass reversal $\kappa \rightarrow \underline{\kappa} = -\kappa$, and

$$\begin{aligned} \psi_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}) &\rightarrow \psi'_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}) \\ &= \lambda \gamma_5 \psi_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}) \end{aligned}$$

and

$$\begin{aligned} U_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}) &\rightarrow U'_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}) \\ &= \lambda' U_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}, \underline{\kappa}), \\ \lambda &= \pm 1, \pm i, \lambda' = \pm 1 \end{aligned}$$

for the fermion and the boson field respectively.

We can easily show that the only permissible transformation for state vector under mass reversion is written as follows:

$$\mathcal{P}[\sigma] \rightarrow \underline{\mathcal{P}}[\sigma] = \mathbf{R} \mathcal{P}[\sigma].$$

The requirements of invariance of canonical field equations under this transformation allow us two types of transformations \mathbf{M}_1 and \mathbf{M}_2 as mass reversal just as the Pauli³⁾ and Wigner⁴⁾ type transformations in the time reversal.

Type \mathbf{M}_1 is obtained without assuming any commutation relations between field variables, and

type \mathbf{M}_2 is obtained by assuming the positive and negative type commutation relations for the fermion and the boson field respectively.

Now we define the sign function Π_M of the physical quantities by the equation

$$F(Q'_\alpha(x, \kappa)) = \Pi_M F(Q_\alpha(x, \kappa)),$$

where Π_M takes the value 1 or -1. The transformation properties of the tensorial quantities of spinor fields under \mathbf{M}_1 or \mathbf{M}_2 are shown in Table.

Table

	\mathbf{M}_1	\mathbf{M}_2
(S) $\bar{\psi}\psi$	-	-
(V) $\bar{\psi}\gamma_\mu\psi$	+	-
(T) $\bar{\psi}\gamma_{[\mu}\gamma_{\nu]}\psi$	-	+
(PS) $\bar{\psi}\gamma_5\psi$	-	-
(PV) $\bar{\psi}\gamma_5\gamma_\mu\psi$	+	+
(PT) $\bar{\psi}\gamma_5\gamma_{[\mu}\gamma_{\nu]}\psi$	-	+

By requiring the invariance of the Schrödinger equation under mass reversal, the interaction Hamiltonians are restricted either in \mathbf{M}_1 or \mathbf{M}_2 to the ones which satisfy

$$H(Q'(x, \underline{\kappa})) = \mathbf{R} H(Q(x, \kappa)) \mathbf{R}^{-1}.$$

We examine the typical interactions in the following.

Case A. The boson field couples linearly with fermion source.

(a). When we reverse all of the masses, we find the following relations between coupling constants f 's and phase factors ρ 's;

$$\begin{aligned} \pm f_1 \rho_a^* \rho_b \rho &= f_1 & \text{for } \mathbf{M}_1, \\ \pm f_2 \rho_a^* \rho_b \rho &= f_2^* & \text{for } \mathbf{M}_2, \end{aligned}$$

respectively, where + is for (V, PV) and - is for (S, T, PS, PT). As the phase factors are arbitrary but fixed, simultaneous coupling with and without derivatives is forbidden in the case \mathbf{M}_1 , but, in the case \mathbf{M}_2 , these relations put no restriction concerning the mixing of the different types of interaction.

(b). When we perform the mass reversion for spinor fields, only \mathbf{M}_1 is allowed by the charge conservation, i.e., we have

$$\pm f_{\rho a}^* \rho_b = f.$$

So we can not take the coupling both with and without derivatives simultaneously. Thus the sources can be mixed only within the group (V, PV) or (S, T, PS, PT) ,⁵⁾ these results seem to conform to the prescription in 5-dimensional theory of Corben.⁶⁾ *Case B.* The fermion fields interact directly with each other (universal Fermi interaction)-

By reversing all the masses of fermions we get the relations, independent of the commutation relations between two pairs,

$$f_{\alpha} = f_{\alpha \rho a}^* \rho_b \rho_c^* \rho_d \quad \text{for } M_1,$$

$$f_{\alpha}^* = f_{\alpha \rho a}^* \rho_b \rho_c^* \rho_d \quad \text{for } M_2.$$

Under M_2 the ratio f_{α}/f_{α}' for all combinations of α and α' must be real.

If we reverse the masses of one pair only, we get

$$\pm f_{\alpha} = f_{\alpha \rho a}^* \rho_b \quad \text{for } M_1.$$

This leads to the same conclusion as what is given by Tiomno, i.e., either a linear combination of (V) and (PV) interaction or a linear combination of (S) , (PS) and (T) interaction is allowed.

Case C. Pseudospinors⁷⁾ exist in the universal Fermi interaction.

We get the relations

$$f_{\alpha} = f_{\alpha \rho a}^* \rho_b \rho_c^* \rho_d \quad \text{for } M_1,$$

$$\pm f_{\alpha}^* = f_{\alpha \rho a}^* \rho_b \rho_c^* \rho_d \quad \text{for } M_2.$$

Under M_2 the interactions are divided into two groups. Different types of interaction can be mixed only within each group.

When we apply this operation to Pais⁸⁾ theory, Meggitt⁹⁾ formalism, or other theories considering the structure of elementary particles, the definition of mass reversal should be revised properly.

It is now in progress to classify all the elementary particles with the help of constant factor λ or λ' and to examine whether it may be possible, by the mass reversal, to divide the whole elementary particles into families or not. Detailed discussions on these points, together with our general method of treating this problem, will be published shortly.

We wish to thank Prof. K. Sakuma and Dr. S. Ogawa for their helpful discussions.

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- 2) H. Umezawa, S. Kamefuchi and S. Tanaka, *Prog. Theor. Phys.* **12** (1954), 383.
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Errata

On the β - γ Angular Correlations of Tm^{170} and Sb^{124} — VT Interaction —

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Prog. Theor. Phys. 13 (1955), 276

- p. 277, 3. line of Formula S and A:
 $\cdots + 3N_{23}P_4(\cos \theta)]$ should be replaced by
 $\cdots + 3N_{-23}P_4(\cos \theta)]$.
- p. 277, 6. line of Formula V and T:
 $\cdots + \{-(1/6)K^2L_{-2} + (1/2)KL_{-1}\cdots$ should be replaced by
 $\cdots + \{-(1/6)K^2L_{-12} + (1/2)KL_{-1}\cdots$.
- p. 278, 4. line of Formula V and A:
 $\cdots [\{(1/15)K^3L + \cdots$ should be replaced by
 $\cdots [\{(1/15)K^3L_0 + \cdots$.
- p. 278, 12. line: Delete, L_{-ij} and N_{-ij} .
- p. 278, 13. line: Add, L_{-ij} and N_{-ij} are as follows: .
- p. 281, in the remark of Fig. 2 and p. 284, in the Note added in proof:
 The set of parameters $x=0.8$, $y=0.96$, $y'=1.01$, $A=1.2$ should be replaced by $x=0.8$,
 $y'=0.96$, $y=1.01$, $A=1.2$.
- p. 283, 4. line: The factor $-(1/96)$ should be replaced by $-(p^2/96)$.
- p. 284, in the Note added in proof: The word "There" should be replaced by "Therefore."
- p. 284, reference 5i): Add, Phys. Rev. Rev. 89 (1953), 16.
- p. 284, reference 9): p. 367 should be replaced by p. 1202.

The Formal Theory of Scattering

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The time-dependent theory of scattering is reformulated. In § 1, it is developed on the basis of a new limiting process which is self-consistent, and the equivalence is shown of Heisenberg's S -matrix and Dyson's one, when the total Hamiltonian permits the existence of bound states. In § 2, a theory of scattering of wave packet is proposed in conformity with the physical picture. The damping factor $e^{-\epsilon|\epsilon|}$ is derived from the amplitude of the wave packet. In § 3, the rearrangement scattering is treated on the basis of the wave packet formalism.

Introduction

The time-dependent formulation of scattering has been investigated by many authors. In their treatments, they are always forced to use some limiting procedures such as the adiabatic "switch-off" procedure of an interaction¹⁾²⁾³⁾ and the averaging procedure over initial states.⁴⁾⁵⁾ These theories, however, seem to be somewhat unsatisfactory as will be shown afterwards. On the other hand, Ma⁶⁾ and Belinfante and Møller³⁾ used the concept of the conditional equality. As this concept is not free from mathematical obscurity, we leave this standpoint out of consideration.

Firstly, let us consider Lippmann-Schwinger's theory¹⁾ which is based on the "switch-off". The total Hamiltonian of a system is given by $H=K+V$ in the Schrödinger representation, where K is the free Hamiltonian and V is the interaction part. Let us denote by Φ_a the eigenfunction of K which belongs to the eigenvalue E_a :

$$K\Phi_a = E_a\Phi_a.$$

Dyson's S -matrix is given by

$$S_{fi} = \Phi_f^* U(\infty, -\infty) \Phi_i = \Phi_f^* \Phi_i - i \Phi_f^* V \Psi_i^{(+)}(E_f),$$

where

$$\Psi_i^{(+)}(E) = \int_{-\infty}^{\infty} dt e^{i(E-K)t} e^{-\epsilon|\epsilon|} U_+(t) \Phi_i. \quad (0 \cdot 1)$$

Here Lippmann and Schwinger have used the adiabatic "switch-off" procedure by introducing the factor $e^{-\epsilon|\epsilon|}$. (ϵ is an infinitesimal positive constant.) From the definition of $\Psi_i^{(+)}(E)$ and the integral equation for $U_+(t)$, they derived the equation:

$$\Psi_i^{(+)}(E) = \int_{-\infty}^{\infty} dt e^{i(E-E_i)t} e^{-\epsilon|\epsilon|} \Phi_i - i \int_0^{\infty} d\tau e^{i(E-K)\tau} e^{-\epsilon\tau} V \Psi_i^{(+)}(E). \quad (0 \cdot 2)$$

Integrating this equation, one obtains

$$\Psi_i^{(+)}(E) = 2\pi\delta(E-E_i)\Phi_i + \frac{1}{E-K+i\epsilon}V\Psi_i^{(+)}(E).$$

$$\text{Writing } \Psi_i^{(+)}(E) = 2\pi\delta(E-E_i)\Psi_i^{(+)}, \quad (0.3)$$

one gets the well-known Lippmann-Schwinger's equation :

$$\Psi_i^{(+)} = \Phi_i + \frac{1}{E_i - K + i\epsilon}V\Psi_i^{(+)}. \quad (0.4)$$

As to the above derivation, we have the following questions ;

- (a) $U_+(t)$ in (0.1) is usually denoted as $U(t, t' \rightarrow -\infty)$, the limit of which as $t \rightarrow \pm \infty$ is made definite by the factor $e^{-\epsilon|t|}$, but that of which as $t' \rightarrow -\infty$ seems to be obscure.
 (b) We cannot derive (0.2) from (0.1) by a lucid calculation. If we use the equation

$$U_+(t) = 1 - i \int_{-\infty}^t V(t') U_+(t') dt'$$

which is the equation (1.12) in Lippmann-Schwinger's paper, we have

$$\Psi_i^{(+)}(E) = \int_{-\infty}^{\infty} dt e^{i(E-F_i)t} e^{-\epsilon|t|} \Phi_i - i \int_0^{\infty} d\tau e^{i(E-K)\tau} V \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'+\tau|} e^{i(E-K)t'} U_+(t') \Phi_i$$

instead of (0.2).

In order to clarify the question (a), let us introduce, as usual, the time-dependent Hamiltonian $H = K + e^{-\epsilon|t|}V$ which assures the adiabatic "switch-off" of the interaction.²⁾ From this Hamiltonian, the integral equation for $U(t, -\infty)$ is given by

$$U(t, -\infty) = 1 - i \int_{-\infty}^t dt' e^{-\epsilon|t'|} V(t') U(t', -\infty). \quad (0.5)$$

$U(t, -\infty)$ is defined by the solution of this equation. It has been attempted by many authors to derive Lippmann-Schwinger's equation from (0.5), by using the iteration method. However, the convergence of the series thus derived by iteration will not generally be assured, when the total Hamiltonian permits the existence of bound states.²⁾⁽³⁾⁽⁶⁾⁽⁷⁾⁽⁸⁾ Actually, if we iterate formally (0.5), we arrive at the conclusion that the wave matrix $\Phi_f^\dagger U(0, -\infty) \Phi_i$ is unitary and the eigenfunctions of the total Hamiltonian belonging to the continuous spectrum constitute the complete set without the contributions from bound states. Thus, the validity of the adiabatic "switch-off" procedure comes to be doubtful for the system with bound states.

On the other hand, Gell-Mann and Goldberger's theory¹⁾ which utilizes the averaging procedure need not use the iteration method in order to derive (0.4). In their theory, $U(+\infty, t)$ and $U(t, -\infty)$ are defined respectively by

$$U(t, -\infty) = \lim_{\epsilon \rightarrow 0} \epsilon \int_{-\infty}^0 dT e^{+\epsilon T} U(t, T) \quad (0.6)$$

and

$$U(\infty, t) = \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{\infty} dT e^{-\epsilon T} U(T, t).$$

With these definitions, we can easily derive Lippmann-Schwinger's equation without iteration, and by virtue of the definitions (0.6) it holds that the wave matrix should be non-unitary in a system with bound states. In this respect, the theory based on the averaging procedure is preferable to that of the adiabatic "switch-off." If we consider the S -matrix, however, we encounter the following difficulty. They define the S -matrix in the following ways :

$$S = U(\infty, 0)U(0, -\infty) \quad (0.7)$$

and
$$S = 1 - i \int_{-\infty}^{\infty} dt' V(t') U(t', -\infty), \quad (0.8)$$

where $U(\infty, t)$ and $U(t, -\infty)$ in the right hand sides are defined by (0.6), and the integral in (0.8) with respect to t' is defined by an appropriate limit that will give the usual meaning to the oscillatory integral. They proved the unitarity of the S -matrix by using the definition (0.7), and proved the equivalence between Heisenberg's S -matrix and Dyson's one by the definition (0.8). Then, the consistency of the theory requires the equivalence between the definitions (0.7) and (0.8). We substitute (0.6) into (0.7) to get an equation corresponding to (0.8), and the resultant equation contains an extra term as will be shown in the appendix I. As we cannot show definitely the equivalence between the definitions (0.7) and (0.8), we are forced to conclude that the averaging procedure is not perfect as a limiting process.

It is for these reasons that we shall introduce in § 1 a limiting procedure that will make us free from the above difficulties. Using this limiting process, we can show the equivalence between Heisenberg's S -matrix and Dyson's one for a system with bound states.

Now, in the preceding discussions, all the limiting procedures are rather mathematical than physical. In fact, in practical experiments the incident waves must be considered as wave packets rather than plane waves. In the infinite past, the wave packet is very far from a target spacially and travels freely. At some instant within some finite time-interval, the wave packet begins to interact with the target, and then the scattered wave packet will travel freely after the interaction. In § 2, we shall construct a time-dependent scattering theory on the basis of such a physical picture.⁸⁾⁹⁾¹⁰⁾ In this treatment, the damping factor $e^{-\epsilon|t|}$ which was introduced in order to make definite the integrals will be derived from the amplitude of the wave packet in a natural way, and we can give a physical meaning to the small positive constant ϵ . We shall be able to show the validity of Lippmann-Schwinger's equation in the limit of the plane wave.

In the last section, we shall treat the scattering process involving bound states with the example of the pick-up process from the standpoint of the wave packet.⁴⁾¹¹⁾¹²⁾

§ 1. The theory based on a new limiting process

The Schrödinger equation is given by

$$i\partial\Psi_*(t)/\partial t = (K+V)\Psi_*(t), \quad (1.1)$$

in which \hbar is taken as unity. In order to transform it to the interaction representation,

let us put

$$\Psi(t) = e^{iKt} \Psi_s(t). \quad (1.2)$$

The new state vector $\Psi(t)$ satisfies

$$i\partial\Psi(t)/\partial t = V(t)\Psi(t), \quad (1.3)$$

where

$$V(t) = e^{iKt} V e^{-iKt}. \quad (1.4)$$

We introduce a unitary operator $U(t, t_0)$ such that

$$\Psi(t) = U(t, t_0) \Psi(t_0). \quad (1.5)$$

$U(t, t_0)$ has two obvious properties :

$$U(t, t) = 1 \quad \text{and} \quad U(t, t_0) = U(t, t') U(t', t_0). \quad (1.6)$$

We can easily derive an explicit formula for $U(t, t_0)$:

$$U(t, t_0) = e^{iKt} e^{-iH(t-t_0)} e^{-iKt_0}. \quad (1.7)$$

Setting t or t_0 equal to zero, we have

$$U(t, 0) = e^{iKt} e^{-iHt} \quad \text{and} \quad U(0, t) = e^{iHt} e^{-iKt}. \quad (1.8)$$

From (1.7), we get the following integral equations :

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' V(t') U(t', t_0), \quad (1.9)$$

and

$$U(t, t_0) = 1 + i \int_t^{t_0} dt' U(t, t') V(t'). \quad (1.10)$$

In the integral equations (1.9) and (1.10), the domains of integration are finite. When either t or t_0 is infinite, we must take care of the limits of the integrals as will be seen from (1.7). We have discussed in the introduction the various methods to define the limits.

Here we adopt the following new procedure. By using (1.9) and (1.10), we take the definitions :

$$U(\pm\infty, t) = 1 - i \lim_{\epsilon \rightarrow 0} \int_t^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') U(t', t), \quad (1.11)$$

and

$$U(t, \mp\infty) = 1 + i \lim_{\epsilon \rightarrow 0} \int_t^{\mp\infty} dt' e^{-\epsilon|t'|} U(t, t') V(t'). \quad (1.12)$$

$U(t, t')$ in the right hand sides of (1.11) and (1.12) is given by the explicit solution (1.7), and the limit $\epsilon \rightarrow 0$ must be taken after all other calculations. Of course, this limiting procedure is different from the ordinary adiabatic "switch-off" procedure. If we differentiate (0.5) with respect to t , we have for finite ϵ

$$i \frac{dU_\epsilon(t, -\infty)}{dt} = e^{-\epsilon|t|} V(t) U_\epsilon(t, -\infty),$$

which means the adiabatic "switch-off" of the interaction. On the other hand, if we differentiate (1.12) with respect to t , we obtain for finite ϵ

$$i \frac{dU_\epsilon(t, -\infty)}{dt} = V(t) U_\epsilon(t, -\infty) + V(t) [e^{-\epsilon|t|} - 1],$$

which does not mean necessarily the "switch-off" of the interaction itself.

Let us now establish the properties of the U -matrices with infinite arguments. It can easily be seen that from (1.11) and (1.12)

$$U(\pm\infty, t) = U(\pm\infty, t') U(t', t), \quad (1.13)$$

and

$$U(t, \pm\infty) = U(t, t') U(t', \pm\infty) \quad (1.14)$$

in the limit as $\epsilon \rightarrow 0$, and by using the unitary character of $U(t, t_0)$,

$$U(\pm\infty, t)^\dagger = U(t, \pm\infty), \quad (1.15)$$

$$U(t, \pm\infty)^\dagger = U(\pm\infty, t), \quad (1.16)$$

where the superscript \dagger means hermite conjugate.

We shall now investigate the equations satisfied by $\Psi_i^{(\pm)}$, which are defined by

$$\Psi_i^{(\pm)} \equiv U(0, \mp\infty) \Phi_i. \quad (1.17)$$

From the definition (1.12), we get

$$\Psi_i^{(\pm)} = U(0, \mp\infty) \Phi_i = \Phi_i + i \lim_{\epsilon \rightarrow 0} \int_0^{\mp\infty} dt' e^{-\epsilon|t'|} U(0, t') V(t') \Phi_i. \quad (1.17)'$$

Using (1.4) and (1.8), we have

$$\begin{aligned} \Psi_i^{(\pm)} &= \Phi_i + i \lim_{\epsilon \rightarrow 0} \int_0^{\mp\infty} dt' e^{-\epsilon|t'|} e^{iHt'} V e^{-iE_i t'} \Phi_i \\ &= \Phi_i + i \lim_{\epsilon \rightarrow 0} \int_0^{\mp\infty} dt' e^{-\epsilon|t'|} e^{i(H-E_i)t'} V \Phi_i \\ &= \Phi_i + \lim_{\epsilon \rightarrow 0} \frac{1}{E_i - H \pm i\epsilon} V \Phi_i. \end{aligned} \quad (1.18)$$

If we apply the identity given by Chew and Goldberger,⁽¹³⁾⁽¹⁴⁾ we obtain the following Lippmann-Schwinger's equations:

$$\Psi_i^{(\pm)} = \Phi_i + \lim_{\epsilon \rightarrow 0} \frac{1}{E_i - K \pm i\epsilon} V \Psi_i^{(\pm)}. \quad (1.19)$$

Obviously, $\Psi_i^{(\pm)}$ are the eigenfunctions of the total Hamiltonian H with the eigenvalue E_i .

We must further investigate the properties of such products as $U(-\infty, 0) U(0, -\infty)$. From (1.17) we get

$$\sum_i \Psi_i^{(\pm)} \phi_i^* = U(0, \mp \infty) \sum_i \phi_i \phi_i^*.$$

In view of the fact that $\sum_i \phi_i \phi_i^* = 1$, we obtain

$$U(0, \mp \infty) = \sum_i \Psi_i^{(\pm)} \cdot \phi_i^*.$$

Similarly, we can see that

$$U(\mp \infty, 0) = U(0, \mp \infty)^\dagger = \sum_i \phi_i \cdot \Psi_i^{(\pm)\dagger}.$$

Thus we have

$$U(0, \mp \infty)^\dagger U(0, \mp \infty) = \sum_{ij} \phi_i \Psi_i^{(\pm)\dagger} \cdot \Psi_j^{(\pm)} \phi_j^* = \sum_i \phi_i \phi_i^* = 1, \quad (1.20)$$

where we have used the normalizable properties of $\Psi_i^{(\pm)}$, which were already shown by Möller,^{1b)} Belinfante-Möller³⁾ and Gell-Mann-Goldberger.¹⁾ Of course, it holds in our theory, too. $U(0, \mp \infty)$ are, however, not unitary for the system with bound states, since

$$\begin{aligned} U(0, \mp \infty) U(0, \mp \infty)^\dagger &= \sum_{ij} \Psi_i^{(\pm)} \phi_i^* \phi_j \Psi_j^{(\pm)\dagger} \\ &= \sum_{i=\text{continuous spectrum}} \Psi_i^{(\pm)} \cdot \Psi_i^{(\pm)\dagger} = 1 - \sum_i \Psi_i \cdot \Psi_i^\dagger, \end{aligned} \quad (1.21)$$

where we have taken account of the contribution from the bound states Ψ_i among the eigenstates of the total Hamiltonian H . If our theory is consistent, the bound states should vanish in the limit as $t \rightarrow \pm \infty$. Then, we shall prove that

$$\lim_{t \rightarrow \pm \infty} \Psi_i(t) = \lim_{t \rightarrow \pm \infty} (U(t, 0) \Psi_i) = U(\pm \infty, 0) \Psi_i = 0,$$

where $\lim_{t \rightarrow \pm \infty}$ means the limit in our sense. From (1.11), (1.4) and (1.8), we have

$$\begin{aligned} \lim_{t \rightarrow \mp \infty} \Psi_i(t) &= U(\mp \infty, 0) \Psi_i = \Psi_i - i \lim_{\epsilon \rightarrow 0} \int_0^{\mp \infty} dt' e^{-\epsilon|t'|} e^{i(K-E_i)t'} V \Psi_i \\ &= \Psi_i + \lim_{\epsilon \rightarrow 0} \frac{1}{K - E_i \mp i\epsilon} V \Psi_i. \end{aligned}$$

By using the equation $(K+V) \Psi_i = E_i \Psi_i$, it becomes

$$= \Psi_i + \lim_{\epsilon \rightarrow 0} \frac{E_i - K}{K - E_i \mp i\epsilon} \Psi_i.$$

In the limit as $\epsilon \rightarrow 0$, it runs

$$= \Psi_i - \Psi_i = 0,$$

by using the assumption that the eigenvalues of K do not overlap with the discrete eigenvalue E_i .

Let us now consider the relation between the S -matrix and the wave matrices. We shall define $U(\pm \infty, \mp \infty)$ in the following ways:

$$U(\pm\infty, \mp\infty) \equiv 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') U(t', \mp\infty), \quad (1.22)$$

by taking the limit $t \rightarrow \pm\infty$ in (1.11), or

$$U'(\pm\infty, \mp\infty) \equiv 1 + i \lim_{\epsilon \rightarrow 0} \int_{\pm\infty}^{\mp\infty} dt' e^{-\epsilon|t'|} U(\pm\infty, t') V(t'), \quad (1.23)$$

by using (1.12), where $U(t, \pm\infty)$ and $U(\pm\infty, t)$ in the right hand sides of (1.22) and (1.23) are given by (1.11) and (1.12) respectively. The equivalence between the definitions (1.22) and (1.23) is not self-evident, so the proof may be given below:

$$\begin{aligned} U'(\pm\infty, \mp\infty) &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} U(\pm\infty, t') V(t') \\ &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} \left[1 - i \lim_{\epsilon' \rightarrow 0} \int_{t'}^{\pm\infty} dt'' e^{-\epsilon'|t''|} V(t'') U(t'', t') \right] V(t') \\ &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') + (i)^2 \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon' \rightarrow 0}} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} \int_{\mp\infty}^{\pm\infty} dt'' e^{-\epsilon'|t''|} \left\{ \frac{\theta(t'' - t')}{\theta(t' - t'')} \right\} \\ &\quad \times V(t'') U(t'', t') V(t'). \end{aligned}$$

Carrying out the transformation $t' \rightleftharpoons t''$ and $\epsilon' \rightleftharpoons \epsilon$ in the third term, we have

$$\begin{aligned} &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') + i^2 \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon' \rightarrow 0}} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} \int_{\mp\infty}^{\pm\infty} dt'' e^{-\epsilon|t''|} \left\{ \frac{\theta(t' - t'')}{\theta(t'' - t')} \right\} \\ &\quad \times V(t') U(t', t'') V(t'') \\ &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') \left[1 - i \lim_{\epsilon' \rightarrow 0} \int_{\mp\infty}^{t'} dt'' e^{-\epsilon'|t''|} U(t', t'') V(t'') \right]. \end{aligned}$$

From (1.12), we get

$$= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') U(t', \mp\infty) = U(\pm\infty, \mp\infty),$$

where we have used the step function $\theta(t)$ defined by

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

We can easily verify the property

$$U(\mp\infty, \pm\infty) = U(\pm\infty, \mp\infty)^\dagger$$

from (1.15), (1.16), (1.22) and (1.23). Furthermore, we can prove the important relation

$$U(\pm\infty, \mp\infty) = U(\pm\infty, t) U(t, \pm\infty), \quad (1.24)$$

as will be mentioned below. This property assures that the definitions of $U(\pm\infty, \mp\infty)$ given by (1.22) and (1.23) are equivalent to those which will be given by the products of (1.11) and (1.12), too. It is this property that could not be assured in Gell-Mann

and Goldberger's theory.

We shall give a proof of (1.24). Because of the fact that

$$\begin{aligned} U(\pm\infty, t)U(t, \mp\infty) &= U(\pm\infty, 0)U(0, t)U(t, \mp\infty) \\ &= U(\pm\infty, 0)U(0, \mp\infty), \end{aligned}$$

let us prove

$$U(\pm\infty, \mp\infty) = U(\pm\infty, 0)U(0, \mp\infty).$$

Now, from (1.22) we have

$$\begin{aligned} U(\pm\infty, \mp\infty) &= 1 - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') U(t', 0) \cdot U(0, \mp\infty) \\ &= 1 - i \lim_{\epsilon \rightarrow 0} \int_0^{\pm\infty} dt' e^{-\epsilon|t'|} V(t') U(t', 0) \cdot U(0, \mp\infty) \\ &\quad - i \lim_{\epsilon \rightarrow 0} \int_{\mp\infty}^0 dt' e^{-\epsilon|t'|} V(t') U(t', 0) \cdot U(0, \mp\infty). \end{aligned}$$

Substituting (1.11) into the above equation, we obtain

$$\begin{aligned} U(\pm\infty, \mp\infty) &= 1 + \{U(\pm\infty, 0) - 1\} U(0, \mp\infty) - \{U(\mp\infty, 0) - 1\} U(0, \mp\infty) \\ &= 1 + U(\pm\infty, 0)U(0, \mp\infty) - U(\mp\infty, 0)U(0, \mp\infty). \end{aligned}$$

By virtue of (1.20), we have

$$= U(\pm\infty, 0)U(0, \mp\infty).$$

The matrix element of the S -matrix may be computed from (1.22) as follows;

$$\begin{aligned} \langle f|S|i\rangle &\equiv \Phi_f^* U(+\infty, -\infty) \Phi_i \\ &= \Phi_f^* \Phi_i - i \Phi_f^* \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} V(t') U(t', 0) \cdot U(0, -\infty) \Phi_i \\ &= \Phi_f^* \Phi_i - i \Phi_f^* \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} e^{iE_f t'} V e^{-iHt'} \cdot U(0, -\infty) \Phi_i \\ &= \Phi_f^* \Phi_i - i \Phi_f^* V \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} e^{-i(E_f - H)t'} \cdot U(0, -\infty) \Phi_i \\ &= \Phi_f^* \Phi_i - i \Phi_f^* V \Psi_i^{(+)}(E_f), \end{aligned} \tag{1.25}$$

where

$$\begin{aligned} \Psi_i^{(+)}(E) &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} e^{i(E - H)t'} \cdot U(0, -\infty) \Phi_i \\ &= 2\pi\delta(E - H) \cdot U(0, -\infty) \Phi_i, \end{aligned} \tag{1.26}$$

This is the equation which corresponds to (0.1) defined by Lippmann and Schwinger. From (1.26) we can see that $\Psi_i^{(+)}(E)$ is the eigenfunction of the total Hamiltonian with the eigenvalue E . In view of the fact that $U(0, -\infty) \Phi_i \equiv \Psi_i^{(+)}$ is the eigenfunction of the total Hamiltonian belonging to the eigenvalue E_i , we can rewrite (1.26) as

$$\Psi_i^{(+)}(E) = 2\pi\delta(E - E_i) \cdot \Psi_i^{(+)} \quad (1.27)$$

Substituting (1.27) into (1.25), we obtain

$$\langle f | S | i \rangle = \Phi_f^* \Phi_i - 2\pi i \delta(E_f - E_i) \Phi_f^* V \Psi_i^{(+)} \quad (1.28)$$

Starting with the definition (1.23), we get another expression:

$$\langle f | S | i \rangle = \Phi_f^* \Phi_i - 2\pi i \delta(E_f - E_i) \Psi_f^{(-)\dagger} V \Phi_i \quad (1.29)$$

Clearly, these S -matrices are equivalent to Heisenberg's S -matrices. Here, we must note that the proof of the equivalence is valid even in the case of the system which permits the existence of the bound states.

Our final task is to prove that the S -matrix is unitary. It is easily seen in the following way. First of all, we see that

$$\begin{aligned} \langle a | S | b \rangle &= \langle a | U(+\infty, 0) U(0, -\infty) | b \rangle = \langle U(+\infty, 0)^\dagger \Phi_a | U(0, -\infty) \Phi_b \rangle \\ &= \langle U(0, +\infty) \Phi_a | U(0, -\infty) \Phi_b \rangle = \langle \Psi_a^{(-)} | \Psi_b^{(+)} \rangle, \end{aligned}$$

and

$$\langle a | S^\dagger | b \rangle = \langle \Psi_a^{(+)} | \Psi_b^{(-)} \rangle.$$

Then, we have

$$\begin{aligned} \sum \langle a | S^\dagger | c \rangle \langle c | S | b \rangle &= \sum_{\substack{c=\text{continuous} \\ \text{spectrum}}} \langle \Psi_a^{(+)} | \Psi_c^{(-)} \rangle \langle \Psi_c^{(-)} | \Psi_b^{(+)} \rangle \\ &= \langle \Psi_a^{(+)} | \Psi_b^{(+)} \rangle - \sum_{\substack{l=\text{discrete} \\ \text{spectrum}}} \langle \Psi_a^{(+)} | \Psi_l \rangle \langle \Psi_l | \Psi_b^{(+)} \rangle = \delta_{ab} - \sum_l \langle \Psi_a^{(+)} | \Psi_l \rangle \langle \Psi_l | \Psi_b^{(+)} \rangle. \end{aligned}$$

The second term vanishes by virtue of the orthogonality between the continuous state and the bound state. Similarly, we can see that

$$\sum_c \langle a | S | c \rangle \langle c | S^\dagger | b \rangle = \delta_{ab}.$$

§ 2. The theory based on the wave packet

So far, we have treated the scattering on the basis of the mathematical limiting process which seems to be unnatural from the physical point of view. This procedure is used for the reason that the incident particle is described as a plane wave and the interaction between the particle and the target does not vanish even at $t = \pm \infty$. Thus, as was mentioned in the introduction, we shall propose a theory based on the standpoint of the wave packet.

i) Preliminary considerations

In order to construct the theory based on the wave packet, let us discuss how to describe it.⁹⁾ The free wave packet $\bar{\Phi}_0(\mathbf{x})$ which spreads around the point $\mathbf{x} = 0$ at $t = 0$ with momentum centered about \mathbf{k}_0 , is described by

$$\begin{aligned}\bar{\varphi}_0(\mathbf{x}) &= \int a(\mathbf{k}-\mathbf{k}_0) e^{i\mathbf{k}\mathbf{x}} d\mathbf{k} = \int a(\mathbf{k}-\mathbf{k}_0) e^{i(\mathbf{k}-\mathbf{k}_0)\mathbf{x}} d\mathbf{k} \cdot e^{i\mathbf{k}_0\mathbf{x}} \\ &= A(\mathbf{x}) e^{i\mathbf{k}_0\mathbf{x}},\end{aligned}\quad (2.1)$$

where

$$A(\mathbf{x}) = \int a(\mathbf{k}-\mathbf{k}_0) e^{i(\mathbf{k}-\mathbf{k}_0)\mathbf{x}} d\mathbf{k} \quad \text{and} \quad A(\mathbf{x}) = A(-\mathbf{x}).$$

$A(\mathbf{x})$ is a function which is localized around $\mathbf{x}=0$, and means the amplitude of the wave packet.

When this wave packet moves under the influence of the free Hamiltonian $K = -\frac{1}{2m}\nabla^2$, it can be expressed as

$$\begin{aligned}e^{-iKt}\bar{\varphi}_0(\mathbf{x}) &= \int a(\mathbf{k}-\mathbf{k}_0) e^{i\mathbf{k}\mathbf{x} - i\mathbf{k}^2\mathbf{x}/2m} d\mathbf{k} \\ &= \int a(\mathbf{Jk}) e^{i\mathbf{Jk}\{\mathbf{x} - (\mathbf{k}_0/m + \mathbf{Jk}/2m)t\}} d\mathbf{Jk} \cdot e^{i(\mathbf{k}_0\mathbf{x} - E_0t)},\end{aligned}\quad (2.2)$$

where $\mathbf{Jk} = \mathbf{k} - \mathbf{k}_0$ and $E_0 = \frac{1}{2m}\mathbf{k}_0^2$. On account of the existence of the factor $\exp\left[-i\frac{\mathbf{Jk}^2}{2m}t\right]$ in the right hand side of (2.2), the breadth of the packet increases as t . At this place, let us assume that $a(\mathbf{Jk})$ has a large value only in the case of $\mathbf{k}_0 \approx \mathbf{Jk}$. From this assumption, we can, for a considerably long time-interval t , neglect the term $\mathbf{Jk}^2/2m \cdot t$ in comparison with the term $\mathbf{Jk} \cdot \frac{\mathbf{k}_0}{m}t$ in the exponent of (2.2). In other words, the form of the wave packet can be held for a long time, if its initial spread is large. (Of course, this argument comes to be precise in the limit of the plane wave by virtue of the fact that $a(\mathbf{Jk})$ tends to $\delta(\mathbf{Jk})$.) In this case, we have

$$\begin{aligned}e^{-iKt}\bar{\varphi}_0(\mathbf{x}) &= \int a(\mathbf{Jk}) e^{i\mathbf{Jk}(\mathbf{x} - \mathbf{v}_0t)} d\mathbf{Jk} \cdot e^{i(\mathbf{k}_0\mathbf{x} - E_0t)} \\ &= A(\mathbf{x} - \mathbf{v}_0t) e^{i(\mathbf{k}_0\mathbf{x} - E_0t)},\end{aligned}\quad (2.3)$$

where $\mathbf{v}_0 = \mathbf{k}_0/m$. Let us now denote by T_c the critical time which is too large to neglect the contribution from the factor $\exp(\mathbf{Jk}^2 \cdot t/2m)$ in spite of the condition $\mathbf{k}_0 \approx \mathbf{Jk}$. T_c can be determined in the following way. $a(\mathbf{Jk})$ is not small only in the case of $1/L > |\mathbf{Jk}| > 0$, where L is the breadth of the packet. Thus, we see that $\mathbf{Jk}^2/2m \cdot t \sim 1/2mL^2 \cdot t$. In order to be able to neglect the contribution from this factor, the condition $t/2mL^2 \ll 1$ must be satisfied. Then, we have $T_c \sim 2mL^2$. The distance over which the wave packet travels during T_c is given by $z_c = |\mathbf{v}_0|T_c$, which is considerably larger than the scale of the usual laboratory. For example, z_c is of the order of 10^4 cm for a wave packet of a proton whose energy is about 1 Mev and whose spread is about 10^{-4} cm.

ii) The theory of scattering

Starting with the Schrödinger representation, the Schrödinger equation is given by

$$i\partial\psi_s(t)/\partial t = (K+V)\psi_s(t),\quad (2.4)$$

where $V=V(\mathbf{x})$ is a potential which falls off rapidly at large distances from $\mathbf{x}=0$ and does not contain velocity.

Now, we consider a wave packet centered around $\mathbf{x}=-\mathbf{v}_0T$ (or $+\mathbf{v}_0T$) at a remote time $-T$ (or T ; $T>0$), where \mathbf{x} is far away from the potential but the distance from $\mathbf{x}=0$ is small in comparison with z_c . Let us give the initial (or final) condition to (2.4) at $t=-T$ (or $+T$) by this wave packet. It can be written from (2.3) by

$$\Psi_s(\mp T) = e^{\pm iKT} \bar{\Phi}_0(\mathbf{x}) = A(\mathbf{x} \pm \mathbf{v}_0T) e^{i(k_0x \pm E_0T)}. \quad (2.5)$$

From (2.4) and (2.5), we have

$$\Psi_s^{(\pm)}(t) = e^{-iH(t \mp T)} \Psi_s(\mp T) = e^{-iH(t \mp T)} e^{\pm iKT} \bar{\Phi}_0(\mathbf{x}).$$

In order to transform it to the interaction representation, let

$$\Psi(t) = e^{iKt} \Psi_s(t).$$

Then, we have

$$\Psi^{(\pm)}(t) = e^{iKt} \Psi_s^{(\pm)}(t) = e^{iKt} e^{-iH(t \mp T)} e^{\pm iKT} \bar{\Phi}_0(\mathbf{x}).$$

At the time $t=\mp T$, we see that

$$\Psi^{(\pm)}(\mp T) = \bar{\Phi}_0(\mathbf{x}).$$

Introducing the operator $U(t, \mp T)$ such as

$$\Psi^{(\pm)}(t) = U(t, \mp T) \Psi^{(\pm)}(\mp T) = U(t, \mp T) \bar{\Phi}_0(\mathbf{x}), \quad (2.6)$$

we get

$$U(t, \mp T) = e^{iKt} e^{-iH(t \pm T)} e^{\pm iKT},$$

$$U(\pm T, t) = 1 - i \int_t^{\pm T} dt' V(t') U(t', t) \quad (2.7)$$

and

$$U(t, \mp T) = 1 + i \int_t^{\mp T} dt' U(t, t') V(t'). \quad (2.8)$$

At first sight, the integrals for $T \rightarrow \infty$ in the right hand sides of (2.7) and (2.8) seem to be indefinite. However, the properties of limits of the integrals in such operator equations as (2.7) and (2.8) should be decided by the behavior of the state to be operated by them. Namely, we have from (2.6)

$$\begin{aligned} \Psi^{(\pm)}(t) &= U(t, \mp T) \Psi^{(\pm)}(\mp T) \\ &= \bar{\Phi}_0(\mathbf{x}) + i \int_t^{\mp T} dt' U(t, t') V(t') \bar{\Phi}_0(\mathbf{x}) \\ &= \bar{\Phi}_0(\mathbf{x}) + ie^{iKt} e^{-iHt} \int_t^{\pm T} dt' e^{iHt'} V(\mathbf{x}) A(\mathbf{x} - \mathbf{v}_0t') e^{i(k_0x - E_0t')}. \end{aligned}$$

Taking the limit $T \rightarrow \infty$ at this place, we can see that the amplitude $A(\mathbf{x} - \mathbf{v}_0t')$ will take the part of the damping factor. We must, however, note that the assumption used in (2.3) comes to be incorrect for $|t'| > T_0$ and the last expression of the above equation for $T \rightarrow \infty$ will be inadequate. However, the product $V(\mathbf{x}) \times$ (the amplitude of the spread

wave packet) for $|t'| > T_c$ vanishes at all \mathbf{x} , since the center of the wave packet lies farther than z_c from the potential in the case of $|t'| > T_c$ and the velocity of expansion of the wave packet is always smaller than that of the center v_0 by virtue of the condition $k_0 \succ \Delta k$. So it is permitted for us to extend T formally to $+\infty$. Thus, we obtain

$$\begin{aligned}\Psi_0^{(\pm)}(t) &= U(t, \mp\infty) \bar{\Phi}_0(\mathbf{x}) \\ &= \bar{\Phi}_0(\mathbf{x}) + i e^{iKt} e^{-iHt} \int_t^{\pm\infty} dt' e^{iHt'} V(\mathbf{x}) A(\mathbf{x} - \mathbf{v}_0 t') e^{i(k_0 x - E_0 t')}.\end{aligned}\quad (2.9)$$

Putting $U(0, \mp\infty) \bar{\Phi}_0 \equiv \bar{\Psi}_0^{(\pm)}$, we have

$$\bar{\Psi}_0^{(\pm)} = \bar{\Phi}_0(\mathbf{x}) + i \int_0^{\mp\infty} dt' e^{iHt'} e^{-iE_0 t'} A(\mathbf{x} - \mathbf{v}_0 t') V(\mathbf{x}) e^{iK_0 x}, \quad (2.10)$$

which corresponds to (1.17)'. In order to develop our computation further, we take the simple amplitude $A(\mathbf{x})$ such as

$$\begin{aligned}A(\mathbf{x}) &= e^{-|x|/L} e^{-|y|/L} e^{-|z|/L} \equiv e^{-|x|/L} \\ \text{and} \quad A(\mathbf{x} - \mathbf{v}_0 t) &= e^{-|x|/L} e^{-|y|/L} e^{-|z - v_0 t|} \equiv e^{-|x - v_0 t|/L},\end{aligned}\quad (2.11)$$

where $v_0 = |\mathbf{v}_0|$ and the identities are for the abbreviated symbols. From (2.11), we have the following normalization:

$$\frac{1}{L^3} \int_{-\infty}^{\infty} \bar{\Phi}_0^* \bar{\Phi}_0 d^3x = 1.$$

Substituting (2.11) into (2.10), we obtain

$$\begin{aligned}\bar{\Psi}_0^{(+)} &= \bar{\Phi}_0(\mathbf{x}) - i \int_{-\infty}^0 dt' e^{i(H-E_0)t'} e^{-|x - v_0 t'|/L} V e^{iK_0 x} \\ &= \bar{\Phi}_0(\mathbf{x}) - i \int_{-\infty}^0 dt' e^{i(H-E_0)t'} e^{-|x|/L} e^{-\epsilon|t'|} V e^{iK_0 x} \\ &\quad - i \int_{-\infty}^0 dt' e^{i(H-E_0)t'} [e^{-|x - v_0 t'|/L} - e^{-|x|/L} e^{-\epsilon|t'|}] V e^{iK_0 x} \\ &= \bar{\Phi}_0(\mathbf{x}) - i \int_{-\infty}^0 dt' e^{i(H-E_0)t'} e^{-\epsilon|t'|} V \bar{\Phi}_0(\mathbf{x}) + (\text{the third term}) \\ &= \bar{\Phi}_0(\mathbf{x}) + \frac{1}{E_0 - H + i\epsilon} V \bar{\Phi}_0(\mathbf{x}) + (\text{the third term}),\end{aligned}\quad (2.12)$$

$$\text{where} \quad \epsilon \equiv v_0/L. \quad (2.13)$$

Namely, we see that the constant ϵ is concerned with the properties of the wave packet. From (2.13) the limiting process $1/\epsilon L^3 \rightarrow 0$ which was used by Gell-Mann and Goldberger comes to be self-evident. Now, let us compute the third term. Changing t' to $-t$, we have

$$(\text{the third term}) = -i \int_0^{+\infty} dt e^{-i(H-E_0)t} [e^{-|x + v_0 t|/L} - e^{-|x|/L} e^{-\epsilon|t|}] V(\mathbf{x}) e^{iK_0 x}.$$

(1) In the case of $z \geq 0$, we have

$$e^{-|z+v_0 t|/L} = e^{-\epsilon/L} e^{-\epsilon|t|},$$

since t is always positive. Thus we see that the third term vanishes.

(2) In the case of $z < 0$, we obtain the following result after the straightforward calculation as will be seen in the appendix II:

$$(\text{the third term}) = \frac{-2i\epsilon}{(H-E_0)^2 + \epsilon^2} \left(\text{Exp} \cdot iH \frac{z}{v_0} \cdot e^{-iE_0 z/v_0} - e^{z/L} \right) V(x) e^{ik_0 x}, \quad (2 \cdot 14)$$

where

$$\text{Exp}(ia \cdot b) = \sum_{n=0}^{\infty} (+i)^n / n! \cdot a^n \cdot b^n$$

in contrast with

$$e^{ia \cdot b} = \sum_{n=0}^{\infty} (+i)^n / n! \cdot (a \cdot b)^n.$$

Substituting (2.14) into (2.12), we have

$$\begin{aligned} \bar{\Psi}_0^{(+)} = \bar{\Phi}_0(x) + \frac{1}{E_0 - H + i\epsilon} V \bar{\Phi}_0(x) - \theta(-z) 2i \frac{\epsilon}{(H-E_0)^2 + \epsilon^2} \\ \times \{ \text{Exp}(iH \cdot z/v_0) e^{-iE_0 z/v_0} - e^{z/L} \} V e^{ik_0 x}. \end{aligned}$$

In the limit as $L \rightarrow \infty$ ($\epsilon \rightarrow 0$), it runs

$$\begin{aligned} \bar{\Psi}_0^{(+)} = \bar{\Phi}_0(x) + \lim_{\epsilon \rightarrow 0} \frac{1}{E_0 - H + i\epsilon} V \bar{\Phi}_0(x) - \theta(-z) 2\pi i \cdot \delta(H-E_0) \\ \times \{ \text{Exp}(iH \cdot z/v_0) e^{-iE_0 z/v_0} - 1 \} V \bar{\Phi}_0(x), \end{aligned}$$

where $\lim_{L \rightarrow \infty} \bar{\Psi}_0^{(+)} = \bar{\Psi}_0^{(+)}$ and $\lim_{L \rightarrow \infty} \bar{\Phi}_0(x) = e^{ik_0 x} = \Phi_0(x)$.

At this place, we shall investigate the third term of the above equation:

$$\begin{aligned} 2\pi i \delta(H-E_0) \{ \text{Exp}(iH \cdot z/v_0) e^{-iE_0 z/v_0} - 1 \} V \Phi_0(x) \\ = 2\pi i \delta(H-E_0) \sum_{n=0}^{\infty} \frac{i^n}{n!} H^n \left\{ \left(\frac{z}{v_0} \right)^n e^{-iE_0 z/v_0} V \Phi_0(x) \right\} - 2\pi i \delta(H-E_0) (H-E_0) \Phi_0(x), \end{aligned} \quad (2 \cdot 15)$$

where we have used the equation

$$(H-V) \Phi_0(x) = E_0 \Phi_0(x).$$

Using the property of δ -function, we see that the second term vanishes. Furthermore, because of the fact that

$$H^n \delta(H-E_0) f(x) = E_0^n \delta(H-E_0) f(x)$$

for an arbitrary function $f(x)$, (2.15) can be written as follows:

$$(2 \cdot 15) = 2\pi i \delta(H-E_0) \sum_{n=0}^{\infty} \frac{i^n}{n!} E_0^n \{ (z/v_0)^n e^{-iE_0 z/v_0} V \Phi_0(x) \}$$

$$\begin{aligned}
 &= 2\pi i \delta(H - E_0) e^{+iE_0 z/v_0} e^{-iE_0 z/v_0} V \phi_0(\mathbf{x}) \\
 &= 2\pi i \delta(H - E_0) (H - E) \phi_0(\mathbf{x}) = 0.
 \end{aligned}$$

Therefore, we have succeeded in deriving Lippmann-Schwinger's equation from the standpoint of the wave packet:

$$\psi_0^{(+)} = \phi_0 + \lim_{\epsilon \rightarrow 0} \frac{1}{E_0 - K + i\epsilon} V(\mathbf{x}) \psi_0^{(+)} \quad (2.16)$$

Similarly, we have the following equation for $\bar{\psi}_0^{(-)} \equiv U(0, +\infty) \bar{\phi}_0(\mathbf{x})$:

$$\psi_0^{(-)} = \phi_0 + \lim_{\epsilon \rightarrow 0} \frac{1}{E_0 - K - i\epsilon} V(\mathbf{x}) \psi_0^{(-)} \quad (2.17)$$

So far, we have used such a particular amplitude as (2.11). However, we can generalize slightly the type of the amplitude in the following way;

$$A(\mathbf{x}) = A(x) A(y) A(z)$$

and
$$A(x) = \int_{-\infty}^{\infty} b(\lambda; L) e^{-|\lambda||x|} d\lambda, \quad \text{etc.},$$

where $b(\lambda; L)$ is an arbitrary function which makes the above integral converge, and which satisfies the following condition,

$$\lim_{L \rightarrow \infty} b(\lambda; L) = \delta(\lambda).$$

By using this generalized amplitude, (2.16) and (2.17) can be derived in the same way as before.

From (2.16) and (2.17), we can easily verify the relations

$$U(\mp\infty, 0) U(0, \mp\infty) = 1 \quad (2.18)$$

and
$$U(0, \mp\infty) U(\mp\infty, 0) \neq 1.$$

In the next place, we consider the S -matrix. Here we define $U(\pm\infty, \mp\infty)$ as follows;

$$U(\pm\infty, \mp\infty) = 1 - i \int_{\mp\infty}^{\pm\infty} dt' V(t') U(t', \mp\infty) \quad (2.19)$$

and
$$U(\pm\infty, \mp\infty) = 1 + i \int_{\pm\infty}^{\mp\infty} dt' U(\pm\infty, t') V(t').$$

In this case, it is apparent that the two definitions of $U(\pm\infty, \mp\infty)$ are equivalent. Furthermore, we can easily see that

$$U(\pm\infty, \mp\infty) = U(\pm\infty, t) U(t, \mp\infty). \quad (2.20)$$

Let us now compute an element of the S -matrix:

$$\langle f | S | i \rangle \equiv \lim_{\epsilon_f, \epsilon_i \rightarrow 0} \langle \phi_f | U(+\infty, -\infty) | \bar{\phi}_i \rangle.$$

Using (2.19) and (2.20), we have

$$\begin{aligned}
 &= \lim_{\epsilon_f, \epsilon_i \rightarrow 0} \langle \bar{\Phi}_f | 1 - i \int_{-\infty}^{\infty} dt' V(t') U(t', 0) \cdot U(0, -\infty) | \bar{\Phi}_i \rangle \\
 &= \langle \Phi_f | \Phi_i \rangle - i \lim_{\epsilon_f, \epsilon_i \rightarrow 0} \langle \bar{\Phi}_f | \int_{-\infty}^{\infty} e^{iKt'} V e^{-iHt'} dt' | \bar{\Psi}_i^{(+)} \rangle \\
 &= \langle \Phi_f | \Phi_i \rangle - i \lim_{\epsilon_f, \epsilon_i \rightarrow 0} \langle \int_{-\infty}^{\infty} dt' e^{iHt'} V e^{-iKt'} \bar{\Phi}_f | \bar{\Psi}_i^{(+)} \rangle \\
 &= \langle \Phi_f | \Phi_i \rangle - i \lim_{\epsilon_f, \epsilon_i \rightarrow 0} \langle \int_{-\infty}^{\infty} dt' e^{i(H-E_f)t'} A(\mathbf{x} - \mathbf{v}_f t') V e^{iK_0 x} | \bar{\Psi}_i^{(+)} \rangle.
 \end{aligned}$$

By the similar calculation to (2.12), it leads to

$$\begin{aligned}
 &= \langle \Phi_f | \Phi_i \rangle - 2\pi i \langle \Phi_f | V \delta(E_f - H) | \Psi_i^{(+)} \rangle \\
 &= \langle \Phi_f | \Phi_i \rangle - 2\pi i \delta(E_f - E_i) \langle \Phi_f | V | \Psi_i^{(+)} \rangle,
 \end{aligned}$$

where we have used the fact that $\Psi_i^{(+)}$ is an eigenfunction of H . Then, we can see that Dyson's S -matrix is equivalent to Heisenberg's one. It is easily seen that the S -matrix is unitary.

§ 3. The scattering involving bound states²⁾¹¹⁾¹²⁾¹⁶⁾

In this section, we shall treat the scattering involving bound states on the basis of the wave packet. As an example of our formalism, we shall present a rigorous theory of the so-called "pick-up" process. It is easy to extend our formalism to other cases.

Now, we consider the following problem: A proton is bound to a fixed nucleus by a potential V_P , and is bombarded by a neutron which interacts with the proton and the nucleus through the potentials V_{PN} and V_N respectively. The neutron and the proton may be bound together by V_{PN} to form a deuteron, and we wish to compute the transition probability per unit time for producing deuterons.

As it is necessary to extend the physical interpretation of Dyson's S -matrix in order to compute such a probability, we shall review the ordinary interpretation of the S -matrix: The transition probability per unit time is obtained from

$$\langle b | S | a \rangle = \Phi_b^* U(+\infty, -\infty) \Phi_a, \quad (3.1)$$

where Φ_a and Φ_b are the eigenstates of the free Hamiltonian. The interpretation of (3.1) is given as follows: At the time $t = -\infty$, an incident particle and a target are very far from each other, and there is no interaction between them. So we shall be able to specify the initial state with the eigenstate of the free Hamiltonian Φ_a . As the lapse of time, the particle will be scattered under the influence of the interaction ($U(+\infty, -\infty)$), and it will be released from the interaction at the time $t = +\infty$. This state $U(+\infty, -\infty) \Phi_a$ will be described by a superposition of the eigenstates of the free Hamiltonian. Then, (3.1) gives the probability amplitude with which we shall find a state Φ_b in $U(+\infty, -\infty) \Phi_a$.

There is an alternative interpretation for the S -matrix. Rewriting (3.1), we have

$$\begin{aligned}\langle b|S|a\rangle &= \Phi_b^* U(\infty, 0) U(0, -\infty) \Phi_a = (U(0, +\infty) \Phi_b)^\dagger \cdot U(0, -\infty) \Phi_a \\ &= \mathcal{P}_b^{(-)\dagger} \cdot \mathcal{P}_a^{(+)}. \quad (3.2)\end{aligned}$$

The rewritten S -matrix (3.2) can be interpreted in the following way: The state $\mathcal{P}_b^{(-)}$ at $t=0$ is a state which will become Φ_b at the infinite future. On the other hand, the state Φ_a at the time $t=-\infty$ grows up to the state $\mathcal{P}_a^{(+)} = U(0, -\infty) \Phi_a$ at $t=0$. Then, (3.2) gives the probability amplitude with which we shall find a state $\mathcal{P}_b^{(-)}$ in the state $\mathcal{P}_a^{(+)}$ at the time $t=0$, where we must note that $\mathcal{P}_a^{(+)}$ and $\mathcal{P}_b^{(-)}$ are the eigenstates of the same total Hamiltonian.

Now, let us define the S -matrix by the form of (3.2), even if the Hamiltonian H_0 in the remote past is different from the Hamiltonian H_0' in the infinite future. Namely, the S -matrix is defined by

$$S_{ba} = (U'(0, +\infty) \Phi_b')^\dagger \cdot (U(0, -\infty) \Phi_a) = \mathcal{P}_b'^{(-)\dagger} \cdot \mathcal{P}_a^{(+)}, \quad (3.3)$$

where Φ_b' and Φ_a are the eigenstates of H_0' and H_0 respectively and $\mathcal{P}_b'^{(-)}$ and $\mathcal{P}_a^{(+)}$ are those of the same total Hamiltonian \mathbf{H} . (This definition (3.3) can be derived readily from the standpoint of § 1.) Using the definition (3.3), we shall compute an element of the S -matrix for the pick-up process.

Starting with the Schrödinger representation, the wave equation of the total system is given by

$$i \partial \Psi_s(t) / \partial t = \mathbf{H} \Psi_s(t), \quad (3.4)$$

where $\mathbf{H} = K + V_P + V_{PN} + V_N$ and $K = K_P + K_N$. K_P and K_N are the free Hamiltonian of the proton and the neutron respectively. We consider now a wave packet of the neutron which is centered around $x=0$, $y=0$ and $z=-v_N T$ at the time $t=-T$, where $T \ll T_c$. At this time T , the interactions V_{PN} and V_N vanish effectively in the same way as in § 2. Then, this wave packet will be composed of the eigenstates Φ_i 's of Hamiltonian $H_0 = K + V_P$ in the following way: The eigenstate Φ_i of $K + V_P$ satisfies

$$(K_N + K_P + V_P) \Phi_i = E_i \Phi_i,$$

so we can write it in the form

$$\Phi_i = e^{i k_N x_N} \varphi_a(\mathbf{x}_P), \quad (3.5)$$

where $E_i = E(k_N) + E_a$ and $\varphi_a(\mathbf{x}_P)$ is the eigenfunction of the bound proton. Using (3.5), we can describe the wave packet which is the initial condition for (3.4) at the time $t=-T$ by

$$\begin{aligned}\Psi_s(-T) &= e^{i(K+V_P)T} \bar{\Phi}_i(\mathbf{x}) = e^{i(K+V_P)T} A(\mathbf{x}_N) \Phi_i(\mathbf{x}) \\ &= A(\mathbf{x}_N + \mathbf{v}_N T) \varphi_a(\mathbf{x}_P) e^{i(k_N x_N + E_i T)},\end{aligned} \quad (3.6)$$

where $\mathbf{x} = (\mathbf{x}_N, \mathbf{x}_P)$. At the time t , the wave packet may be given by

$$\Psi_s(t) = e^{-i\mathbf{H}(t+T)} \Psi_s(-T) = e^{-i\mathbf{H}(t+T)} e^{i(K+V_P)T} \bar{\Phi}_i(\mathbf{x}). \quad (3.7)$$

Now, let us introduce the interaction representation for $0 > t > -T$ as follows ;

$$\Psi(t) = e^{i(K+V_P)t} \Psi_s(t).$$

From (3.6) and (3.7), we have

$$\Psi(t) = e^{+i(K+V_P)t} e^{-iH(t+T)} e^{+i(K+V_P)T} \bar{\Phi}_i(\mathbf{x}) \quad (3.8)$$

and

$$\Psi(-T) = e^{+i(K+V_P)T} \Psi_s(-T) = \bar{\Phi}_i(\mathbf{x}). \quad (3.9)$$

Writing

$$\Psi(t) = U(t, -T) \Psi(-T),$$

we obtain

$$U(t, -T) = e^{i(K+V_P)t} e^{-iH(t+T)} e^{+i(K+V_P)T}. \quad (3.10)$$

If we differentiate (3.10) with respect to $-T$ and then integrate, we have

$$U(t, -T) = 1 + i \int_t^{-T} dt' U(t, t') [V_{PN}(t') + V_N(t')], \quad (3.11)$$

where

$$V_{PN} = e^{i(K+V_P)t} V e^{-i(K+V_P)t} \quad \text{and} \quad V_N(t) = e^{i(K+V_P)t} V_N e^{-i(K+V_P)t}.$$

From (3.8), (3.10) and (3.11), we get

$$\Psi(t) = U(t, -T) \bar{\Phi}_i = \Phi_i + i \int_t^{-T} dt' U(t, t') [V_{PN}(t') + V_N(t')] \bar{\Phi}_i.$$

As a result of the discussion in § 2, we can put $T \rightarrow +\infty$ in this equation and get the following important equation :

$$\bar{\Psi}_i^{(+)} \equiv U(0, -\infty) \bar{\Phi}_i = \bar{\Phi}_i + i \int_0^{-\infty} dt' U(0, t') [V_{PN}(t') + V_N(t')] \bar{\Phi}_i. \quad (3.12)$$

In the next place, we consider a wave packet of the deuteron concerning the motion of its center of mass for the time $t > 0$. At a time T' where $T_{c.c.} T' \rightarrow 0$, the interactions V_P and V_N vanish. The wave packet can be described as follows : The eigenstate Φ_f' of $H_0' = K + V_{PN}$ satisfies the equation

$$(K + V_{PN}) \Phi_f' = E_f \Phi_f',$$

and is expressed as

$$\Phi_f'(X, \mathbf{r}) = e^{iK_f \cdot X} \varphi_b(\mathbf{r})$$

where $\mathbf{r} = \mathbf{x}_P - \mathbf{x}_N$, $X = \frac{1}{2}(\mathbf{x}_P + \mathbf{x}_N)$ and $E_f = E(K_f) + E_b$. Thus, we can describe the wave packet at the time $t = +T'$ by

$$\Psi_s(T') = e^{-i(K+V_{PN})T'} \bar{\Phi}_f'(X, \mathbf{r}) = A(X - \mathbf{v}_f T') \varphi_b(\mathbf{r}) e^{i(K_f X - E_f T')}. \quad (3.13)$$

In the same way as before, we can derive the following equation :

$$\bar{\Psi}_f'^{(-)} \equiv U'(0, +\infty) \Phi_f' = \bar{\Phi}_f' + i \int_0^{\infty} dt U(0, t) (V_P'(t) + V_N'(t)) \bar{\Phi}_f', \quad (3.14)$$

where

$$U'(t, T') = e^{i(K+V_{PN})t} e^{-iH(t-T')} e^{-i(K+V_{PN})T'},$$

$$V'_P(t) = e^{i(K+V_{PN})t} V_P e^{-i(K+V_{PN})t} \quad \text{and} \quad V'_N(t) = e^{i(K+V_{PN})t} V_N e^{-i(K+V_{PN})t}.$$

From (3.12) and (3.14), we obtain Lippmann-Schwinger's equations in the same way as in § 2 in the limit as $\epsilon_i \equiv v_i/L_i \rightarrow 0$ and $\epsilon_f \equiv v_f/L_f \rightarrow 0$ respectively :

$$\Psi_i^{(+)} = \Phi_i + \lim_{\epsilon_i \rightarrow 0} \frac{1}{E_i - (K + V_P) + i\epsilon_i} (V_{PN} + V_N) \Psi_i^{(+)}, \quad (3.15)$$

and

$$\Psi_f^{(-)} = \Phi_f' + \lim_{\epsilon_f \rightarrow 0} \frac{1}{E_f - (K + V_{PN}) - i\epsilon_f} (V_P + V_N) \Psi_f^{(-)}, \quad (3.16)$$

where we can easily see that both $\Psi_i^{(+)}$ and $\Psi_f^{(-)}$ are the eigenstates of the total Hamiltonian \mathbf{H} .

From the definition (3.3), we can compute the transition amplitude for producing deuterons in the following way :

$$\begin{aligned} S_{fi} &= \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \langle U'(0, +\infty) \bar{\Phi}_f' | U(0, -\infty) \bar{\Phi}_i \rangle \\ &= \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \langle U'(0, +\infty) \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \rangle, \end{aligned} \quad (3.17)$$

From (3.14), we have

$$\begin{aligned} &= \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \langle \bar{\Phi}_f' + i \int_0^\infty dt U'(0, t) (V_P'(t) + V_N'(t)) \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \rangle \\ &= \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \{ \langle \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \rangle - i \langle \int_0^\infty dt e^{i\mathbf{H}t} (V_P + V_N) e^{-i(K+V_{PN})t} \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \rangle \} \\ &= \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \{ \langle \bar{\Phi}_f' | \bar{\Psi}_i \rangle - i \langle \int_0^\infty dt e^{i(\mathbf{H}-E_f)t} A(\mathbf{X}-\mathbf{v}_f t) (V_P + V_N) \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \rangle \}. \end{aligned}$$

After the same calculation as (2.12), we obtain

$$\begin{aligned} &= \langle \Phi_f' | \Psi_i^{(+)} \rangle + \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \left\langle \frac{1}{E_f - \mathbf{H} - i\epsilon_f} \cdot (V_P + V_N) \bar{\Phi}_f' | \bar{\Psi}_i^{(+)} \right\rangle \\ &= \langle \Phi_f' | \Psi_i^{(+)} \rangle + \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \left\langle \bar{\Phi}_f' | (V_P + V_N) \frac{1}{E_f - \mathbf{H} + i\epsilon_f} | \bar{\Psi}_i^{(+)} \right\rangle \\ &= \langle \Phi_f' | \Psi_i^{(+)} \rangle + \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \left(\frac{1}{E_f - E_i + i\epsilon_f} \right) \langle \Phi_f' | V_P + V_N | \Psi_i^{(+)} \rangle. \end{aligned} \quad (3.18)$$

Now, using (3.15), we can compute the first term :

$$\langle \Phi_f' | \Psi_i^{(+)} \rangle = \lim_{\epsilon_i \rightarrow 0} \left\langle \Phi_f' | \Phi_i + \frac{1}{E_i - \mathbf{H} + i\epsilon_i} (V_{PN} + V_N) \Phi_i \right\rangle$$

$$\begin{aligned}
&= \lim_{\epsilon_i \rightarrow 0} \left\langle \Phi'_f | \Phi_i + \frac{1}{E_i - (K + V_{PN}) + i\epsilon_i} (V_{PN} + V_N) \Phi_i \right. \\
&\quad \left. + \frac{1}{E_i - (K + V_{PN}) + i\epsilon_i} (V_P + V_N) \frac{1}{E_i - H + i\epsilon_i} (V_{PN} + V_N) \Phi_i \right\rangle \\
&= \lim_{\epsilon_i \rightarrow 0} \left\{ \langle \Phi'_f | \Phi_i \rangle + \frac{1}{E_i - E_f + i\epsilon_i} \langle \Phi'_f | V_{PN} + V_N | \Phi_i \rangle \right. \\
&\quad \left. + \frac{1}{E_i - E_f + i\epsilon_i} \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} - \Phi_i \rangle \right\} \\
&= \langle \Phi'_f | \Phi_i \rangle + \lim_{\epsilon_i \rightarrow 0} \frac{1}{E_i - E_f + i\epsilon_i} \{ \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} \rangle + \langle \Phi'_f | V_{PN} - V_P | \Phi_i \rangle \}.
\end{aligned} \tag{3.19}$$

In view of $(K + V_P) \Phi_i = E_i \Phi_i$ and $(K + V_{PN}) \Phi'_f = E_f \Phi'_f$, we find that

$$\langle \Phi'_f | V_{PN} - V_P | \Phi_i \rangle = (E_f - E_i) \langle \Phi'_f | \Phi_i \rangle. \tag{3.20}$$

Substituting (3.20) into (3.19), we have

$$\begin{aligned}
\langle \Phi'_f | \Psi_i^{(+)} \rangle &= \langle \Phi'_f | \Phi_i \rangle + \lim_{\epsilon_i \rightarrow 0} \frac{E_f - E_i}{E_i - E_f + i\epsilon_i} \langle \Phi'_f | \Phi_i \rangle \\
&\quad + \lim_{\epsilon_i \rightarrow 0} \frac{1}{E_i - E_f + i\epsilon_i} \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} \rangle.
\end{aligned} \tag{3.21}$$

Let us consider the second term. From the fact that

$$\frac{E_f - E_i}{E_i - E_f + i\epsilon_i} = -1 + \frac{i\epsilon_i}{E_i - E_f + i\epsilon_i},$$

we see that

$$\text{the second term of (3.21)} = \begin{cases} -\langle \Phi'_f | \Phi_i \rangle & \text{for } E_i \neq E_f, \\ 0 & E_i = E_f. \end{cases}$$

In the same calculation, however, Gell-Mann and Goldberger⁴⁾ say that

$$\text{the second term} = -\langle \Phi'_f | \Phi_i \rangle \text{ for all } E_i \text{ and } E_f,$$

which is apparently incorrect in the case of $E_i = E_f$. Therefore, we have

$$\langle \Phi'_f | \Psi_i^{(+)} \rangle = \langle \Phi'_f | \Phi_i \rangle_{E_i = E_f} + \lim_{\epsilon_i \rightarrow 0} \frac{1}{E_i - E_f + i\epsilon_i} \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} \rangle. \tag{3.22}$$

Substituting (3.22) into (3.18), we have

$$\begin{aligned}
S_{fi} &= \langle \Phi'_f | \Phi_i \rangle_{E_i = E_f} + \lim_{\epsilon_i \rightarrow 0} \left(\frac{1}{E_i - E_f + i\epsilon_i} + \frac{1}{E_f - E_i + i\epsilon_f} \right) \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} \rangle \\
&= \langle \Phi'_f | \Phi_i \rangle_{E_i = E_f} - 2\pi i \delta(E_f - E_i) \langle \Phi'_f | V_P + V_N | \Psi_i^{(+)} \rangle.
\end{aligned} \tag{3.23}$$

If we start with

$$S_{fi} = \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_f \rightarrow 0}} \langle \psi_f'^{(-)} | U(0, -\infty) \bar{\psi}_i \rangle$$

in contract with (3.17), we obtain

$$S_{fi} = \langle \psi_f' | \psi_i \rangle_{E_f=E_i} - 2\pi i \delta(E_f - E_i) \langle \psi_f'^{(-)} | V_{PN} + V_N | \psi_i \rangle. \quad (3.24)$$

Comparing (3.24) with (3.23), we can see the reciprocity character.

It will be difficult to investigate the properties of wave matrices $U'(+\infty, 0)$ and $U(0, -\infty)$, corresponding to (2.18).

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Appendix I.

Inconsistency between (0.7) and (0.8)

We shall compute (0.7) and show that the resultant equation contains an extra term. From (0.6) and (0.7), we have

$$\begin{aligned} U(\infty, -\infty) &= U(\infty, 0)U(0, -\infty) \\ &= \epsilon \epsilon' \int_0^\infty dT \int_{-\infty}^0 dT' e^{-\epsilon T} e^{+\epsilon' T'} U(T, T') \\ &= \epsilon \epsilon' \int_0^\infty dT e^{-\epsilon T} \int_{-\infty}^0 dT' e^{+\epsilon' T'} \cdot (1 - i \int_{T'}^T dt V(t) U(t, T')) \\ &= 1 - i \epsilon \epsilon' \int_0^\infty dT e^{-\epsilon T} \int_{-\infty}^0 dT' e^{+\epsilon' T'} \int_{-\infty}^\infty dt \theta(T-t) \theta(t-T') V(t) U(t, T'). \end{aligned}$$

Exchanging the sequence of integration, we see that

$$= 1 - i \epsilon \epsilon' \int_{-\infty}^\infty dt V(t) \int_0^\infty dT e^{-\epsilon T} \theta(T-t) \int_{-\infty}^0 dT' e^{+\epsilon' T'} \theta(t-T') U(t, T').$$

From the fact that

$$\epsilon \int_0^\infty dT e^{-\epsilon T} \theta(T-t) = \begin{cases} e^{-\epsilon t} & \text{for } t > 0, \\ 1 & \text{for } t < 0, \end{cases}$$

we get

$$\begin{aligned} U(\infty, -\infty) &= 1 - i \epsilon \epsilon' \int_0^\infty dt V(t) e^{-\epsilon t} \int_{-\infty}^0 dT' e^{+\epsilon' T'} \theta(t-T') U(t, T') \\ &\quad - i \epsilon' \int_{-\infty}^0 dt V(t) \int_{-\infty}^0 e^{\epsilon' T'} \theta(t-T') U(t, T') dT'. \end{aligned}$$

Since t is always positive in the second term, it runs

$$\begin{aligned}
 &= 1 - i\epsilon' \int_0^\infty dt V(t) e^{-\epsilon' t} \int_{-\infty}^0 dT' e^{\epsilon' T'} U(t, T') - i\epsilon' \int_{-\infty}^0 dt V(t) \int_{-\infty}^t e^{\epsilon' T'} U(t, T') dT' \\
 &= 1 - i \int_{-\infty}^\infty e^{-\epsilon' |t|} V(t) U(t, -\infty) dt \\
 &\quad - i \int_{-\infty}^0 dt (1 - e^{\epsilon' t}) V(t) \epsilon' \int_{-\infty}^0 e^{\epsilon' T'} U(t, T') dT' - i \int_0^\infty dt V(t) \epsilon' \int_0^t e^{\epsilon' T'} U(t, T') dT'.
 \end{aligned}$$

Thus, it is necessary that the extra terms should vanish definitely. Substituting (1.4) and (1.7) in them, we have

$$\begin{aligned}
 \text{the extra terms} &= -i \int_{-\infty}^0 dt (1 - e^{\epsilon' t}) e^{iKt} V e^{-iHt} \epsilon' \int_{-\infty}^0 e^{\epsilon' T'} e^{iHT'} e^{-iKT'} dT' \\
 &\quad - i \int_{-\infty}^0 dt e^{iKt} V e^{-iHt} \epsilon' \int_0^t e^{\epsilon' T'} e^{iHT'} e^{-iKT'} dT'. \quad (\text{A} \cdot 1)
 \end{aligned}$$

In order to carry out the integration with respect to T' , let us multiply (A.1) by Φ_f^* and Φ_i from the respective sides. Then we have

$$\begin{aligned}
 \langle \Phi_f | \text{the extra terms} | \Phi_i \rangle &= \Phi_f^* V \int_{-\infty}^0 dt [e^{\epsilon' t} e^{i(E_f - H)t} - e^{\epsilon' t} e^{i(E_f - E_i)t}] \frac{\epsilon'}{H - E_i - i\epsilon'} \Phi_i \\
 &= -\frac{1}{i} \frac{\epsilon'}{E_f - E_i - i\epsilon'} \Phi_f^* V \{H - E_i - i(\epsilon - \epsilon')\} \frac{1}{H - E_f + i\epsilon} \frac{1}{H - E_i - i\epsilon'} \Phi_i.
 \end{aligned}$$

Using the equation $(H - V)\Phi_f = E_f\Phi_f$, we get

$$= -\frac{1}{i} \frac{\epsilon'}{E_f - E_i - i\epsilon'} \Phi_f^* (H - E_f) (H - E_i - i(\epsilon' - \epsilon)) \frac{1}{H - E_f + i\epsilon} \frac{1}{H - E_i - i\epsilon'} \Phi_i.$$

If we take the limit $\epsilon' \rightarrow 0$ after $\epsilon \rightarrow 0$, we have

$$= -\langle \Phi_f - \Psi_f^{(+)} | \Phi_i \rangle_{E_i = E_f}.$$

On the other hand, if we take the limit $\epsilon = \epsilon' \rightarrow 0$, we obtain

$$= -\langle \Phi_f - \Psi_f^{(+)} | \Phi_i - \Psi_i^{(+)} \rangle_{E_i = E_f}.$$

Appendix II.

Calculation of (2.14)

Let us prove (2.14). At first, we consider the term

$$-i \int_0^\infty dt e^{-i(H - E_0)t} e^{-1/L_* |z + \mathbf{r}_0 t|} V(\mathbf{x}) e^{iK_0 x}.$$

Expanding this in the power series of H , we have

$$= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} H^n \int_0^\infty dt t^n e^{iE_0 t} e^{-1/L_* |z + \mathbf{r}_0 t|} V(\mathbf{x}) e^{iK_0 x}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} H^n \left[\int_{-z/v_0}^{\infty} dt t^n e^{i(E_0 + i\epsilon)t} e^{-z/L} + \int_0^{-z/v_0} dt t^n e^{i(E_0 - i\epsilon)t} e^{z/L} \right] V(x) e^{ik_0 x} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} H^n \left\{ \left[\frac{e^{iE_0 z/L} e^{-\epsilon z}}{i(E_0 + i\epsilon)} \left(t^n + n \frac{1}{-i(E_0 + i\epsilon)} t^{n-1} \right. \right. \right. \\
&\quad \left. \left. \left. + n(n-1) \frac{1}{(-i)^2 (E_0 + i\epsilon)^2} t^{n-2} + \dots \right) \right] e^{-z/v_0} \right. \\
&\quad \left. + \left[\frac{e^{iE_0 z/L} e^{\epsilon z}}{i(E_0 - i\epsilon)} \left(t^n + n \frac{1}{-i(E_0 - i\epsilon)} t^{n-1} + \dots + \frac{n!}{(-i)^n (E_0 - i\epsilon)^n} \right) \right] e^{z/v_0} \right\} V(x) e^{ik_0 x} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} H^n \left\{ \left[-\frac{e^{iE_0 z/v_0} e^{z/L}}{i(E_0 + i\epsilon)} \left(\left(-\frac{z}{v_0} \right)^n + n \frac{1}{-i(E_0 + i\epsilon)} \left(-\frac{z}{v_0} \right)^{n-1} + \dots \right) e^{-z/L} \right. \right. \\
&\quad \left. \left. + \frac{e^{-iE_0 z/v_0} e^{-z/L}}{i(E_0 - i\epsilon)} \left(\left(-\frac{z}{v_0} \right)^n + n \frac{1}{-i(E_0 - i\epsilon)} \left(-\frac{z}{v_0} \right)^{n-1} + \dots \right) e^{z/L} \right. \right. \\
&\quad \left. \left. - \frac{1}{i(E_0 - i\epsilon)} \frac{n!}{(-i)^n (E_0 - i\epsilon)^n} e^{z/L} \right] V(x) e^{ik_0 x} \right\}. \quad (\text{A} \cdot 2)
\end{aligned}$$

Here, we shall compute the first and second terms of (A·2) as follows :

$$\begin{aligned}
\text{the } \left\{ \begin{array}{l} \text{first} \\ \text{second} \end{array} \right\} \text{ term of (A} \cdot 2) &= \sum_{n=0}^{\infty} \pm \frac{(-i)^n}{n!} H^n \frac{1}{E_0 \pm i\epsilon} e^{-iE_0 z/v_0} \\
&\quad \times \left[\left(-\frac{z}{v_0} \right)^n + \frac{n}{-i(E_0 \pm i\epsilon)} \left(-\frac{z}{v_0} \right)^{n-1} + \dots \right] V(x) e^{ik_0 x} \\
&= \pm \left\{ \frac{1}{E_0 \pm i\epsilon} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} H^n \left(-\frac{z}{v_0} \right)^n + \frac{H}{(E_0 \pm i\epsilon)^2} \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} H^{n-1} \left(-\frac{z}{v_0} \right)^{n-1} + \dots \right\} \\
&\quad \times e^{-iE_0 z/v_0} V(x) e^{ik_0 x} \\
&= \pm \left(\frac{1}{E_0 \pm i\epsilon} + \frac{H}{(E_0 \pm i\epsilon)^2} + \frac{H^2}{(E_0 \pm i\epsilon)^3} + \dots \right) \text{Exp} \left(iH \frac{z}{v_0} \right) e^{-iE_0 z/v_0} V(x) e^{ik_0 x} \\
&= \pm \frac{1}{E_0 \pm i\epsilon} \frac{1}{1 - H/(E_0 \pm i\epsilon)} \text{Exp} \left(iH \frac{z}{v_0} \right) e^{-iE_0 z/v_0} V(x) e^{ik_0 x} \\
&= \mp \frac{1}{H - E_0 \mp i\epsilon} \text{Exp} \left(iH \frac{z}{v_0} \right) e^{-iE_0 z/v_0} V(x) e^{ik_0 x}. \quad (\text{A} \cdot 3)
\end{aligned}$$

In the next place, we shall compute the third term of (A·2) in the following way :

$$\begin{aligned}
\text{the third term of (A} \cdot 2) &= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} H^n \frac{n!}{(-i)^{n+1} (E_0 - i\epsilon)^{n+1}} e^{z/L} V(x) e^{ik_0 x} \\
&= \frac{1}{E_0 - i\epsilon} \sum_{n=0}^{\infty} \left(\frac{H}{E_0 - i\epsilon} \right)^n e^{z/L} V(x) e^{ik_0 x}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{E_0 - i\epsilon} \frac{1}{1 - H/(E_0 - i\epsilon)} e^{z/L} V(x) e^{ik_0 x} \\
&= \frac{1}{E_0 - H - i\epsilon} V(x) e^{z/L} e^{ik_0 x}.
\end{aligned} \tag{A.4}$$

Now, we can easily see that

$$\begin{aligned}
&+ i \int_{-\infty}^0 dt' e^{i(H-E_0)t'} e^{-|z|/L} e^{-\epsilon|t'|/L} V(x) e^{ik_0 x} \\
&= - \frac{1}{E_0 - H + i\epsilon} V(x) e^{-|z|/L} e^{ik_0 x}.
\end{aligned} \tag{A.5}$$

Collecting (A.3), (A.4) and (A.5), we have
the third term of (2.12) for $z < 0$

$$= \frac{-2i\epsilon}{(H-E_0)^2 + \epsilon^2} \{ \exp(iH \cdot z/v_0) e^{-iE_0 z/v_0} - e^{z/L} \} V(x) e^{ik_0 x}. \quad \text{Q.E.D.}$$

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Phenomenological Analysis of Elastic $\pi^- - p$ Scattering at 1.4 Bev

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Pion-proton scattering at 1.4 Bev observed by Eisberg et al.¹⁾ is analyzed in a phenomenological way. Interpreting the peak in forward direction as "shadow scattering" and analyzing this angular distribution, we obtain the result which agrees fairly well with the experimental one. By this method, moreover, it is found that the partial waves of $l=0, 1, 2$ and $l=1, 2, 3, 4$ are mainly responsible to the elastic and production cross sections, respectively.

§ 1. Introduction

Recent experiments on pion-nucleon¹⁾ and nucleon-nucleon²⁾ collisions have revealed that the shadow scattering plays an important role in these processes. Eisberg et al. have tried to analyze their experiment on elastic pion-nucleon scattering in terms of a phenomenological potential, as customarily employed in interpreting nuclear reactions.

As is well known, the shadow effect results from the opening of new channels which have been closed at lower energies. As it is closely related to the multiple production of pions, some knowledge about the multiple pion production is required for the analysis of the scattering data in a Bev region. The experiments at the present stage, however, do not seem to supply enough data to such an approach. In these circumstances, we intend to attempt another approach, that is, estimation of the cross section for the multiple pion production from the knowledge about the shadow scattering. When we notice the result of Eisberg et al., it will be expected that it may be possible to separate, without large error, the contribution of the shadow scattering from the angular distribution of elastic scattering. Taking advantage of this, we have tried to estimate the cross section for pion production through the analysis of this shadow scattering.

In § 2, the general theory of scattering including the shadow effect is developed for the most simplified model. In § 3, the shadow part of the differential cross section obtained by Eisberg et al. is analyzed in terms of partial waves. From this result, the cross section of pion production is calculated for every partial waves. In spite of this somewhat crude estimation, the obtained total cross section of pion production shows fairly good agreement with the observed one.

§ 2. Shadow scattering and S-matrix theory

Below the threshold of the pion production, the unitarity of S-matrix is guaranteed in

the one pion subspace (0) of the pion configuration, and may be expressed as

$$S^{+(0)}S^{(0)}=1.* \quad (2.1)$$

In a Bev region, however, the cross section for the multiple pion production σ_{prod} is observed as slightly larger than that for the elastic scattering σ_{scatt} . In such a case, the unitarity has to be expressed as

$$S^{+(0)}S^{(0)}+S^{+(1)}S^{(1)}+\dots=1 \quad (2.2)$$

where, $S^{(i)}$'s represent the submatrices relevant to $(i+1)$ -pions production.

In the first place, let us simply ignore the spin of nucleon and analyze (2.2) into partial waves. Then

$$\begin{aligned} S_l &= S_l^{(0)} + S_l^{(1)} + S_l^{(2)} + \dots \equiv S_l^{(0)} + M_l, \\ |S_l|^2 &= |S_l^{(0)}|^2 + |M_l|^2 = 1 \end{aligned} \quad (2.3)$$

for every partial wave.

From this relation, $S_l^{(0)}$ may be written as

$$S_l^{(0)} = \sqrt{1 - |M_l|^2} e^{2i\delta_l} \equiv \cos A_l \cdot \exp(2i\delta_l), \quad (2.4)$$

where

$$M_l \equiv \exp(i\hat{s}_l) \cdot \sin A_l.$$

Then the scattering and the production cross sections are expressed respectively as

$$d\sigma_{\text{scatt}}/d\Omega = (1/4k_0^2) \left| \sum_l (2l+1) \{ \exp(2i\delta_l) \cdot \cos A_l - 1 \} P_l(\cos \theta) \right|^2, \quad (2.5)$$

$$d\sigma_{\text{prod}}/d\Omega = (1/4k_0^2) \left| \sum_l (2l+1) \exp(i\hat{s}_l) \cdot \sin A_l \cdot P_l(\cos \theta) \right|^2. \quad (2.6)$$

Here it must be noticed that the phase factor $\exp(i\hat{s}_l)$ is not important so far as the total cross section σ_{prod} is discussed.

Even when the spin of nucleon is taken into account, a similar procedure can easily be carried out. In this case, $R_l (= S_l^{(0)} - 1)$ becomes as follows :

$$R_l \rightarrow R_l^+ \frac{l+1+(l\sigma)}{2l+1} + R_l^- \frac{l-(l\sigma)}{2l+1}. \quad (2.7)$$

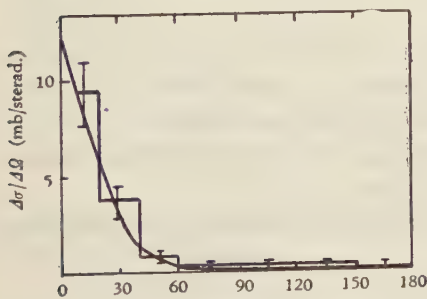
§ 3. Analysis of experiment

Eisberg et al. have obtained the following cross sections

$$\begin{aligned} \sigma_{\text{scatt}} &= 10.0 \pm 0.8 \text{ mb} \\ \sigma_{\text{prod}} &= 24.6 \text{ mb} \end{aligned} \quad (3.1)$$

and the angular distribution for the elastic scattering illustrated in Fig. 1.

As they have pointed out, the strong forward



C. M. S. Scattering angle θ
Fig. 1.

* Here we discard weakly interacting processes such as the radiative capture of pions.

peak may be attributed to the shadow scattering. Adopting this interpretation, the angular distribution is analyzed. For simplicity, we shall set the crude assumptions of $R_l^+ = R_l^- \equiv -T_l$ and of $\delta_l' = 0^*$.

With such a simplification, the scattering amplitude can be expressed as follows:

$$f(\theta) = (i/2k_0) \sum_l (2l+1) T_l P_l(\cos \theta) = (i/2) \sum_l (2l+1) Q_l' P_l(\cos \theta), \quad (3.2)$$

$$T_l = 1 - \cos \Delta_l \text{ and } Q_l' = T_l/k_0.$$

Here the phase shift analysis is made considerably simpler by the fact that Q_l' 's in eq. (3.2) are real numbers. Moreover, if the experimental values of $d\sigma/d\Omega$ are expressed by an empirical formula, the analysis may be carried out more easily.

As an empirical formula, we adopt

$$d\sigma/d\Omega = 0.2/(1.13 - \cos \theta)^2 \text{ mb/sterad.}, \quad (3.3)$$

which fits fairly well to the experiment as is shown in Fig. 1**. From (3.3), the scattering amplitude can generally be written as

$$f(\theta) = e^{i\eta} \sqrt{0.2}/(1.13 - \cos \theta). \quad (3.4)$$

The unknown phase factor $e^{i\eta}$ may be determined on account of the dispersion relation holding at $\theta=0^\circ$. (c.f. (3.2)). Thus

$$f(\theta) = i\sqrt{0.2}/(1.13 - \cos \theta). \quad (3.4)'$$

From the eqs. (3.2) and (3.4)'

$$\sqrt{0.8}/(1.13 - x) = \sum_l (2l+1) Q_l' P_l(x). \quad (3.5)$$

Q_l' in eq. (3.5) is readily obtained by

$$Q_l' = \frac{1}{2} \int_{-1}^1 \frac{\sqrt{0.8}}{1.13 - x} P_l(x) dx = \sqrt{0.8} Q_l, \quad (3.6)$$

$$Q_l = \frac{1}{2} \int_{-1}^1 \frac{P_l(x)}{1.13 - x} dx. \quad (3.6)'$$

According to the well known Neumann-Heine's formula,*** the right hand side of (3.6)' is $Q_l(1.13)$, where $Q_l(x)$ is the Legendre function of second kind. Their numerical values are easily obtained from a table as listed in the 2nd column of Table 1. From these values of Q_l' , the partial scattering cross sections $\sigma_l^{\text{scatt}} = \pi(2l+1)Q_l'^2$ are calculated and they are also listed in the 3rd column of the Table.

The scattering cross section thus obtained as $\sum_l \pi(2l+1)Q_l'^2 = 9.1 \text{ mb}$ agrees fairly well with the experimental value (3.1).

* Note that we consider only the shadow part of elastic scattering.

** As only the shadow scattering is taken into account, it will be rather natural that our empirical formula gives somewhat smaller value of cross section than the experimental value of elastic scattering.

*** c. f. Whittaker and Watson' *A Course of Modern Analysis* (Fourth ed.) pp. 320-322, (1935)

Table 1

l	Q_l'	σ_l^{scatt} mb	σ_l^{prod} mb
0	1.25	4.91	1.79
1	0.519	2.54	5.80
2	0.254	1.01	5.79
3	0.133	0.39	4.59
4	0.0716	0.15	3.31
5	0.0396	0.05	2.28
6	0.0224	0.02	1.54
total		9.1 mb	25.1 mb

The partial production cross sections

$$\sigma_l^{\text{prod}} = (\pi/k_0^2) (2l+1) \sin^2 \mathcal{A}_l \quad (3.7)$$

are evaluated by inserting the values of \mathcal{A}_l 's calculated from the relation $T_l = 1 - \cos \mathcal{A}_l = k_0 Q_l'$. The results are listed in the 4th column of Table 1. The production cross section calculated up to $l=6$ ($\sigma_{\text{prod}} = \sum_{l=0}^6 (\pi/k_0^2) (2l+1) \sin^2 \mathcal{A}_l = 25.1$ mb) shows a fairly good agreement with the experimental value (3.1), in spite of the rather crude estimation. These results are illustrated in Fig. 2.

In order to examine our approximation of neglecting the contributions from the orbital angular momenta higher than $l=6$, let us evaluate the total cross section ($\sigma_T = \sigma_{\text{scatt}} + \sigma_{\text{prod}}$) by the well-known relation of

$$4\pi \text{Im } f(0^\circ) = k_0 \sigma_T \quad (3.8)$$

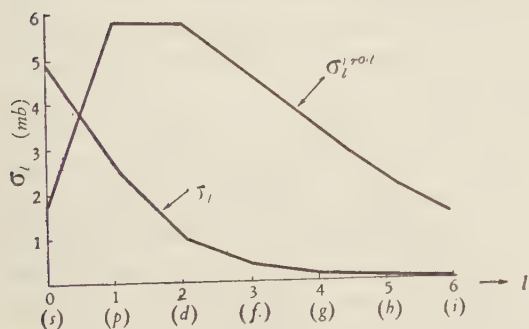


Fig. 2

s , p and d waves in the case of the elastic scattering, and from p , d and f waves in the case of the production.

Since the mean value of l 's responsible for production in the case of $1/k_0 = 2.7 \times 10^{-14}$ cm is estimated to be 3 or 4, we may consider that the collision is taking place in the region of 4~5 times nucleon Compton wave length (mean value of collision parameter). This result is consistent with the experimental fact that the momentum transfer in the reaction is small.

Inserting the value at 0° of our empirical formula (3.4)' into the left hand side of (3.8), we obtain the result of $\sigma_T = 36.8$ mb. This value is not so different from that of 34.2 mb in Table 1.

§ 4. Concluding remarks

Although the above results have to be regarded as qualitative, it may be said that dominant contributions come from

It is important to examine whether or not the above conclusions depend sensitively upon the assumed form of the empirical formula. Within the range of the experimental errors, we have examined it in two extreme cases; that is, the steepest case $0.1/(1.08 - \cos \theta)^2$ and the broadest case $0.36/(1.2 - \cos \theta)^2$. It may be seen that our qualitative conclusions are not sensitively affected by the assumed forms of the empirical formulae.

The authors would like to express their thanks to Prof. T. Miyazima, Prof. S. Hayakawa and Dr. K. Nishijima for their valuable discussions.

Table 2.

empirical formula	$d\sigma/d\Omega = 0.1/(1.08 - \cos \theta)^2$			$d\sigma/d\Omega = 0.36/(1.2 - \cos \theta)^2$		
l	Q_l'	σ_l^{scatt} mb	σ_l^{prod} mb	Q_l'	σ_l^{scatt} mb	σ_l^{prod} mb
0	1.03	3.34	2.19	1.44	6.50	1.21
1	0.480	2.17	5.55	0.527	2.61	5.85
2	0.263	1.09	5.96	0.228	0.82	5.30
3	0.153	0.51	5.22	0.106	0.25	3.72
4	0.0919	0.24	4.19	0.0508	0.07	2.38
5	0.0564	0.11	3.21	0.0251	0.02	1.46
6	0.0350	0.05	2.39	0.0128	0.01	0.89
total		7.5 mb	28.7 mb		10.3 mb	20.8 mb

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Solutions of a Bethe-Salpeter Equation for Scattering States

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A complete set of scattering-state solutions of a Bethe-Salpeter equation is obtained in this paper. This equation is the one for which Wick and Cutkosky succeeded to obtain the bound-state solutions. For bound-state solutions, they have found the same degeneracy as that of the non-relativistic hydrogen atom. This result is very suggestive for the present case, and indeed the scattering-state solutions are obtained without decomposing them into partial waves as is the case for Rutherford scattering.

Adopting the integral transformation method, the four dimensional integral equation is reduced to a simple ordinary differential equation. It is interesting to see that this equation is identical with Cutkosky's one for $n=0$. Finally by a close inspection of the resulting equation and boundary condition, it is shown that the virtual-state solutions, if any, can be obtained by modifying the inhomogeneous boundary condition for scattering states into a homogeneous one. Furthermore, it is proved that an isolated virtual level, if it really exists, gives rise to a resonance scattering.

§ 1. Introduction

It is generally anticipated that the completely relativistic formulation of the two-body problem will provide us with a more powerful tool than those achieved by the non-relativistic approach. For instance, the non-separability of the equation between centre of mass and relative coordinates shows in a direct way that a two-body system in a composite state suffers a Lorentz contraction when it is put in motion. The apparent covariance makes one readily believe its renormalizability, its ease in taking account of various relativistic effects, and so on.

Nevertheless, the equation has hardly been utilized for the solutions of actual problems except for the case of quantum electrodynamics mainly because of the difficulty related to the appearance of a surplus variable, the relative time or the relative energy. Many authors have endeavoured for the elimination of this redundant variable so as to reduce the equation into familiar forms, but it is successful only for quantum electrodynamics in which the instantaneous Coulomb interaction is dominant.

For this reason, illustration of the relativistic solutions of the equation seems to be especially instructive at the present stage. A real progress towards this direction was first achieved by Wick¹⁾ and Cutkosky²⁾. These authors have investigated the Bethe-Salpeter equation (abbreviated as B-S hereafter) for two scalar particles interacting through a massless scalar field and obtained a complete set of bound-state solutions. It was shown that the degeneracy of the eigenvalues are the same as that in the non-relativistic hydrogen atom. This result suggests us that the scattering-state solutions of this special equation will possess, in many respects, close similarities to the non-relativistic Rutherford scattering.

Invoking that it is really the case, the integral transformation method is applied to this equation with a more complicated ansatz on the form of the parametric representation than those used for bound states. Then after some calculations it is shown that the B-S equation for scattering states can be reduced to an ordinary differential equation

$$g''(v) - 2v(1-v^2)^{-1}g'(v) + \lambda(1-v^2)^{-1}(1-\eta^2 + \eta^2 v^2 - i\varepsilon)^{-1}g(v) = 0, \quad (1.1)$$

with the boundary condition $g(1) = g(-1) = \text{const.} \neq 0$. In this equation, ε represents an infinitesimal positive quantity.

It is worth while to mention that the above equation bears a strong resemblance to Cutkosky's one for bound states

$$g_n''(z) + 2(n-1)z(1-z^2)^{-1}g_n'(z) - n(n-1)(1-z^2)^{-1}g_n(z) + \lambda(1-z^2)^{-1}(1-\eta^2 + \eta^2 z^2)^{-1}g_n(z) = 0, \quad (1.2)$$

with the boundary condition $g_n(1) = g_n(-1) = 0$. Indeed, eq. (1.1) is the special case of eq. (1.2) for $n=0$ although Cutkosky's derivation requires $n > 0$ for bound states. For scattering states we have $\eta^2 > 1$, so that there are additional poles in eq. (1.1) that have never appeared in the case of bound states. The occurrence of these new poles is due to the energy momentum conservation in the intermediate states and the quantity $-i\varepsilon$ designates the contour to be taken at these poles so as to satisfy the outgoing wave condition. Thus it may be feasible to solve the simple differential eq. (1.1).

It is remarkable that we have now a complete set of solutions of a simple B-S equation for bound states as well as scattering states just like for the case of the non-relativistic Schrödinger equation for a Coulomb potential.

Furthermore, it can be seen that if we replace the inhomogeneous boundary condition $g(1) = g(-1) = \text{const.} \neq 0$ by a homogeneous one $g(1) = g(-1) = 0$ the eq. (1.1) turns out to represent an eigenvalue problem for η . A precise argument shows that the eigenvalues are those for virtual levels. This reasoning provides us with a general means to formulate the theory of virtual levels or resonance scattering.

§ 2. The integral transformation method

The B-S equation for two scalar particles interacting through a massless scalar field, in the ladder approximation, takes the form

$$\left(p^2 + pP + \mu^2 + \frac{P^2}{4}\right) \left(p^2 - pP + \mu^2 + \frac{P^2}{4}\right) \phi(p) = -\frac{i\lambda\mu^2}{\pi^2} \int \frac{\phi(q)}{(p-q)^2 - i\varepsilon} (dq), \quad (2.1)$$

where P , p and μ denote the total and relative four momenta and the rest mass of the two scalar particles, respectively, and $(dq) = dq_1 dq_2 dq_3 dq_0$.

For bound states, we can rewrite eq. (2.1) as

$$\phi(p) = -\frac{i\lambda\mu^2}{\pi^2} \left(p^2 + pP + \mu^2 + \frac{P^2}{4} - i\varepsilon\right)^{-1} \left(p^2 - pP + \mu^2 + \frac{P^2}{4} - i\varepsilon\right)^{-1} \int \frac{\phi(q)}{(p-q)^2 - i\varepsilon} (dq). \quad (2.2a)$$

This equation was thoroughly investigated by Wick and Cutkosky.

In the present paper, we shall be concerned with scattering states for which the eq. (2.1) can be written as

$$\psi(p) = \varphi(p) - \frac{i\lambda\mu^2}{\pi^2} \left(p^2 + pP + \mu^2 + \frac{P^2}{4} - i\epsilon \right)^{-1} \left(p^2 - pP + \mu^2 + \frac{P^2}{4} - i\epsilon \right)^{-1} \int \frac{\psi(q)}{(p-q)^2 - i\epsilon} (dq), \quad (2.2b)$$

where $\varphi(p)$ represents the incident wave and conveniently we shall fix it as

$$\varphi(p) = \delta(p - p_0), \quad (2.3)$$

p_0 is the relative four momentum in the initial state.

For the solution of eq. (2.2a), Wick utilized a parametric representation of the form

$$\phi(p) = \int_{-1}^1 \frac{g(z) dz}{(p^2 + zpP + \mu^2 + P^2/4 - i\epsilon)^3}, \quad (2.4)$$

and reduced the complicated eq. (2.2a) to a simple equation for $g(z)$

$$g''(z) + \lambda(1-z^2)^{-1} Q(z)^{-1} g(z) = 0, \quad (2.5)$$

with a boundary condition $g(1) = g(-1) = 0$. $Q(z)$ and η are defined respectively by

$$Q(z) = 1 - \eta^2 + \eta^2 z^2, \text{ and } P^2 = -(2\mu\eta)^2.$$

Then Cutkosky extended Wick's method so as to obtain a complete set of bound-state solutions. With a more complicated ansatz on the form of the parametric representation than (2.4), he has generalized the eigenvalue equation (2.5) as

$$g''(z) + 2(n-1)z(1-z^2)^{-1} g'_n(z) - n(n-1)(1-z^2)^{-1} g_n(z) + \lambda(1-z^2)^{-1} Q(z)^{-1} g_n(z) = 0, \quad (2.6)$$

with a boundary condition $g_n(1) = g_n(-1) = 0$.

Since the integral transformation method enabled us to much simplify the bound-state equation, we may invoke that it will also be the case for scattering problem. For this purpose, however, it gives rise to a discussion whether such a parametric representation is always possible. Recently this problem was resolved positively by Nambu³⁾ on the basis of the microscopic causality condition. Hence we can utilize this method with confidence, and furthermore we may interpret this result as that *the microscopic causality condition imposes a new kind of boundary condition on the relative energy dependence of the Feynman amplitudes so as to supplement the conventional outgoing wave condition.*

In order to apply the integral transformation method to eq. (2.2b), it is more convenient to express $\psi(p) - \varphi(p)$ parametrically than to do with $\psi(p)$ itself. We shall denote $\psi(p) - \varphi(p)$ newly by $\psi(p)$, then the new $\psi(p)$ satisfies an integral equation

$$\psi(p) = -\frac{i\lambda\mu^2}{\pi^2} \left(p^2 + pP + \mu^2 + \frac{P^2}{4} - i\epsilon \right)^{-1} \left(p^2 - pP + \mu^2 + \frac{P^2}{4} - i\epsilon \right)^{-1} \times \left[\frac{1}{(p-p_0)^2 - i\epsilon} + \int \frac{\psi(q)}{(p-q)^2 - i\epsilon} (dq) \right]. \quad (2.7)$$

In accordance with the microscopic causality condition, we have adopted the following representation to $\psi(p)$:

$$\psi(p) = \int dz d\zeta dM^2 \frac{g(z, \zeta, M^2)}{[p^2 + \mu^2 + P^2/4 + zpP - 2\zeta p p_0 + M^2 - i\epsilon]^3}, \quad (2.8)$$

which is quite general and covers the case of bound states. Inserting this expression into eq. (2.7), one sees at once that if $M^2 = \zeta(p_0^2 - \mu^2 - P^2/4)$, the mass term reproduces itself. Thus we set

$$g(z, \zeta, M^2) = g(z, \zeta) \delta[M^2 + \zeta(\mu^2 + P^2/4 - p_0^2)]. \quad (2.9)$$

We shall call this function $g(z, \zeta)$ the density function. The equation for the density function is obtained by inserting (2.8) and (2.9) into eq. (2.7) and then trying to reduce the result again to the form (2.8). By utilizing the well-known relation

$$\begin{aligned} \int \frac{(dq)}{(p-q)^2 - i\epsilon} \cdot \frac{1}{(q^2 + 2Qq + M^2 - i\epsilon)^3} \\ = \frac{i\pi^2}{2} \cdot \frac{1}{M^2 - Q^2 - i\epsilon} \cdot \frac{1}{(q^2 + 2Qq + M^2 - i\epsilon)} \end{aligned} \quad (2.10)$$

the equation for the density function is given by

$$\begin{aligned} g(z, \zeta) = g^0(z, \zeta) + (i\lambda\mu^2/2) \int_{-1}^1 dx \int_0^1 dy \delta[z - (xy + \zeta'(1-y))] \delta[\zeta - \zeta'(1-y)] \\ \times \left[\mu^2 + \frac{P^2}{4} - \zeta'(\mu^2 + \frac{P^2}{4} - p_0^2) - \frac{1}{4}(\zeta'P - 2\zeta'p_0)^2 - i\epsilon \right]^{-1} g(\zeta', \zeta'), \end{aligned} \quad (2.11)$$

where

$$g^0(z, \zeta) = -\frac{i\lambda\mu^2}{\pi^2} \int_{-1}^1 dx \int_0^1 dy \delta(z - xy) \delta[\zeta - (1-y)]. \quad (2.12)$$

The derivation is completely analogous to Wick's one.

§ 3. Reduction of the parametric integral equation

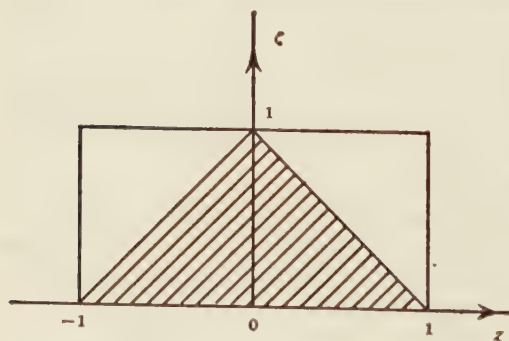


Fig. 1.

The domain of the density function. The density function $g(z, \zeta)$ differs from zero only inside the shaded triangle.

The equation (2.11) is still considerably complicated and we shall try to reduce it into a differential equation. For this purpose, we shall first eliminate the delta-functions appearing in eqs. (2.11) and (2.12).

The integration of (2.12) readily yields

$$\begin{aligned} g^0(z, \zeta) = -i\lambda\mu^2/\pi^2 \times \\ \times \begin{cases} 1, & \text{if the point } (z, \zeta) \text{ is in the} \\ & \text{domain } \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.1)$$

The triangular domain \mathcal{A} is indicated in

Fig. 1 by shaded lines and defined by the inequalities

$$1 \geq \zeta \geq 0, \text{ and } 1 \geq z/(1-\zeta) \geq -1. \quad (3.2)$$

The delta-functions in the eq. (2.11) require

$$\zeta = \zeta'(1-y), \text{ and } z/(1-\zeta) = (xy + z'(1-y))/(y + (1-\zeta')(1-y)). \quad (3.3)$$

Hence if the point (z', ζ') moves in the domain Δ , the point (z, ζ) will also move in the same domain, since $y, (1-y)$, and $(1-\zeta')$ are positive and

$$1 \geq x, \quad z'/(1-\zeta') \geq -1.$$

Thus we can prove by induction that the density function $g(z, \zeta)$ differs from zero only in the triangular domain Δ .

Eliminating x and y from (3.3), we have inequalities which restrict the domain of (z', ζ') for given values of z and ζ .

$$\zeta' \geq \zeta, \text{ and } \zeta' - \zeta \geq z \zeta' - z' \zeta \geq \zeta - \zeta'. \quad (3.4)$$

Thus we can transform the integration appearing in the eq. (2.11) as

$$\begin{aligned} & \int_{-1}^1 dx \int_0^1 y \, dy \int_{\Delta} dz' \, d\zeta' \, \partial[z - (xy + z'(1-y))] \partial[\zeta - \zeta'(1-y)] \dots \\ &= \int_{\zeta}^1 \frac{d\zeta'}{\zeta'} \int_{\text{Min}(1, (1+\varepsilon)/R-1)}^{\text{Min}(1, (1+\varepsilon)/R-1)} dz' \dots = \int_{\zeta}^1 \frac{d\zeta'}{\zeta'} \int_{1-(1-\varepsilon)/R}^{(1+\varepsilon)/R-1} dz' \, \rho(z') \dots, \end{aligned} \quad (3.5)$$

where R is the ratio $\zeta/\zeta' \leq 1$, and $\rho(z') = 1$ for $|z'| \leq 1$ and $\rho(z') = 0$ otherwise. Let us put

$$Z_+ = (1+z)/R - 1 = (1+z)\zeta'/\zeta - 1, \quad Z_- = 1 - (1-z)/R = -(1-z)\zeta'/\zeta + 1, \quad (3.6)$$

and

$$x = z/\zeta, \quad y = 1/\zeta, \quad (3.7)$$

then we can rewrite (3.5)

$$\int_{\zeta}^1 \frac{d\zeta'}{\zeta'} \int_{Z_-}^{Z_+} dz' \, \rho(z') \dots, \quad (3.5')$$

and one readily sees

$$D_+ Z_- \equiv \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) Z_- = 0, \quad D_- Z_+ \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) Z_+ = 0. \quad (3.8)$$

Let us consider an integral

$$G(z, \zeta) = \int_{\zeta}^1 d\zeta' \int_{Z_-}^{Z_+} dz' \, F(z', \zeta'), \quad (3.9)$$

and apply the operation $D_- D_+$ on G . Noticing that $Z_+ = Z_- = z$ for $\zeta' = \zeta$, one obtains

$$\begin{aligned} D_+ G(z, \zeta) &= \int_{\zeta}^1 d\zeta' \, D_+ \int_{Z_-}^{Z_+} dz' \, F(z', \zeta') \\ &= \int_{\zeta}^1 d\zeta' \, F(Z_+, \zeta') (D_+ Z_+), \end{aligned}$$

and

$$\begin{aligned} D_- D_+ G(z, \zeta) &= - (D_- \zeta) [F(Z_+, \zeta') (D_+ Z_+)]_{\zeta'=\zeta} \\ &= -F(z, \zeta) (D_- \zeta) (D_+ Z_+)_{\zeta'=\zeta}. \end{aligned} \quad (3 \cdot 10)$$

Inserting into (3·10) the relations

$$D_- \zeta = \zeta^2, \quad (D_+ Z_+)_{\zeta'=\zeta} = 2\zeta,$$

we have

$$D_- D_+ G(z, \zeta) = -2\zeta^3 F(z, \zeta), \quad (3 \cdot 11)$$

i.e. the integral equation (3·9) can be transformed into a differential form (3·11). By applying this formula to the eq. (2·11), we have with reference to (3·5') a hyperbolic type differential equation

$$D_- D_+ g(z, \zeta) + \lambda \mu^2 \zeta^2 Q(z, \zeta)^{-1} g(z, \zeta) = 0, \quad (\text{in the domain } \mathcal{A}) \quad (3 \cdot 12)$$

where

$$Q(z, \zeta) = \mu^2 + P^2/4 - \zeta(\mu^2 + P^2/4 - p_0^2) - (1/4)(zP - 2\zeta p_0)^2 - i\varepsilon. \quad (3 \cdot 13)$$

The boundary condition for this equation can be inferred from the original eq. (2·11). An inspection of the eqs. (3·5') and (2·11) leads to the result that $g(z, \zeta) - g^0(z, \zeta)$ is continuous at the boundary lines

$$z/(1-\zeta) = \pm 1. \quad (3 \cdot 14)$$

Since the density functions $g(z, \zeta)$ and $g^0(z, \zeta)$ vanish outside the triangular domain \mathcal{A} , one sees at once that

$$g(z, \zeta) = g^0, \quad \text{const. at } z = \pm(1-\zeta). \quad (3 \cdot 15)$$

This is a sufficient boundary condition to settle the solution of the differential equation (3·12) uniquely.

Let us put

$$\zeta^2/Q(z, \zeta) = [(\mu^2 + P^2/4)\gamma^2 - (\mu^2 + P^2/4 - p_0^2)\gamma - (1/4)(xP - 2p_0)^2 - i\varepsilon]^{-1} \equiv q(x, \gamma)^{-1},$$

then the eq. (3·12) can be written as

$$(\partial^2/\partial x^2 - \partial^2/\partial \gamma^2 + \lambda \mu^2/q(x, \gamma))g = 0. \quad (3 \cdot 16)$$

The solution of this equation with the boundary condition (3·15) represents the Green function for the scattering if the four vector p_0 is unspecified. However, if we impose an initial condition on p_0 that the incident scalar particles are free, i.e.,

$$(P/2 \pm p_0)^2 + \mu^2 = 0, \quad (3 \cdot 17)$$

the eq. (3·16) becomes separable. In this case p_0 is a spatial vector with vanishing time component in the centre of mass system, and we have

$$p_0 P = 0, \quad \mu^2 + P^2/4 = -p_0^2,$$

and consequently

$$q(x, y) = (p_0^2 + \mu^2)x^2 - p_0^2(1-y)^2 - i\varepsilon.$$

If we make a variable transformation $y' = y - 1$, the eq. (3.16) takes the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y'^2} + \frac{\lambda \mu^2}{(p_0^2 + \mu^2)x^2 - p_0^2 y'^2 - i\varepsilon} \right) g = 0. \quad (3.18)$$

The triangular domain \mathcal{A} is transformed into a region in the (x, y') plane defined by the inequalities :

$$1 \geq x/y' \geq -1, \text{ and } y' \geq 0, \quad (3.19)$$

which is indicated in Fig. 2 by shaded lines.

The boundary condition is now reduced to

$$g = g^0 \text{ at } x = \pm y'. \quad (3.20)$$

An inspection of (3.18) and (3.19) suggests us a transformation of variables

$$\begin{aligned} x &= r \sinh \varphi, \\ y' &= r \cosh \varphi, \quad (r > 0) \end{aligned} \quad (3.21)$$

by which the inequalities (3.19) are automatically satisfied for all positive values of r , and all real values of φ . The boundary condition (3.20) is simplified as

$$g = g^0 \text{ at } \varphi \rightarrow \pm \infty, \quad (3.22)$$

and the eq. (3.18) turns out to be of the form :

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} - \frac{2\lambda}{r^2 (\cosh 2\varphi - \cosh 2\varphi_0 - i\varepsilon)} \right) g = 0, \quad (3.23)$$

where φ_0 is defined by

$$p_0^2 / (p_0^2 + \mu^2) = \tanh^2 \varphi_0 = v_0^2,$$

v_0 denotes the initial velocity of the two scalar particles in the centre of mass system. It is worth noticing that the eq. (3.23) is of a separable type, and indeed by putting

$$g = R(r) \cdot \Phi(\varphi), \quad (3.24)$$

one finds a couple of separated equations

$$r (d/dr) (r dR/dr) = kR, \quad d^2 \Phi / d\varphi^2 + 2\lambda \Phi / (\cosh 2\varphi - \cosh 2\varphi_0 - i\varepsilon) = k\Phi, \quad (3.25)$$

where k is a constant of integration to be determined subject to the boundary condition (3.22). The solution of the first equation is readily found to be

$$R = C_1 r^n + C_2 r^{-n} \text{ with } k = n^2. \quad (3.26)$$

Since, however, the inhomogeneous boundary condition (3.22) is independent of r , only the case $n=0$ is accessible, i.e., g is a function of φ alone. Then making a further variable transformation

$$v = \tanh \varphi = x/y' = z/(1-\zeta), \quad (3.27)$$

one can transform the second equation into

$$\frac{d}{dv}(1-v^2) \frac{d}{dv} g(v) + \lambda(1-v_0^2) \cdot \frac{g(v)}{v^2 - v_0^2 - i\varepsilon} = 0 \quad (3.28)$$

with the boundary condition

$$g(1) = g(-1) = g^0 = -i\lambda\mu^2/\pi^2.$$

Since

$$v_0^2 = (4/P^2)(\mu^2 + P^2/4) = -(1/\tau^2) + 1,$$

we can also write the eq. (3.28) as

$$g''(v) - 2v(1-v^2)^{-1} g'(v) + \lambda(1-v^2)^{-1} (1-\tau^2 + \tau^2 v^2 - i\varepsilon)^{-1} g(v) = 0. \quad (3.29)$$

This equation has a strong resemblance to Cutkosky's one (1.2) for bound states, and indeed it is identical with Cutkosky's equation for $n=0$ although his argument requires $n \geq 0$ for bound states*. The scattering equation (3.28) or (3.29) has a pair of new poles at $v = \pm v_0$ which do not appear in the bound-state equation (1.2). The occurrence of these new poles, characteristic of scattering problems, is due to the energy momentum conservation in the intermediate states and the quantity $-i\varepsilon$ designates the contour to be taken at these poles so as to automatically select the outgoing waves.

We shall not further discuss the eq. (3.29), but it may be susceptible of solutions if any.

§ 4 The virtual levels

Until now the theory of virtual levels in terms of Bethe-Salpeter equation was not available, but a close investigation of the especially simple B-S eq. (2.1) makes it possible to formulate the problem in a completely general form.

Suppose that a B-S equation for scattering states is transformed into a parametric integral equation as has been done in § 2,

$$g = g^0 + Kg, \quad (4.1)$$

where g and g^0 are the density functions and the integral operator K involves an infinitesimal quantity $-i\varepsilon$ so as to automatically select outgoing scattered waves alone. Let the masses of the two colliding particles be m_a and m_b , then the initial condition for scattering requires

* Wick's equation corresponds to the case $n=1$.

$$\begin{aligned}(\mu_a P + p_0)^2 + m_a^2 &= 0, \\ (\mu_b P - p_0)^2 + m_b^2 &= 0,\end{aligned}\tag{4.2}$$

where P , p_0 , μ_a and μ_b are the total four momentum, relative four momentum in the initial state, and the mass ratios defined by

$$\mu_a = m_a / (m_a + m_b), \quad \mu_b = m_b / (m_a + m_b).$$

The relations (4.2) enable us to eliminate p_0^2 and $p_0 P$ in favour of $P^2 = -E^2$, where E is the total energy of the system in the centre of mass system.

Owing to the appropriate choice of the operator K , the scattering state solution will have a form

$$\sim (\text{incident wave}) + (\text{outgoing scattered wave}).\tag{4.3}$$

Now consider a homogeneous equation instead of (4.1)

$$g = Kg,\tag{4.4}$$

then this equation has non-vanishing solutions only if E is equal to one of the characteristic values

$$E_1, E_2, \dots.\tag{4.5}$$

These values are of course complex in general since the eq. (4.4) is not self-adjoint. Furthermore it must be noticed that the eq. (4.4) is by no means equivalent to the homogeneous B-S equation for bound states, since the insertion of the solution of (4.4) in general brings about divergent integrals in the configuration space.

The characteristic values (4.5) represent the virtual levels as we shall see in the following discussions.

In the first place, the solution of (4.4) will have a form

$$\sim (\text{outgoing scattered wave}),\tag{4.6}$$

since there is no incident wave *i.e.*, the solutions consist of outgoing waves only. This is one of the well-known properties of the virtual levels or decaying states in the theory of nuclear reactions⁴⁾.

In the second place, such levels can give rise to resonance scattering if the energy of the system is close to one of the levels (4.5), say E_n . If the total energy E is close to E_n , then on comparing eqs. (4.1) and (4.4) one sees that the density function η will not be small even if the amplitude of the incident wave g^0 is small, *i.e.*, the incident wave is very much amplified near the virtual levels. This statement can also be verified mathematically.

Suppose that there exists such a differential operator D that satisfies

$$Dg^0 = 0, \quad (\text{in a region } R)\tag{4.7}$$

then the eqs. (4.1) and (4.4) are transformed into a single equation

$$Lg = 0, \quad (L = D - DK),\tag{4.8}$$

and the difference between the eqs. (4.1) and (4.4) remains only in the boundary conditions. The density function g satisfies an inhomogeneous boundary condition for scattering states as is the case for (3.29)

$$g=f \text{ (at the boundary } B), \quad (4.9a)$$

and a homogeneous one for virtual states

$$g=0 \text{ (at the boundary } B), \quad (4.9b)$$

Since g is a function of a set of variables $\hat{\xi}_1, \hat{\xi}_2, \dots$ and a parameter E , we shall explicitly write the eq. (4.8) as

$$L(E)g(\hat{\xi}, E)=0, \quad (4.10)$$

and (4.9a) as

$$g(\hat{\xi}, E)=f(\hat{\xi}, E) \text{ at the boundary } B. \quad (4.11)$$

For simplicity we shall tentatively assume that $f(\hat{\xi}, E)$ can be factorized:

$$f(\hat{\xi}, E)=f(E)b(\hat{\xi}),$$

where $f(E)$ is a function of the parameter E alone and $b(\hat{\xi})$ is a function of the variables $\hat{\xi}_1, \hat{\xi}_2, \dots$ defined at the boundary B .

Next we shall define such a function $G(\hat{\xi}, E)$ for values of E in the neighbourhood E_n that satisfies

$$L(E)G(\hat{\xi}, E)=0,$$

and the boundary condition

$$G(\hat{\xi}, E_n)=0, \quad \partial G(\hat{\xi}, E)/\partial E=f(E)b(\hat{\xi}) \text{ at the boundary } B. \quad (4.13)$$

Then we can show that in the neighbourhood of E_n , $g(\hat{\xi}, E)$ is related to $G(\hat{\xi}, E)$ by

$$g(\hat{\xi}, E)=c(E)G(\hat{\xi}, E)=(f(E)/\int_{E_n}^E f(E)dE)G(\hat{\xi}, E). \quad (4.14)$$

It is clear that $c(E)G(\hat{\xi}, E)$ satisfies the same equation with $g(\hat{\xi}, E)$ and from the relation

$$\begin{aligned} c(E)G(\hat{\xi}, E)_{\text{at } B} &= c(E)(G(\hat{\xi}, E_n) + (E-E_n)(\partial/\partial E)G(\hat{\xi}, E_n) + \dots)_{\text{at } B} \\ &= c(E)((E-E_n)f(E_n) + (1/2)(E-E_n)^2 f'(E_n) + \dots)b(\hat{\xi}) \\ &= f(E)b(\hat{\xi}), \end{aligned} \quad (4.15)$$

we see that $c(E)G(\hat{\xi}, E)$ satisfies the same boundary condition with $g(\hat{\xi}, E)$. This establishes the equality (4.14). Then expanding the right hand side of (4.14) in the neighbourhood of E_n , one has

$$\begin{aligned} g(\hat{\xi}, E) &= (E-E_n)^{-1}G(\hat{\xi}, E_n) + \left(\frac{1}{2} \frac{f'(E_n)}{f(E_n)} G(\hat{\xi}, E_n) + \frac{\partial}{\partial E} G(\hat{\xi}, E_n) \right) \\ &\quad + \dots \end{aligned} \quad (4.16)$$

This relation explicitly shows that near an isolated virtual level E_n the so-called resonance scattering

takes place as is clear from the occurrence of the characteristic energy denominator $(E - E_n)^{-1}$ provided that the imaginary part of E_n is not large.

Finally it can be inferred that for the eigenvalues (4.5) one has

$$\text{Im}E_n < 0, \quad (n=1, 2, \dots). \quad (4.17)$$

The virtual levels (4.5) are the poles of the scattering matrix in the complex E -plane as seen from (4.16). The causality condition⁵⁾ requires that the scattering matrix has no pole in the upper half plane, which establishes the inequality (4.17). Thus we have completed the proof that the eigenvalues (4.5) represent the virtual levels.

Although our reasoning stands on the inspection of the eq. (3.29), this equation will not possess such a virtual level solution that causes a resonance scattering, since it is not the case for Rutherford scattering. The above arguments remain to be valid only for short range interactions.

In conclusion, the author expresses his sincere thanks to Prof. G. C. Wick, Dr. R. E. Cutkosky, and Prof. Y. Nambu for their preprints and to Dr. T. Nakano for his helpful discussions.

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- 2) R. E. Cutkosky, Phys. Rev. **96** (1954), 1135.
- 3) Y. Nambu, Phys. Rev. **98** (1955), 803.
- 4) J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics*, Chap. VIII, 412.
- 5) See for instance, Gell-Mann, Goldberger and Thirring, Phys. Rev. **95** (1954), 1612.

Note added in proof: In a private communication to the author, Mr. S. Okubo pointed out that the parametric eq. (3.29) under the given boundary condition has no solution. This difficulty is due to the confluence of the singularity and boundary of the parametric eq. (3.16) for the Green function in the free limit (3.17) as seen from the separability of the eq. (3.18), and corresponds to the fact that the wave function for the Rutherford scattering cannot be expanded in plane waves because of the long range character of the force. The author is indebted to Mr. S. Okubo on this point.

The Energy-Levels and Transition Probabilities for a Bounded Linear Harmonic Oscillator

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The first five energy levels and the oscillator strengths for transitions involving the first three states have been numerically computed for a bounded linear harmonic oscillator for various values of the boundary parameter. It is found that for $l > 3l_0$, where l is the length of the box within which the oscillator is confined and l_0 is the classical amplitude of the oscillator when it has energy $h\nu$, the bounded oscillator behaves more or less like a free oscillator (in the first few energy levels), while for $l < l_0$ it has properties closely approaching those of a free particle enclosed in a box.

Introduction

The study of artificially bounded atomic systems on the basis of quantum mechanics was initiated by Sommerfeld and Welker¹⁾ in 1938. They dealt with the problem of the hydrogen atom confined in a spherical enclosure of radius R . The wave function instead of vanishing at infinity, as usual, now vanishes on the surface of the sphere. They found that for $R > 1.835a$ where a is the radius of the first Bohr orbit, the electron is bound to the nucleus while for $R < 1.835a$, it is not (energy positive). The corresponding problem of the bounded linear harmonic oscillator has been attacked by Auluck and Kothari²⁾, and they have obtained expressions for the energy levels of the oscillator in two extreme cases, viz., $l \ll l_0$ and $l \gg l_0$, where l is the length of the box within which the oscillator is confined and l_0 is the classical amplitude of the oscillator when it has energy $h\nu$ (ν = classical frequency of the oscillator). In this paper we have numerically evaluated the energy levels of the bounded oscillator for intermediate values of l , and have also considered the problem of the transition probabilities of the oscillator. In the case of a free oscillator (wave function vanishing at ∞), transitions can occur only between adjacent levels. For a bounded oscillator, however, there is a non-zero probability of transition between any two states of different parities (we label the states as 1 (lowest energy), 2, 3, ...; the 'odd' states 1, 3, 5, ... and the 'even' states 2, 4, 6, ..., are referred to as states of different parities). We have numerically worked out the oscillator strengths for all transitions involving the first three states (Tables III, IV, and V). It is of interest to note that recently Corson & Kaplan³⁾ have referred to a possible role of the bounded oscillator in the theory of the specific heats of solids. It might also find an application in the study of the second order phase transitions⁴⁾. We hope to discuss these questions in a subsequent paper.

§ 1. The wave equation

For a particle of mass m moving in a field of force of potential $(1/2) \alpha x^2$, the Schrödinger wave-equation is

$$d^2\psi/dx^2 + 2m/\hbar^2 (E - 1/2 \cdot \alpha x^2) \psi = 0. \quad (1)$$

Putting

$$E = (n + 1/2) \hbar \omega,$$

and

$$x = (\hbar/2m\omega)^{1/2} \xi, \quad (2)$$

where $\omega = (\alpha/m)^{1/2}$ is the 'classical' angular frequency of the oscillator, we get

$$\frac{d^2\psi}{d\xi^2} + (n + 1/2 - 1/4 \cdot \xi^2) \psi = 0. \quad (3)$$

This equation has solutions

$$\psi(\xi) = \xi^{-1/2} M_{n/2+1/4, \pm 1/4}(\xi^2/2),$$

where the M 's are the confluent hypergeometric functions given by⁵⁾

$$2^{1/4} \xi^{-1/2} M_{n/2+1/4, -1/4}(\xi^2/2) = e^{-\xi^2/4} \left[1 - \frac{n}{2!} \xi^2 + \frac{n(n-2)}{4!} \xi^4 - \dots \right], \quad (4)$$

$$2^{3/4} \xi^{-1/2} M_{n/2+1/4, +1/4}(\xi^2/2) = e^{-\xi^2/4} \left[\xi - \frac{(n-1)}{3!} \xi^3 + \frac{(n-1)(n-3)}{5!} \xi^5 - \dots \right]. \quad (5)$$

In what follows we shall denote the two solutions (4) and (5) by $W_{n, -1/4}(\xi)$ and $W_{n, +1/4}(\xi)$ respectively.*

We assume the oscillator to be enclosed between infinitely high and steep potential walls at $x = \pm l/2$. The energy eigenvalues are then determined from the condition

$$\psi(x) \longrightarrow 0 \text{ as } x \longrightarrow \pm l/2,$$

or in terms of ξ

$$\psi(\xi) \longrightarrow 0 \text{ as } \xi \longrightarrow \pm \xi_0,$$

where

$$\xi_0 = l/2 \cdot (2m\omega/\hbar)^{1/2} = l/l_0; \quad (6)$$

$l_0 = (2\hbar/m\omega)^{1/2}$ is the 'classical amplitude' of the oscillator when it has energy $\hbar\omega$.

The zeros of (4) and (5) can be evaluated numerically (this is done in Section III). Zeros of $W_{n, -1/4}(\xi)$ give the first (lowest), third, fifth... energy levels, while those of $W_{n, +1/4}(\xi)$ determine the even energy levels.

An approximate formula for the energy levels in terms of ξ_0 can be obtained by treating the term $(1/2) \alpha x^2$ in the Hamiltonian as a small perturbation. When this term is neglected, we have the Hamiltonian for a free particle, and the energy levels are given by

$$E_q = \pi^2 q^2 / 4 \xi_0^2 \cdot \hbar \omega, \quad q = 1, 2, 3, \dots$$

*) These W -function are not identical with those defined by Whittaker and Watson. Reader should be careful not to confuse them.

The first order correction term according to the ordinary perturbation theory is

$$(q|1/2 \cdot \alpha x^2|q) = \frac{\xi_0^2}{12} \left(1 - \frac{6}{\pi^2 q^2}\right) \hbar \omega,$$

and the second order term is

$$\sum_{p \neq q} \frac{|(q|1/2 \cdot \alpha x^2|p)|^2}{(E_q - E_p)} = \frac{256}{\pi^6} \xi_0^6 \sum_{p \neq q} \frac{q^2 p^2}{(q^2 - p^2)^5} \hbar \omega,$$

where p takes on odd or even integral values according as q is odd or even. Thus we have for the energy of the q -th level,

$$E_q = (n_q + 1/2) \hbar \omega \\ = \frac{\pi^2 q^2}{4 \xi_0^2} \left[1 + \frac{\xi_0^4}{3 \pi^2 q^2} \left(1 - \frac{6}{\pi^2 q^2}\right) + \frac{1024}{\pi^8} \xi_0^6 \sum_{p \neq q} \frac{p^2}{(q^2 - p^2)^5} \right] \hbar \omega. \quad (7)$$

This formula gives correct values of energy, (correct to 1 in 10^5) for the first few levels up to $\xi_0 \leq 1.5$. For higher levels the approximation is still better.

We list below a few recurrence relations between the W 's that are of help in evaluating the matrix elements (Sec. II). These formulae may be verified by direct substitution.

$$nW_{n-1,1/4} + \hat{\xi} W_{n,-1/4} - (n+1)W_{n+1,1/4} = 0, \quad (8a)$$

$$W_{n+1,-1/4} + \hat{\xi} W_{n,1/4} - W_{n-1,-1/4} = 0, \quad (8b)$$

$$W'_{n,1/4} - (1/2) \hat{\xi} W_{n,1/4} - W_{n+1,-1/4} = 0, \quad (8c)$$

$$W'_{n,-1/4} - (1/2) \hat{\xi} W_{n,-1/4} + (n+1)W_{n+1,1/4} = 0. \quad (8d)$$

('Dash' denotes differentiation with respect to $\hat{\xi}$).

§ 2. The matrix elements

The dipole matrix element between two states q and p is

$$(q|x|p) = c_{n_q, \pm 1/4} \cdot c_{n_p, \pm 1/4} \int_{-l/2}^{l/2} \psi_q x \psi_p dx \\ = c_{n_q, \pm 1/4} \cdot c_{n_p, \pm 1/4} \left(\frac{\hbar}{2m\omega} \right) \int_{-\xi_0}^{\xi_0} W_{n_q, \pm 1/4} \hat{\xi} W_{n_p, \pm 1/4} d\hat{\xi}, \dots \quad (9)$$

where the c 's are the normalisation constants given by

$$c_{n_q, \pm 1/4}^2 = (2m\omega/\hbar)^{1/2} \int_{-\xi_0}^{\xi_0} W_{n_q, \pm 1/4}^2 d\hat{\xi}. \quad (10)$$

Since $W_{n_q, \pm 1/4}$ is an odd function of $\hat{\xi}$ and $W_{n_q, -1/4}$ is an even function of $\hat{\xi}$

$$\int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} \hat{\xi} W_{n_q, 1/4} d\hat{\xi} = \int_{-\xi_0}^{\xi_0} W_{n_q, -1/4} \hat{\xi} W_{n_q, -1/4} d\hat{\xi} = 0.$$

Hence transitions between states labelled by integers of the same parity are forbidden. Next we consider $\int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} \hat{\xi} W_{n_q, -1/4} d\hat{\xi}$. We have

$$\frac{d^2 W_{n_q, -1/4}}{d\hat{\xi}^2} + (n_q + 1/2 - 1/4 \cdot \hat{\xi}^2) W_{n_q, 1/4} = 0, \quad (i)$$

$$\frac{d^2 W_{n_p, -1/4}}{d\hat{\xi}^2} + (n_p + 1/2 - 1/4 \cdot \hat{\xi}^2) W_{n_p, -1/4} = 0. \quad (ii)$$

From (i) and (ii) we easily obtain

$$(n_p - n_q) \int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} \hat{\xi} W_{n_p, -1/4} d\hat{\xi} = 2 \int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} W'_{n_p, -1/4} d\hat{\xi},$$

or using recurrence relation (8d),

$$(n_p - n_q - 1) \int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} \hat{\xi} W_{n_p, -1/4} d\hat{\xi} = -2(n_p + 1) \int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} W_{n_p+1, 1/4} d\hat{\xi}.$$

Again

$$(n_p - n_q + 1) \int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} W_{n_p+1, 1/4} d\hat{\xi} = 2W_{n_p+1, 1/4}(\hat{\xi}_0) W_{n_q+1, -1/4}(\hat{\xi}_0),$$

hence

$$\int_{-\xi_0}^{\xi_0} W_{n_q, 1/4} \hat{\xi} W_{n_p, -1/4} d\hat{\xi} = -\frac{4(n_p + 1)}{(n_p - n_q)^2 - 1} W_{n_p+1, 1/4}(\hat{\xi}_0) W_{n_q+1, -1/4}(\hat{\xi}_0). \quad (11)$$

The normalisation constants can be evaluated as follows: it may be shown from the wave equation²⁾ that

$$\int_{-\xi_0}^{\xi_0} u^2(n_q, \hat{\xi}) e^{-\xi^2/2} d\hat{\xi} = 2e^{-\xi_0^2/2} (\partial u / \partial n)_{n_q, \xi_0} (\partial u / \partial \hat{\xi})_{n_q, \xi_0}$$

where

$$u(n, \hat{\xi}) = e^{\xi^2/4} W_{n, \pm 1/4}(\hat{\xi}).$$

From $u(n_q, \hat{\xi}_0) = 0$, we have

$$dn_q / d\hat{\xi}_0 = (-\partial u / \partial \hat{\xi})_{n_q, \xi_0} / (\partial u / \partial n)_{n_q, \xi_0}, \quad (13)$$

so that

$$\int_{-\xi_0}^{\xi_0} u^2(n_q, \hat{\xi}) e^{-\xi^2/2} d\hat{\xi} = -2e^{-\xi_0^2/2} [(\partial u / \partial \hat{\xi})_{n_q, \xi_0}]^2 / (dn_q / d\hat{\xi}_0).$$

From this equation and the recurrence formulae (8c) and (8d), it is readily shown that

$$c_{n_q, 1/4} = (m\omega / 2\hbar)^{1/4} \frac{(-dn_q / d\hat{\xi}_0)}{W_{n_q+1, -1/4}(\hat{\xi}_0)}, \quad (14)$$

$$c_{n_p, -1/4} = (m\omega / 2\hbar)^{1/4} \frac{(-dn_p / d\hat{\xi}_0)}{(n_p + 1) W_{n_p+1, 1/4}(\hat{\xi}_0)}. \quad (15)$$

Substituting from (11), (14) and (15) in (9), we obtain

$$(q|x|p) = (\hbar/2m\omega)^{1/2} \frac{2}{(n_q - n_p)^2 - 1} (dn_q/d\hat{\zeta}_0)^{1/2} (dn_p/d\hat{\zeta}_0)^{1/2}, \quad (16)$$

q and p being of different parities. The oscillator strengths f_{qp} are given by

$$\begin{aligned} f_{qp} &= \frac{2m}{\hbar^2} (E_p - E_q) |(q|x|p)|^2 \\ &= \frac{4(n_p - n_q)}{[(n_p - n_q)^2 - 1]^2} (dn_q/d\hat{\zeta}_0) (dn_p/d\hat{\zeta}_0). \end{aligned} \quad (17)$$

The f 's satisfy the Thomas-Kuhn rule

$$\sum_p f_{qp} = 1. \quad (18)$$

§ 3. Numerical calculations

In order to calculate the oscillator strengths f_{qp} , we need the values of n_q 's and $(dn_q/d\hat{\zeta}_0)$'s. These values have been computed in this paper for $\hat{\zeta}_0 \leq 3$. For $q > 5$, the values of n obtained from formula (7) are correct to 1 in 10^5 for $\hat{\zeta}_0 \leq 3$. For the lower levels (7) gives correct n_q 's for $\hat{\zeta}_0 \leq 1.5$ only, and for larger values of $\hat{\zeta}_0$ the n_q 's have to be evaluated directly from the equations $W_{n, \pm 1/4}(\hat{\zeta}_0) = 0$. Equation (7) is still useful in as much as it provides us with the rough values of the zeros of $W_{n, \pm 1/4}(\hat{\zeta}_0)$. The correct roots are computed by applying Newton's rule which states that if n' is an approximate root of $f(n) = 0$, a better value for the root is

$$n'' = n' - \frac{f(n')}{(df/dn)_{n'}}. \quad (19)$$

In our case the functions $W_{n, \pm 1/4}(\hat{\zeta}_0)$ correspond to $f(n)$. This procedure was adopted to calculate the first three roots of $W_{n, -1/4}(\hat{\zeta}_0)$ and the first two roots of $W_{n, +1/4}(\hat{\zeta}_0)$ for $1.5 < \hat{\zeta}_0 \leq 3$. Usually one Newton approximation was sufficient to give the roots correct to 1 in 10^5 . The derivatives $dn_q/d\hat{\zeta}_0$ were obtained from (7) for $q > 5$. For the lower levels, they were calculated from the relations (for $\hat{\zeta}_0 > 1.5$)

$$\left. \begin{aligned} dn_q/d\hat{\zeta}_0 &= \frac{(n_q + 1) W_{n_q + 1, 1/4}(\hat{\zeta}_0)}{(\partial W_{n, -1/4}/\partial n)_{n_q, \hat{\zeta}_0}}, \quad q = 1, 3, 5, \dots, \\ &= \frac{W_{n_q - 1, -1/4}(\hat{\zeta}_0)}{(\partial W_{n, 1/4}/\partial n)_{n_q, \hat{\zeta}_0}}, \quad q = 2, 4, 6, \dots, \end{aligned} \right\} \quad (20)$$

which follows from (13) with the help of recurrence formulae. (For $\hat{\zeta}_0 < 1.5$, (7) was again used). The calculation of the derivatives does not involve much additional labour since $\partial W_{n, \pm 1/4}/\partial n$ become available from the calculation of the n_q 's. The overall accuracy of the calculations is 1 in 10,000.

Table 1.
 $n_q = (E_q/\hbar\omega - 1/2)$ as function of $\xi_0 = l/l_0$ for $q=1, 2, 3, 4$ and 5 .

$\xi_0 \backslash q$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	61.186	14.926	6.3657	3.3762	2.0000	1.2603	0.82236	0.54624
2	246.24	61.196	26.941	14.966	9.4402	6.4555	4.6735	3.5350
3	554.66	138.30	61.213	34.247	21.784	15.0332	10.982	8.3736
4	986.46	246.25	109.191	61.236	39.059	27.0311	19.7993	15.1268
5	1541.6	385.04	170.877	95.935	61.266	42.454	31.1314	23.8041

$\xi_0 \backslash q$	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	0.36496	0.24300	0.16002	0.10355	0.065476	0.040247	0.023946	
2	2.7728	2.24566	1.87342	1.60747	1.41680	1.28064	1.18450	
3	6.6061	5.36323	4.4654	3.80463	3.3125	2.94404	2.66842	
4	11.9451	9.69148	8.0468	6.81909	5.88730	5.17207	4.61953	
5	18.8026	15.2475	12.6402	10.6806	9.17941	8.01278	7.0966	

Table 2.
 $(dn_q/d\xi_0)$ as function of $\xi_0 = l/l_0$ for $q=1, 2, 3, 4$ and 5 .

$\xi_0 \backslash q$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	616.84	77.080	22.807	9.5861	4.8699	2.7785	1.7093	1.1048
2	2467.4	308.37	91.300	38.440	19.598	11.254	6.9979	4.5974
3	5551.6	693.89	205.52	86.620	44.258	25.515	15.967	10.594
4	9869.6	1233.6	365.44	154.08	78.796	45.500	28.549	19.019
5	15421.3	1927.6	571.06	240.83	123.207	71.200	44.732	29.858

$\xi_0 \backslash q$	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	0.73657	0.49945	0.34041	0.23089	0.15444	0.10107	0.064267	
2	3.1381	2.1979	1.5639	1.1208	0.80288	0.57058	0.39945	
3	7.3345	5.2395	3.8265	2.8371	2.1210	1.5889	1.1853	
4	13.248	9.5454	7.0570	5.3186	4.0639	3.1326	2.4230	
5	20.859	15.093	11.2235	8.5265	6.5853	5.1497	4.0604	

Table III.

The oscillator strengths f_{1p} ($p=2, 4, 6\cdots$) as functions of $\xi_0=l/l_0$.
Initial level 1 (LOWEST)

ξ_0 p	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
2	0.9607	0.9607	0.9608	0.9610	0.9613	0.9620	0.9632	0.9649
4	0.0307	0.0307	0.0307	0.0305	0.0302	0.0296	0.0287	0.0274
6	0.0054	0.0054	0.0054	0.0054	0.0054	0.0053	0.0051	0.0049
8	0.0017	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0015
10	0.0007	0.0007	0.0007	0.0007	0.0007	0.0006	0.0006	0.0006
12	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
14	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0004
(to ∞ Total)								
Asymptotic Formula	$6.5 \times p^{-4}$	$6.5 \times p^{-4}$	$6.5 \times p^{-4}$	$6.4 \times p^{-4}$	$6.4 \times p^{-4}$	$6.3 \times p^{-4}$	$6.2 \times p^{-4}$	$5.9 \times p^{-4}$
Sum	1.0000	1.0000	1.0000	1.0000	1.00000	0.9999	1.0000	1.0000

ξ_0 p	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
2	0.9673	0.9702	0.9738	0.9778	0.9823	0.9864	0.9904	
4	0.0255	0.0231	0.0202	0.0170	0.0135	0.0101	0.0071	
6	0.0046	0.0042	0.0038	0.0032	0.0027	0.0021	0.0015	
8	0.0014	0.0013	0.0012	0.0010	0.0008	0.0007	0.0005	
10	0.0006	0.0005	0.0005	0.0004	0.0003	0.0003	0.0002	
12	0.0003	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001	
14	0.0004	0.0004	0.0004	0.0003	0.0003	0.0002	0.0002	
(to ∞ Total)								
Asymptotic Formula	$5.6 \times p^{-4}$	$5.2 \times p^{-4}$	$4.8 \times p^{-4}$	$4.2 \times p^{-4}$	$3.6 \times p^{-4}$	$2.9 \times p^{-4}$	$2.3 \times p^{-4}$	
Sum	1.0001	0.9999	1.0001	0.9999	1.0001	0.9999	1.0000	

Table IV.

The oscillator strengths f_{2p} ($p=1, 3, 5\cdots$) as functions of $\xi_0=l/l_0$.
Initial level =2.

ξ_0 p	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	-0.9607	-0.9607	-0.9608	-0.9610	-0.9613	-0.9620	-0.9632	-0.9649
3	1.8677	1.8677	1.8678	1.8681	1.8690	1.8704	1.8730	1.8766
5	0.0700	0.0700	0.0699	0.0698	0.0694	0.0688	0.0678	0.0663
7	0.0139	0.0139	0.0139	0.0139	0.0138	0.0137	0.0135	0.0132
9	0.0046	0.0046	0.0046	0.0046	0.0046	0.0045	0.0045	0.0044
11	0.0020	0.0020	0.0020	0.0020	0.0019	0.0019	0.0019	0.0019
13	0.0026	0.0026	0.0026	0.0026	0.0025	0.0025	0.0025	0.0025
(to ∞)								
Asymptotic Formula	$25.9 \times p^{-4}$	$25.9 \times p^{-4}$	$25.9 \times p^{-4}$	$25.9 \times p^{-4}$	$25.7 \times p^{-4}$	$25.5 \times p^{-4}$	$25.2 \times p^{-4}$	$24.7 \times p^{-4}$
Sum	1.0001	1.0001	1.0000	1.0000	0.9999	0.9998	1.0000	1.0000

ξ_0 p	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	-0.9673	-0.9702	-0.9738	-0.9778	-0.9822	-0.9864	-0.9904	
3	1.8818	1.8888	1.8973	1.9076	1.9195	1.9322	1.9451	
5	0.0641	0.0611	0.0572	0.0524	0.0467	0.0403	0.0333	
7	0.0128	0.0122	0.0115	0.0106	0.0095	0.0083	0.0070	
9	0.0042	0.0041	0.0038	0.0035	0.0032	0.0028	0.0024	
11	0.0018	0.0017	0.0016	0.0015	0.0014	0.0012	0.0010	
13	0.0024	0.0023	0.0021	0.0020	0.0018	0.0016	0.0014	
(to ∞) Asymptotic Formula	$24.0 \times p^{-4}$	$23.1 \times p^{-4}$	$21.9 \times p^{-4}$	$20.3 \times p^{-4}$	$18.5 \times p^{-4}$	$16.5 \times p^{-4}$	$14.2 \times p^{-4}$	
Sum	0.9998	1.0000	0.9998	0.9998	0.9999	1.0000	0.9998	

Table V.
The oscillator strengths f_{3p} ($p=2, 4, 6, \dots$) as functions of $\xi_0=l/l_0$.
Initial level=3.

ξ_0 p	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
2	-1.8676	-1.8676	-1.8678	-1.8681	-1.8690	-1.8704	-1.8730	-1.8766
4	2.7226	2.7227	2.7228	2.7232	2.7244	2.7264	2.7301	2.7353
6	0.1067	0.1067	0.1067	0.1065	0.1063	0.1058	0.1050	0.1038
8	0.0224	0.0224	0.0224	0.0224	0.0224	0.0223	0.0221	0.0219
10	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0076	0.0075
12	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0033
14	0.0048	0.0048	0.0048	0.0048	0.0048	0.0047	0.0047	0.0047
(to ∞) Total	$58.3p^{-1} \times$	$58.3p^{-1} \times$	$58.3p^{-1} \times$	$58.3p^{-1} \times$	$58.1p^{-1} \times$	$57.9p^{-1} \times$	$57.6p^{-1} \times$	$57.0p^{-1} \times$
Asymptotic	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$	$[1+27p^{-2}]$
Formula								
Sum	1.0000	1.0001	1.0000	1.0001	1.0000	0.9999	0.9999	0.9999

ξ_0 p	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
2	-1.8818	-1.8888	-1.8973	-1.9076	-1.9195	-1.9322	-1.9451	
4	2.7430	2.7532	2.7660	2.7821	2.8012	2.8229	2.8462	
6	0.1021	0.0996	0.0964	0.0921	0.0867	0.0801	0.0722	
8	0.0215	0.0210	0.0203	0.0195	0.0184	0.0170	0.0155	
10	0.0074	0.0073	0.0070	0.0068	0.0064	0.0059	0.0054	
12	0.0033	0.0032	0.0031	0.0030	0.0028	0.0026	0.0024	
14	0.0046	0.0045	0.0044	0.0042	0.0040	0.0037	0.0034	
(to ∞) Total	$56.2p^{-1} \times$	$55.1p^{-1} \times$	$53.6p^{-1} \times$	$51.5p^{-1} \times$	$49.0p^{-1} \times$	$45.8p^{-1} \times$	$42.0p^{-1} \times$	
Asymptotic	$[1+27p^{-2}]$	$[1+26p^{-2}]$	$[1+26p^{-2}]$	$[1+26p^{-2}]$	$[1+25p^{-2}]$	$[1+24p^{-2}]$	$[1+24p^{-2}]$	
Formula								
Sum	1.0001	1.0000	0.9999	1.0001	1.0000	1.0000	1.0000	

Table I gives the first five energy levels as functions of ξ_0 from $\xi_0=0$ to $\xi_0=3$ at sub-intervals of 0.2. The higher levels for the same interval ($0 < \xi_0 \leq 3$) can be computed from (7) and are consequently not tabulated. Figure 1 is a plot of the first three energy levels against ξ_0 . Table II gives $dn_q/d\xi_0$ as a function of ξ_0 for $q=1, 2, 3, 4$, and 5. The oscillator strengths f_{1p} , f_{2p} and f_{3p} are tabulated (tables III, IV, and V). Asymptotic expressions for the f_{qp} , when p is large, are obtained from (17) by assuming $(n_p - n_i)^2 \gg 1$ and substituting for $dn_p/d\xi_0$ the approximate value

$$(dn_p/d\xi_0)_{p \text{ large}} = -\frac{\pi^2 p^2}{2\xi_0^3} \left(1 - \frac{\xi_0^4}{3\pi^2 q^2}\right).$$

This gives

$$f_{qp} (p \text{ large}) = \frac{128 \xi_0^3}{\pi^4} \frac{dn_q}{d\xi_0} \frac{1}{p^4} \left[1 + \frac{12\xi_0^2}{\pi^2 p^2} \left(n_q + \frac{1}{2} - \frac{\xi_0^2}{9}\right)\right], \quad (21)$$

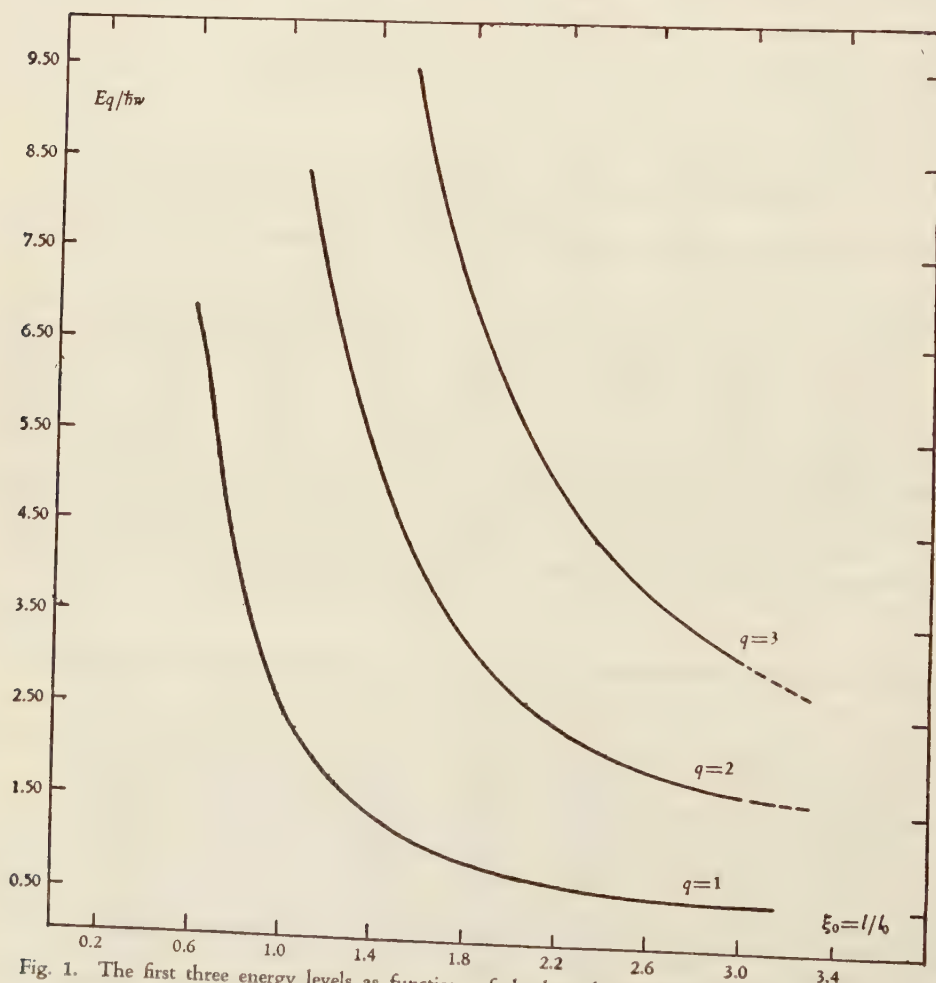


Fig. 1. The first three energy levels as functions of the boundary parameter $\xi_0 = l/l_0$.

$dn_q/d\hat{\xi}_0$ and n_q being obtained directly from the tables. The asymptotic formulae for f_{1p} , f_{2p} , and f_{3p} are listed in the corresponding tables. The Thomas-Kuhn rule, $\sum_p f_{qp} = 1$, is seen to be satisfied in all the three cases and this serves as a check on the calculations. The oscillator strengths f_{12} and f_{14} are plotted against $\hat{\xi}_0$ in Fig. 2. It will be noted that while the energy levels become increasingly sensitive to variations in $\hat{\xi}_0$ as the latter decreases, the oscillator strengths become sensitive to changes in $\hat{\xi}_0$ for relatively large values of $\hat{\xi}_0$ ($\hat{\xi}_0 > 1.5$). In fact, for $0 < \hat{\xi}_0 < 1$, the results are almost the same as those for a free particle enclosed in a box. The effects of the potential begin to show up as $\hat{\xi}_0$ increases beyond 1, and for $\hat{\xi}_0 > 3$, the bounded oscillator behaves more or less like a free oscillator (in the first few energy states.)

In conclusion, we wish to express our thanks to Prof. D. S. Kothari for valuable advice, and to Prof. F. C. Auluck for continued interest and guidance during the course of this work.

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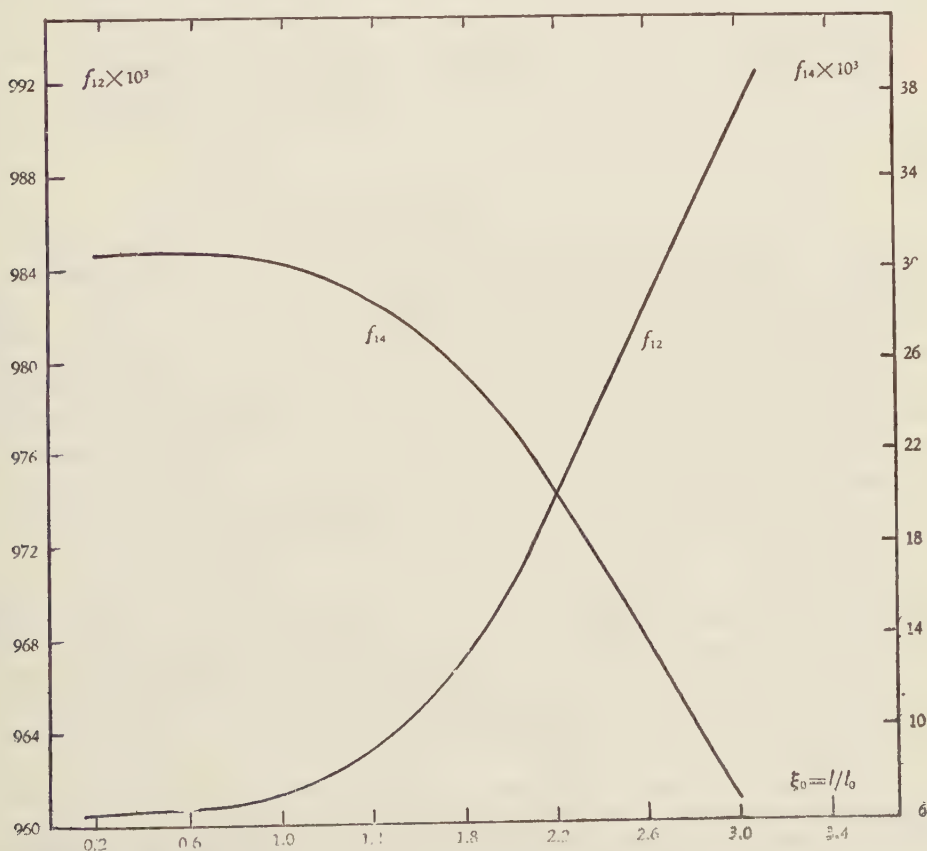


Fig. 2. f_{12} and f_{14} as functions of the boundary parameter $\hat{\xi}_0 = l/l_0$.

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On the Jauch Field

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As an approach to theoretical understanding of neutral particles a new kind of field recently suggested by Jauch, referred to as Jauch field below, is discussed in detail. Lagrangian and Hamiltonian formulation of Jauch field is obtained, which leads to the required anticommutators given by Jauch. The Jauch field, in appearance, shows the intermediate nature between Dirac and Majorana fields, but it is shown that contrary to Jauch's conclusion it can be a charged field with gauge invariant electromagnetic interaction. Gauge and Lorentz transformations of this field are examined, which again lead to quite contrary results to Jauch. Interaction terms are introduced and as an example the application to β -decay neutrino is discussed.

It is concluded that Jauch field is essentially equivalent to Dirac field and so in general the problem of this field should be considered from the view-point of the structure of interactions of Dirac field. It is pointed out, however, that in some circumstances the concept of Jauch field may more suitably describe the physical reality than that of Dirac field. In the Appendix some remarks are given on the quantization of Majorana field.

§ 1. Introduction and summary

Recent experimental observations have made clear the fact that in addition to several charged particles there exist in nature various kinds of neutral particles, that is, neutrino, π^0 -meson, Λ^0 -meson, θ^0 -meson and so on. On the other hand, most of the theoretical studies done in the past were mainly concerned with properties of charged particles and it seems for us that there remain many problems to be investigated concerning the theoretical understanding of neutral particles. In view of their physical reality we believe that we are now in a position to face these problems not only from the academic interests but also from the more realistic standpoint. The investigations recently made by Sakata¹⁾ and Pais-Gell-Mann²⁾ in connection with neutrino and θ^0 -meson are very interesting in this sense.

Now, let us restrict our consideration to neutral fermions. It has been well known that the theoretical formulation allows two kinds of possibilities, namely, Dirac and Majorana particles.*¹⁾ But, in connection with the problem of spinor field quantization, Jauch³⁾ has recently suggested a new possibility of field, which in appearance has characteristic features intermediate between Dirac and Majorana cases. Thus, there immediately arises a question as to whether this field in fact corresponds to a new physical reality or

*) Corresponding to the fermion case, there are also two kinds of neutral bosons, so to speak, Dirac boson and Majorana boson. The former is described by a complex field or two real fields, while the latter by a real field. In this paper we shall be concerned with the fermion case only, but we can easily see that the same argument may also apply to the boson case.

not. In view of theoretical approach to the problem of neutral particles we aim in this paper to make clear the physical meanings of this field and further to examine in what way such a field can be applied to the concrete example of neutral particles found in nature.

Jauch has given the following generalized anticommutators for a spinor field $\psi(x)$;

$$\begin{aligned}\{\psi(x), \psi^*(x')\} &= \delta^3(x-x'), \\ \{\psi(x), \psi(x')\} &= \rho \delta^3(x-x'), \quad 0 \leq \rho \leq 1.\end{aligned}\tag{A}^*)$$

In this paper, the field which satisfies the above relations, will be called "Jauch field" and the suffix J be attached together with the corresponding ρ -value, when necessary. For special values of parameter $\rho=0$ or $\rho=1$, Jauch field reduces to Dirac or Majorana field, respectively. Jauch's main conclusions may be summarized as follows: i) Jauch field for different ρ -values are not equivalent, since they can never be combined with each other through a canonical transformation, ii) this field is necessarily neutral since in the case $\rho=1$ the relation (A) can not remain invariant under the gauge transformation, iii) this field has C, D type transformation properties for space reflection and A, B type for time reversal.** In the following sections we shall examine Jauch's conclusion and show that they are not necessarily correct.

In § 2, we shall give the Lagrangian of Jauch field, from which we can obtain the anticommutator (A) by following the usual prescription of field quantization. The Lagrangian for $\rho \neq 0, 1$, however, is shown to be just equal to that of $\rho=0$, that is, the usual Dirac field Lagrangian. Thus, contrary to the Jauch's result i), we can derive the conclusion that as far as the free field is concerned the Jauch field $\mathcal{U}_J(\rho \neq 1)$ is merely a different representation of Dirac field. On the other hand, this Lagrangian becomes singular at $\rho=1$ and is found to be equal to the Lagrangian of two Majorana fields.

In § 3 the free field properties are further analysed in detail. We show that the particle picture is very obscure in the representation of Jauch field and the interpretation as the field quanta is only possible when going over into the Dirac representation. We can also introduce the gauge transformation, which has somewhat different from the usual one, but keeps the Lagrangian invariant and further show that just as in the case of Dirac field the Jauch field $\psi_J(\rho \neq 1)$ has four possible transformation types A, B, C and D under space inversion and time reversal. These results are quite contrary to Jauch's conclusions ii) and iii).

In § 4 general considerations are given on the interaction of the Jauch field. For the Jauch field $\psi_J(\rho \neq 1)$ the gauge invariant electromagnetic interaction is introduced, that is, this field can describe the charged particle. Detailed considerations are given further concerning the application of neutral Jauch field to the β -decay neutrino. We can also generally

*) Throughout this paper we shall use the Majorana representation for γ -matrices. * and \sim mean Hermitian conj. and transpose, respectively.

) As to the classification of transformation types A, B, C and D, refer to the papers: C. N. Yang and J. Tiomno, Phys. Rev. **79 (1950), 495; E. R. Caianiello, Nuovo Cimento **8** (1951), 749, **9** (1952), 336.

show that a Jauch field in interaction can be reduced to a Dirac field with generalized interactions, their coupling constants being dependent on the ρ -values. Consequently, we are led to conclude that at the present stage of the theory the problem of Jauch field should be discussed from the view-point of the structure of interaction of Dirac field. We shall suggest, however, the possibility that in some cases the concept of Jauch field may give the more suitable image than that of Dirac field. This will be elucidated concerning an example of β -decay neutrino. In the Appendix some remarks are given as to the quantization of the Majorana field.

§ 2. Lagrangian and Hamiltonian formalism

The Jauch field $\psi_J(\mathbf{x}; \rho)$, which satisfies the anticommutation relation (A), can be decomposed in the usual way into two Hermitian spinor fields $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ in the form

$$\left. \begin{aligned} \psi_J(\mathbf{x}; \rho) &= c_1(\rho)\phi_1(\mathbf{x}) + ic_2(\rho)\phi_2(\mathbf{x}), \\ \psi_J^*(\mathbf{x}; \rho) &= c_1(\rho)\phi_1(\mathbf{x}) - ic_2(\rho)\phi_2(\mathbf{x}), \end{aligned} \right\} \quad (2.1)$$

or in matrix notation

$$\Psi_J(\mathbf{x}; \rho) = C\Phi(\mathbf{x}), \quad (2.1)'$$

where

$$\Psi_J(\mathbf{x}; \rho) \equiv \begin{pmatrix} \psi_J(\mathbf{x}; \rho) \\ \psi_J^*(\mathbf{x}; \rho) \end{pmatrix}, \quad \Phi(\mathbf{x}) \equiv \begin{pmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \end{pmatrix}, \quad C \equiv \begin{pmatrix} c_1(\rho), & ic_2(\rho) \\ c_1(\rho), & -ic_2(\rho) \end{pmatrix}.$$

In the above, without loss of generality we can regard $c_i(\rho)$'s as real functions of ρ .

Now, as the general Lagrangian to describe the behavior of two spinor fields ϕ_1 and ϕ_2 we can assume the following one

$$L_0 = -\frac{1}{2} \bar{\Phi}(\mathbf{x}) A \begin{pmatrix} f_1(\rho) & 0 \\ 0 & f_2(\rho) \end{pmatrix} \Phi(\mathbf{x}), \quad (2.2)$$

where $A = \gamma_4(\gamma_\mu \partial_\mu + \kappa)$ and $f_i(\rho)$ ($i=1, 2$) are real from the Hermiticity requirement. At first sight the Lagrangian (2.2) appears to lack the generality and so we can take the more general one with the cross terms

$$L_0' = -\frac{1}{2} \bar{\Phi}(\mathbf{x}) \begin{pmatrix} f_1 A & g A \\ -g A^T & f_2 A \end{pmatrix} \Phi(\mathbf{x}), \quad (2.3)$$

where $A^T = (-\gamma_\mu^T \partial_\mu + \kappa) \gamma_4^T = -A$ in the Majorana representation. But, even if we start from (2.3), we can easily see that it is brought into the form of (2.2) by performing a real linear transformation of ϕ_i 's. Thus, it is sufficient, for our further consideration, to take the Lagrangian as in (2.2).

We shall now proceed to the quantization of Jauch field $\Psi_J(\mathbf{x}; \rho)$ starting with the Lagrangian (2.2). This can be achieved by quantizing ϕ_i 's with the orthodox method, but some cautions are necessary¹⁾ when picking out the canonical variables since ϕ_i 's are real (Hermitian) field quantities. As shown in the Appendix ϕ_i itself cannot be the canonical variable, but the relation between them is given by

$$\phi_i = \begin{pmatrix} \phi_i^1 \\ \phi_i^2 \\ \phi_i^3 \\ \phi_i^4 \end{pmatrix} = \frac{1}{\sqrt{2f_i}} \begin{pmatrix} I_2 & -iI_2 \\ iI_2 & -I_2 \end{pmatrix} \begin{pmatrix} \hat{\xi}_i^1 \\ \hat{\xi}_i^2 \\ \pi_i^1 \\ \pi_i^2 \end{pmatrix}, \quad i=1, 2, \quad (2.4)$$

where I_2 is 2×2 unit matrix and the superscript μ ($\mu=1, 2, 3, 4$) attached to ϕ_i denotes the spinor suffix. $\hat{\xi}_i$'s and π_i 's ($i=1, 2$) are canonical variables, which satisfy the anti-commutation relations

$$\left. \begin{aligned} \{\hat{\xi}_i^a(\mathbf{x}), \pi_j^b(\mathbf{x}')\} &= i\delta_{ij}\delta_{ab}\delta^3(\mathbf{x}-\mathbf{x}'), \\ \{\hat{\xi}_i^a(\mathbf{x}), \hat{\xi}_j^b(\mathbf{x}')\} &= 0, \quad \{\pi_i^a(\mathbf{x}), \pi_j^b(\mathbf{x}')\} = 0, \\ i, j &= 1, 2; \quad a, b = 1, 2. \end{aligned} \right\} \quad (2.5)$$

From (2.4) and (2.5) those of ϕ_i 's take the form

$$\{\phi_i^\mu(\mathbf{x}), \phi_j^\nu(\mathbf{x}')\} = \frac{1}{f_i} \delta_{ij} \partial_{\mu\nu} \delta^3(\mathbf{x}-\mathbf{x}'). \quad (2.6)$$

Consequently, the anticommutators of the Jauch field given by (2.1) finally read

$$\left. \begin{aligned} \{\psi_J(\mathbf{x}), \tilde{\psi}_J^*(\mathbf{x}')\} &= (c_1^2/f_1 + c_2^2/f_2) \delta^3(\mathbf{x}-\mathbf{x}'), \\ \{\psi_J(\mathbf{x}), \tilde{\psi}_J(\mathbf{x}')\} &= (c_1^2/f_1 - c_2^2/f_2) \delta^3(\mathbf{x}-\mathbf{x}'). \end{aligned} \right\} \quad (2.7)$$

In order that the above ψ_J coincides with $\psi_J(\mathbf{x}; \rho)$ we see from the comparison of (2.7) with (A) that c_i 's must satisfy the relations

$$\left. \begin{aligned} c_1^2/f_1 + c_2^2/f_2 &= 1 \\ c_1^2/f_1 - c_2^2/f_2 &= \rho \end{aligned} \right\} \quad \text{or} \quad \left. \begin{aligned} c_1^2 &= (1+\rho)/2 \cdot f_1 \\ c_2^2 &= (1-\rho)/2 \cdot f_2 \end{aligned} \right\}. \quad (2.8)^*$$

As seen from the above relation, the signs of c_i 's are indefinite but they are quite trivial, and so we shall take them as positive in the following.

From (2.1)', (2.4) and (2.8) the Jauch field $\Psi_J(\mathbf{x}; \rho)$ can be written in terms of canonical variables in the following way;

$$\Psi_J(\mathbf{x}; \rho) = \begin{pmatrix} \sqrt{\frac{1+\rho}{2}} I_4 & i\sqrt{\frac{1-\rho}{2}} I_4 \\ \sqrt{\frac{1+\rho}{2}} I_4 & -i\sqrt{\frac{1-\rho}{2}} I_4 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \pi_1 \\ \pi_2 \end{pmatrix}, \quad (2.9)$$

* As shown by Jauch we see that $|\rho| \leq 1$, and so $f_i \geq 0$ from (2.8). The latter relation is required from (2.6), too.

where $K = \begin{pmatrix} I_2 & -iI_2 \\ iI_2 & I_2 \end{pmatrix}$, $\xi_i = \begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix}$, $\pi_i = \begin{pmatrix} \pi_i^1 \\ \pi_i^2 \end{pmatrix}$, ($i=1, 2$) and I_4 is 4×4 unit matrix.

When writing the Jauch field in terms of ξ_i 's and π_i 's, we find that there exists another Jauch field Ψ_J , which satisfies the relation (A) but is different from (2.9). This field Ψ_J is just given by the defining equation (2.1) but with the exchange of roles of ϕ_1 and ϕ_2 . That is

$$\Psi_J(x) = C' \Phi(x), \quad (2.10)$$

with

$$C' = \begin{pmatrix} ic_1' & c_2' \\ -ic_1' & c_2' \end{pmatrix}.$$

The relation corresponding to (2.8) becomes in this case

$$\left. \begin{aligned} c_1'^2/f_1 + c_2'^2/f_2 &= 1 \\ -c_1'^2/f_1 + c_2'^2/f_2 &= \rho \end{aligned} \right\} \quad \text{or} \quad \left. \begin{aligned} c_1'^2 &= (1-\rho)/2 \cdot f_1 \\ c_2'^2 &= (1+\rho)/2 \cdot f_2 \end{aligned} \right\}. \quad (2.11)$$

The relation between Ψ_J and canonical variables is given by

$$\Psi_J(x; \rho) = \begin{pmatrix} i\sqrt{\frac{1-\rho}{2}} I_4 & \sqrt{\frac{1+\rho}{2}} I_4 \\ -i\sqrt{\frac{1-\rho}{2}} I_4 & \sqrt{\frac{1+\rho}{2}} I_4 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \xi_1 \\ \pi_1 \\ \xi_2 \\ \pi_2 \end{pmatrix}. \quad (2.12)$$

Next, we shall rewrite the Lagrangian (2.2) in terms of $\Psi_J(x; \rho)$ or $\Psi_J(x; \rho)$, using (2.1) and (2.8) or (2.10) and (2.11).

$$\begin{aligned} L_0 &= -\frac{1}{2} \tilde{\Psi}_J(x; \rho) \tilde{C}^{-1} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} C^{-1} \Lambda \Psi_J(x; \rho) \\ &= -\frac{1}{2(1-\rho^2)} \tilde{\Psi}_J(x; \rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \Lambda \Psi_J(x; \rho) \end{aligned} \quad (2.13)^*$$

or

$$\begin{aligned} L_0 &= -\frac{1}{2} \tilde{\Psi}_J(x; \rho) \tilde{C}'^{-1} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \tilde{C}'^{-1} \Lambda \Psi_J(x; \rho) \\ &= -\frac{1}{2(1-\rho^2)} \tilde{\Psi}_J(x; \rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \Lambda \Psi_J(x; \rho), \end{aligned} \quad (2.13)'$$

where

$$C = \begin{pmatrix} \sqrt{\frac{1+\rho}{2}} \sqrt{f_1} & i\sqrt{\frac{1-\rho}{2}} \sqrt{f_2} \\ \sqrt{\frac{1+\rho}{2}} \sqrt{f_1} & -i\sqrt{\frac{1-\rho}{2}} \sqrt{f_2} \end{pmatrix}, \quad C' = \begin{pmatrix} i\sqrt{\frac{1-\rho}{2}} \sqrt{f_1} & \sqrt{\frac{1+\rho}{2}} \sqrt{f_2} \\ -i\sqrt{\frac{1-\rho}{2}} \sqrt{f_1} & \sqrt{\frac{1+\rho}{2}} \sqrt{f_2} \end{pmatrix}.$$

* It is to be noticed that the Lagrangian (2.13) is of the most general form built with the spinor field quantity Ψ_J .

Conversely, starting with this Lagrangian (2.13) or (2.13)' we can directly quantize the field Ψ_J or $\Psi_{J'}$ by the Schwinger's method,⁵⁾ for instance. This prescription just corresponds to follow the reversed procedure as developed above and as a matter of course we can obtain again the anticommutator (A) as required*. This fact shows that (2.13) (or (2.13)') is the one and only one possible Lagrangian for a Jauch field $\Psi_J(\mathbf{x}; \rho)$ (or $\Psi_{J'}(\mathbf{x}; \rho)$).

In this connection it is to be noticed that the Lagrangian (2.13) becomes singular at $\rho=1$ and so loses its physical meaning. In this case, however, it is better to rewrite it into an alternative form with the use of both ϕ_J and $\phi_{J'}$.

Namely,

$$L_0 = \rho/2 \cdot \{ \tilde{\phi}_J(\mathbf{x}; \rho) A \phi_J(\mathbf{x}; \rho) + \tilde{\phi}_{J'}(\mathbf{x}; \rho) A \phi_{J'}(\mathbf{x}; \rho) \} \\ - i\sqrt{1-\rho^2} \{ \tilde{\phi}_J(\mathbf{x}; \rho) A \phi_{J'}(\mathbf{x}; \rho) + \tilde{\phi}_{J'}(\mathbf{x}; \rho) A \phi_J(\mathbf{x}; \rho) \}. \quad (2.13)''$$

In the case $\rho \neq 1$, $\Psi_J(\mathbf{x}; \rho)$ and $\Psi_{J'}(\mathbf{x}; \rho)$ are not mutually independent, but they are connected by the relation

$$\left. \begin{aligned} \Psi_{J'}(\mathbf{x}; \rho) &= \frac{-i}{\sqrt{1-\rho^2}} \begin{pmatrix} \rho & -1 \\ 1 & -\rho \end{pmatrix} \Psi_J(\mathbf{x}; \rho), \\ \text{or what is the same thing,} \\ \Psi_J(\mathbf{x}; \rho) &= \frac{-i}{\sqrt{1-\rho^2}} \begin{pmatrix} \rho & -1 \\ 1 & -\rho \end{pmatrix} \Psi_{J'}(\mathbf{x}; \rho), \end{aligned} \right\} \quad (2.14)$$

and moreover anticommutators between them are

$$\left. \begin{aligned} \{ \phi_J(\mathbf{x}; \rho), \phi_{J'}(\mathbf{x}'; \rho) \} &= i\sqrt{1-\rho^2} \delta^3(\mathbf{x}-\mathbf{x}'), \\ \{ \phi_J(\mathbf{x}; \rho), \phi_{J'}^*(\mathbf{x}'; \rho) \} &= 0. \end{aligned} \right\} \quad (2.15)$$

We are now in a position to discuss to some extent the fundamental features of Jauch fields. The cases $\rho=1$ and $\rho \neq 1$ are to be considered separately.

i) The case $\rho=1$.

In this case the matrix which connects Ψ_J (or $\Psi_{J'}$) with ξ 's and π 's in (2.9) (or (2.12)) becomes singular matrix and its inverse does not exist. This means that $\Psi_J(\mathbf{x}; \rho=1)$ (or $\Psi_{J'}(\mathbf{x}; \rho=1)$) is equivalent to the variables (ξ_1, π_1) (or (ξ_2, π_2)), so that the Lagrangian can not be written in terms of Ψ_J or $\Psi_{J'}$ only. Between two components of Ψ_J or $\Psi_{J'}$ there exist the relations

* If we apply the Umezawa-Takahashi's formula⁶⁾ to the Lagrangian (2.13) $L = \frac{1}{2} \tilde{\Psi}(\mathbf{x}; \rho) A \Psi_J(\mathbf{x}; \rho)$ with $A = -\frac{1}{1-\rho^2} \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} A$, the covariant anticommutator is immediately obtained by $\{ \Psi_J(\mathbf{x}; \rho), \tilde{\Psi}_J(\mathbf{x}'; \rho) \} = iR\Delta(\mathbf{x}-\mathbf{x}')$, where R , defined by $RA = \square - \kappa^2$, is equal to $-\begin{pmatrix} \rho & 1 \\ 1 & \rho \end{pmatrix} (\gamma\partial - \kappa) \gamma_4$. This leads to (A) or (3.18) below. The quantization of (2.13) was also discussed in detail by Y. Takahashi.⁷⁾

$$\left. \begin{aligned} \psi_J(x; \rho=1) &= \psi_J^*(x; \rho=1) = f_1 \psi_1(x), \\ \psi_{J'}(x; \rho=1) &= \psi_{J'}^*(x; \rho=1) = f_2 \psi_2(x), \end{aligned} \right\} \quad (2.16)$$

and further from (2.15)

$$\{\psi_J(x; \rho=1), \psi_{J'}(x'; \rho=1)\} = 0, \quad (2.17)$$

that is, ψ_J and $\psi_{J'}$ are mutually independent. From (2.16), (2.17) (2.13)'' it is evident $\mathcal{V}_J(x; \rho=1)$ and $\mathcal{V}_{J'}(x; \rho=1)$ become two mutually independent and Majorana fields, respectively.

ii) The case $\rho \neq 1$.

In this case the matrix which connects \mathcal{V}_J (or $\mathcal{V}_{J'}$) with ξ 's and π 's in (2.9) (or (2.12)), is non-singular and $\mathcal{V}_J(x; \rho \neq 1)$ or $\mathcal{V}_{J'}(x; \rho \neq 1)$ is equivalent to the variables $(\xi_1, \pi_1, \xi_2, \pi_2)$. Thus, we may say that as far as $\rho \neq 1$, all the Jauch fields are mutually equivalent and to be regarded merely as different representation describing the behavior of the same system. In other words, all $\mathcal{V}_J(x; \rho \neq 1)$'s describe the behavior of the Dirac field since $\mathcal{V}_J(x; \rho=0)$ corresponds to a Dirac field as seen from (2.13). This fact may also be understood from the circumstance that the Lagrangians (2.13) with $\rho \neq 1$ are all equivalent to the Lagrangian (2.2) and further the latter can be reduced to the Dirac field Lagrangian when renormalizing $\phi_i(x)$'s by the factor $\sqrt{f_i(\rho)}$'s.

Our conclusion makes a remarkable contrast to Jauch's i) (§ 1). He has based his argument on the point that the anticommutators (A) are invariant under canonical transformations and therefore Jauch fields with different ρ -values are mutually inequivalent. But, as shown in (2.12) $\mathcal{V}_J(x; \rho \neq 1)$'s themselves can not be canonical variables and the transformation connecting two Jauch fields corresponding to different ρ -values is not a canonical one, leading therefore to different anticommutators.

Summarizing the content of this section we may say as follows: as far as the free Jauch field is concerned, all the Jauch fields with $\rho \neq 1$ are equivalent to a Dirac field, while the field with $\rho=1$ is nothing but a Majorana field.

§ 3. Free Jauch field

In this section we shall consider the energy-momentum and various transformation properties of Jauch field. In the case $\rho=1$ the physical meaning of this field is already evident and so we shall restrict the following consideration to the case $\rho \neq 1$ only.

Canonical energy-momentum tensor is derived in the usual way from the Lagrangian (2.13)

$$T_{\mu\nu} = \frac{1}{2(1-\rho^2)} \tilde{\Psi}_J(\rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \{\gamma_4 \gamma_\mu \partial_\nu - \gamma_4 (\gamma \partial + \kappa) \delta_{\mu\nu}\} \Psi_J(\rho), \quad (3.1)$$

which gives energy and momentum expressions

$$H = - \int d^3x T_{44} = \int d^4x \frac{1}{2(1-\rho^2)} \tilde{\Psi}_J(\rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \gamma_4 (\gamma_k \partial_k + \kappa) \Psi_J(\rho), \quad (3.2)$$

$$G_k = \int d^3x \frac{1}{i} T_{4k} = \int d^3x \frac{1}{2(1-\rho^2)} \tilde{\Psi}_J(\rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \frac{\partial}{\partial x_k} \Psi_J(\rho). \quad (3.3)$$

Now, as the Jauch field satisfies the Dirac's wave equation, it can be expanded in the following way;

$$\left. \begin{aligned} \psi_J(x; \rho) &= \frac{1}{\sqrt{V}} \sum_k \sum_{r=1,2} \left\{ a_r(\mathbf{k}; \rho) u_r(\mathbf{k}) e^{i(kx - k_0 t)} + b_r^*(\mathbf{k}; \rho) v_r(\mathbf{k}) e^{-i(kx - k_0 t)} \right\}, \\ \psi_J^*(x; \rho) &= \frac{1}{\sqrt{V}} \sum_k \sum_{r=1,2} \left\{ a_r^*(\mathbf{k}; \rho) u_r^*(\mathbf{k}) e^{-i(kx - k_0 t)} + b_r(\mathbf{k}; \rho) v_r^*(\mathbf{k}) e^{i(kx - k_0 t)} \right\}, \end{aligned} \right\} \quad (3.4)$$

where u and v are usual Dirac eigen-spinors. Inserting (3.4) into (3.2) and (3.3) we obtain the following expressions;

$$H = \sum_{\mathbf{k}, r} E_k \left[\frac{1}{1-\rho^2} \left\{ N_r^+(\mathbf{k}; \rho) + N_r^-(\mathbf{k}; \rho) - \rho (a_r^*(\mathbf{k}; \rho) b_r(\mathbf{k}; \rho) + b_r^*(\mathbf{k}; \rho) a_r(\mathbf{k}; \rho)) \right\} - 1 \right], \quad (3.5)$$

$$G_k = \sum_{\mathbf{k}, r} k_k \left[\frac{1}{1-\rho^2} \left\{ N_r^+(\mathbf{k}; \rho) + N_r^-(\mathbf{k}; \rho) - \rho (a_r^*(\mathbf{k}; \rho) b_r(\mathbf{k}; \rho) + b_r^*(\mathbf{k}; \rho) a_r(\mathbf{k}; \rho)) \right\} - 1 \right], \quad (3.6)$$

where

$$E_k = \sqrt{k^2 + \kappa^2}, \quad (3.7)$$

and

$$N_r^+(\mathbf{k}; \rho) = b_r^*(\mathbf{k}; \rho) b_r(\mathbf{k}; \rho), \quad N_r^-(\mathbf{k}; \rho) = a_r^*(\mathbf{k}; \rho) a_r(\mathbf{k}; \rho). \quad (3.8)$$

As shown by Jauch, Fourier coefficients a_r 's, b_r 's satisfy the following anticommutation relations:

$$\left. \begin{aligned} \{a_r(\mathbf{k}; \rho), a_s(\mathbf{k}'; \rho)\} &= \{b_r(\mathbf{k}; \rho), b_s(\mathbf{k}'; \rho)\} = 0, \\ \{a_r(\mathbf{k}; \rho), a_s^*(\mathbf{k}'; \rho)\} &= \{b_r(\mathbf{k}; \rho), b_s^*(\mathbf{k}'; \rho)\} = \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}, \\ \{a_r(\mathbf{k}; \rho), b_s^*(\mathbf{k}'; \rho)\} &= \rho \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}, \\ \{a_r(\mathbf{k}; \rho), b_s(\mathbf{k}'; \rho)\} &= 0. \end{aligned} \right\} \quad (3.9)$$

As is seen above, except the case $\rho=0$, a_r 's and b_r 's do not satisfy Jordan-Wigner's canonical relations and so $N_r^\pm(\mathbf{k}; \rho)$'s do not have integer eigenvalues and moreover we have the relation

$$\{N_r^-(\mathbf{k}; \rho), N_r^+(\mathbf{k}'; \rho)\} = \rho \partial_{r_s} \partial_{k_s} \{a_r^*(\mathbf{k}; \rho) b_s(\mathbf{k}'; \rho) - b_s^*(\mathbf{k}'; \rho) a_r(\mathbf{k}; \rho)\}. \quad (3.10)$$

This means that the Fourier coefficients of the Jauch field with $\rho=0$ or 1 have the physical

meaning as creation and annihilation operators of field quanta, while this is not the case for Jauch field with $0 < \rho < 1$. Between the coefficients a_r 's and b_r 's we have from (3.2) the following interrelations:

$$\left. \begin{aligned} a_r(\mathbf{k}; \rho) &= \frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} a_r(\mathbf{k}; \rho=0) + \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} b_r(\mathbf{k}; \rho=0), \\ b_r(\mathbf{k}; \rho) &= \frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} b_r(\mathbf{k}; \rho=0) + \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} a_r(\mathbf{k}; \rho=0). \end{aligned} \right\} \quad (3.11)$$

This is nothing but the relation which Jauch has introduced in order to transform the variables $a_r(\mathbf{k}; \rho \neq 0)$ and $b_r(\mathbf{k}; \rho \neq 0)$ satisfying (4.9), into the variables satisfying the Jordan-Wigner relations. Thus, we may say that the Jauch field as it is does not have clear physical meaning, since it can not represent the property of field quanta. The particle interpretation is only possible if we go over into the operators $a_r(\mathbf{k}; \rho=0)$, $b_r(\mathbf{k}; \rho=0)$, i.e., the Dirac representation. From this standpoint of the particle picture we may conclude that a Jauch field with $\rho \neq 1$ is essentially equivalent to a Dirac field.

Next, we shall examine whether it is possible or not to introduce the gauge transformations which keep the theory invariant. Contrary to Jauch's argument we can expect that since the Jauch field is equivalent to the Dirac field, the gauge transformations of the former may be defined through the intermediary of those of the latter. We shall show below that this is actually the case.

By use of (2.9) we can give the explicit connection of $\Psi_J(x; \rho \neq 1)$ with $\Psi_J(x; \rho=0)$,

$$\Psi_J(x; \rho=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & iI_4 \\ I_4 & -iI_4 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \xi_1(x) \\ \pi_1(x) \\ \xi_2(x) \\ \pi_2(x) \end{pmatrix}, \quad (3.12)$$

or

$$\begin{pmatrix} \xi_1(x) \\ \pi_1(x) \\ \xi_2(x) \\ \pi_2(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1} \end{pmatrix} \begin{pmatrix} I_4 & I_4 \\ -iI_4 & iI_4 \end{pmatrix} \Psi_J(x; \rho=0). \quad (3.12)'$$

From (2.9) and (3.12)' we have

$$\Psi_J(x; \rho) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1+\rho}{2}}, & i\sqrt{\frac{1-\rho}{2}} \\ \sqrt{\frac{1+\rho}{2}}, & -i\sqrt{\frac{1-\rho}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \Psi_J(x; \rho=0) = J \Psi_J(x; \rho=0), \quad (3.13)$$

where

$$J = \frac{1}{2} \begin{pmatrix} \sqrt{1+\rho} + \sqrt{1-\rho}, & \sqrt{1+\rho} - \sqrt{1-\rho} \\ \sqrt{1+\rho} - \sqrt{1-\rho}, & \sqrt{1+\rho} + \sqrt{1-\rho} \end{pmatrix}. \quad (3.14)$$

On the other hand, for the field $\Psi_J(x; \rho=0)$ the gauge transformation, keeping the Lagrangian (2.13) with $\rho=0$ invariant, is given as usual by

$$\Psi_J'(x; \rho=0) = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} \Psi_J(x; \rho=0). \quad (3.15)$$

Hence, from (3.13) and (3.15) we finally obtain the gauge transformation for Jauch field with the form

$$\Psi_J'(x; \rho) = G(\chi; \rho) \Psi_J(x; \rho), \quad (3.16)$$

where $G(\chi; \rho)$ is a 2×2 -matrix depending on χ and ρ and given by

$$G(\chi; \rho) = J \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} J^{-1} = \begin{pmatrix} \cos \chi + \frac{i}{\sqrt{1-\rho^2}} \sin \chi, & \frac{-i\rho}{\sqrt{1-\rho^2}} \sin \chi \\ \frac{i\rho}{\sqrt{1-\rho^2}} \sin \chi, & \cos \chi - \frac{i}{\sqrt{1-\rho^2}} \sin \chi \end{pmatrix}. \quad (3.17)$$

That the Lagrangian (2.13) remains invariant under this gauge transformation (3.17) is explicitly proved in the next section and here we shall only show that the anticommutators (A) are invariant for $G(\chi; \rho)$.

In terms of $\Psi_J(x; \rho)$, (A) can be rewritten in the form

$$\{\Psi_J(x; \rho), \tilde{\Psi}_J(x'; \rho)\} = S \delta^3(x - x'), \quad (3.18)$$

with

$$S = \begin{pmatrix} \rho & 1 \\ 1 & \rho \end{pmatrix}. \quad (3.19)$$

After the transformation (A) becomes

$$\{\Psi_J(x; \rho), \Psi_J'(x'; \rho)\} = G(\chi; \rho) S \tilde{G}(\chi; \rho) \delta^3(x - x')$$

and it is quite easy to see that

$$G(\chi; \rho) S \tilde{G}(\chi; \rho) = S. \quad (3.20)$$

Thus, it is possible to make gauge transformations for Jauch field if we define them in somewhat general forms. This result suggests the possibility of introducing the gauge invariant electromagnetic interaction for Jauch field, which will be examined in detail in the next section.

We now proceed to discuss the transformation property under space reflection. This is defined in just the same way as in the gauge transformation. The Dirac field $\Psi_J(x; \rho=0)$ is subject to the following change under space reflection

$$\Psi_J^s(x'; \rho=0) = \begin{pmatrix} \gamma_4 & 0 \\ 0 & -\gamma_4^* \end{pmatrix} \Psi_J(x; \rho=0), \quad (3.21)$$

where η is the well-known phase factor and takes ± 1 or $\pm i$. From (3.13), (3.14) and (3.21) we get the following transformation for Jauch field;

$$\begin{aligned}\Psi_J^s(x'; \rho \neq 1) &= J \begin{pmatrix} \eta \gamma_4 & 0 \\ 0 & -\eta^* \gamma_4 \end{pmatrix} J^{-1} \Psi_J(x; \rho) \\ &= \frac{1}{2\sqrt{1-\rho^2}} \begin{pmatrix} \eta + \eta^* + (\eta - \eta^*) \sqrt{1-\rho^2} & -(\eta + \eta^*) \\ \eta + \eta^* & -(\eta + \eta^*) + (\eta - \eta^*) \sqrt{1-\rho^2} \end{pmatrix} \gamma_4 \Psi_J(x; \rho \neq 1).\end{aligned}$$

In order that the Jauch field is irreducible we must impose the condition

$$\eta + \eta^* = 0, \quad \text{i. e.,} \quad \eta = \pm i \quad (\text{C or D type}). \quad (3.23)$$

In this case (3.22) takes the simple form

$$\Psi_J^s(x'; \rho \neq 1) = \begin{pmatrix} \eta \gamma_4 & 0 \\ 0 & \eta \gamma_4 \end{pmatrix} \Psi_J(x; \rho \neq 1). \quad (3.24)$$

Jauch's restriction iii) (§ 1) is nothing but the above condition (3.23), but in general we can allow, of course, the transformation (3.22), which, as is easily proved, keeps the Lagrangian (2.13) invariant.

On the other hand, in the case $\rho=1$ we obtain in a similar way

$$\begin{aligned}\phi_J(x; \rho=1) (= \phi_J^*(x; \rho=1)) &= 1/\sqrt{2} \cdot \{\phi_J(x; \rho=0) + \phi_J^*(x; \rho=0)\}, \\ \phi_J^s(x'; \rho=1) &= 1/\sqrt{2} \cdot \{\eta \gamma_4 \phi_J(x; \rho=0) - \eta^* \gamma_4 \phi_J^*(x; \rho=0)\} \\ &= \eta \gamma_4 \phi_J(x; \rho=1) \\ &\quad \text{if } \eta = -\eta^*, \quad (\text{C, D}) \\ &= \eta \gamma_4 / \sqrt{2} \cdot \{\phi_J(x; \rho=0) - \phi_J^*(x; \rho=0)\} (= i \eta \gamma_4 \phi_J(x; \rho=1)) \\ &\quad \text{if } \eta = \eta^*. \quad (\text{A, B})\end{aligned} \quad (3.25)$$

From this result we find that, for the Majorana field the transformation type corresponding to A, B type of Dirac field, amounts to the exchange of two Majorana fields if they are present, while it is necessarily restricted to C or D type if a single Majorana field is present.

The time reversal transformation is discussed along a similar line as above and in general we can admit the four kinds of transformation properties for Jauch field.

§ 4. Interaction of Jauch field

In general, the fundamental nature of any field is to be discussed in relation to its interaction with other field. For instance, as far as the free field is concerned, the question is almost meaningless whether the usual Dirac spinor field really describes one kind of Dirac particle or two kinds of Majorana particles. For this reason we expect that the specific features of Jauch field, if any, would be reflected in its interaction with other fields.

We shall first show that Jauch field is not necessarily neutral, that is, it is possible

to introduce the gauge invariant electromagnetic interaction. This task can easily be achieved if we remember that the field $\Psi_J(x; \rho=0)$ is a Dirac field, the electromagnetic interaction of which is well known, and the general Jauch field $\Psi_J(x; \rho \neq 0, 1)$ is connected with $\Psi_J(x; \rho=0)$ by equations (3.13) and (3.14). The interaction Lagrangian, which reduces to the usual interaction term as $\rho \rightarrow 0$, is found to be

$$L_{\text{int}} = \frac{ie}{2\sqrt{1-\rho^2}} \bar{\Psi}_J(\rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \gamma_\mu \Psi_J(\rho) A_\mu. \quad (4.1)$$

The total Lagrangian of charged Jauch field is thus given by (2.13) and (4.1), that is,

$$L = L_0 + L_{\text{int}}. \quad (4.2)$$

We shall now show that the above L is invariant under the gauge transformation of Jauch field (3.16) but with x -dependent χ and

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + 1/e \cdot \partial_\mu \chi(x). \quad (4.3)$$

Now, the transformed Lagrangian L'_0 becomes

$$\begin{aligned} L'_0 &= -\frac{1}{2(1-\rho^2)} \bar{\Psi}_J(\rho) \tilde{G}(\chi; \rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} A G(\chi; \rho) \Psi_J(\rho) \\ &= -\frac{1}{2(1-\rho^2)} \bar{\Psi}_J(\rho) \tilde{G}(\chi; \rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} G(\chi; \rho) A \Psi_J(\rho) \\ &\quad - \frac{1}{2(1-\rho^2)} \bar{\Psi}_J(\rho) \tilde{G}(\chi; \rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \gamma_4 \gamma_\mu [\partial_\mu G(\chi; \rho)] \Psi_J(\rho). \end{aligned} \quad (4.4)$$

Bearing in mind the equation (3.20) and remarking that $\tilde{G}(\chi; \rho) = G(\chi; -\rho)$, we see that the first term on the right hand side in (4.4) is equal to the free part of Lagrangian before transformation, i.e., $L_0(\Psi_J)$. The second term in (4.4) is rewritten by use of the relation

$$\partial_\mu [G(x; \rho)] = \begin{pmatrix} -\sin \chi + \frac{i}{\sqrt{1-\rho^2}} \cos \chi, & -\frac{i\rho}{\sqrt{1-\rho^2}} \cos \chi \\ \frac{i\rho}{\sqrt{1-\rho^2}} \cos \chi, & -\sin \chi - \frac{i}{\sqrt{1-\rho^2}} \cos \chi. \end{pmatrix} \partial_\mu \chi.$$

After all we find

$$L'_0 = L_0 - \frac{i}{2\sqrt{1-\rho^2}} \bar{\Psi}_J(\rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \gamma_\mu \Psi_J(\rho) \partial_\mu \chi. \quad (4.5)$$

On the other hand the transformed interaction Lagrangian

$$L'_{\text{int}} = -\frac{ie}{\sqrt{1-\rho^2}} \bar{\Psi}_J(\rho) \tilde{G}(\chi; \rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \gamma_\mu G(\chi; \rho) \Psi_J(\rho) \left(A_\mu + \frac{1}{e} \partial_\mu \chi \right) \quad (4.6)$$

is rewritten into the form

$$L'_{\text{int}} = L_{\text{int}} + \frac{i}{\sqrt{1-\rho^2}} \tilde{\Psi}_J(\rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \gamma_\mu \Psi_J(\rho) \partial_\mu \chi \quad (4.7)$$

by virtue of the relation

$$\tilde{G}(\chi; \rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} G(\chi; \rho) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

From (4.5) and (4.7) we get finally

$$L' = L, \quad (4.8)$$

that is, the Lagrangian (4.2) remains invariant under the gauge transformation (3.16) and (4.3).

From the gauge invariancy we can expect as usual the conservation of electric current. But, as to the definition of current density we should remark that as seen from (2.13) and (4.1) the interaction term was not introduced by the usual replacement in the free Lagrangian L_0 , namely $\partial L / \partial(-ieA_\mu) \neq \partial L / \partial(\partial\psi / \partial x_\mu) \cdot \psi$. Therefore, we can no longer use the usual definition of the current $\partial L / \partial(\partial\psi / \partial x_\mu) \cdot \psi$, but must construct it directly from $\partial L / \partial A_\mu$. The result is

$$j_\mu(x) = -\frac{ie}{\sqrt{1-\rho^2}} \tilde{\Psi}(\rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \gamma_\mu \Psi(\rho). \quad (4.9)$$

By virtue of the equation of motion derived from (4.2) we can easily show that $\partial_\mu j_\mu = 0$.

Other types of coupling of Jauch field in interaction with a real (neutral) field are introduced in a similar way. Source densities of the Jauch field interacting via S, A, P, V and T couplings, which reduce to usual expressions when $\rho \rightarrow 0$, are given by

$$\frac{g_\alpha}{2(1-\rho^2)} \tilde{\Psi}_J(\rho) \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \gamma_4 \Omega_\alpha \Psi_J(\rho) \quad \text{for } S, A, P \text{ coupling}, \quad (4.10)$$

or

$$\frac{g_\alpha}{2\sqrt{1-\rho^2}} \tilde{\Psi}_J(\rho) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_4 \Omega_\alpha \Psi_J(\rho) \quad \text{for } V, T \text{ coupling},$$

respectively, where Ω_α 's are usual covariant γ -matrices.

As the source function in interaction term of Dirac field it is usual to assume the most simple form such as $\tilde{\psi}_J^*(x; \rho=0) \dots \dots \psi_J(x; \rho=0)$. But, more generally we have no reason to exclude the source functions such as $\tilde{\psi}_J(x; \rho=0) \dots \dots \psi_J(x; \rho=0)$ or $\tilde{\psi}_J^*(x; \rho=0) \dots \dots \psi_J^*(x; \rho=0)$. Corresponding to this circumstance, the most general interaction Lagrangian of Jauch field is written as

$$L_{\text{int}} = \frac{1}{4} \tilde{\Psi}_J(\rho) \gamma_4 \Omega_\alpha \left[\frac{1}{1+\rho} G_\alpha^{11} + \frac{1}{i\sqrt{1-\rho^2}} G_\alpha^{12} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{1}{i\sqrt{1-\rho^2}} G_\alpha^{21} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right. \\ \left. + \frac{1}{1-\rho} G_\alpha^{22} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right] \Psi_J(\rho), \quad (4.11)$$

where

$$G_{\alpha}^{ij} = g_{\alpha}^{ij} U_{\alpha}^{ij} \quad (i, j = 1, 2)$$

and U_{α}^{ij} are neutral (real or complex) fields with the transformation property α and further g_{α}^{ij} 's are coupling constants. (for $\alpha = V, T, g_{\alpha}^{11} = g_{\alpha}^{22} = 0$). Bilinear expressions containing two different spinors are constructed in the same way.

In (4.10) and (4.11) the ρ -dependent factors are explicitly separated for convenience so that the coupling constants become ρ -independent when these terms are written in terms of Dirac field $\Psi_J(x; \rho=0)$ and g_{α} 's are assumed ρ -independent. On the contrary, if we use the ρ -independent coupling constants in the interaction terms of Jauch field, these terms as a matter of course have ρ -dependent coupling constants when regarded as those of Dirac field $\Psi_J(x; \rho=0)$.

We have so far discussed the case that the Jauch field corresponds to a source "particle". In the following we shall consider the case that Ψ_J enters the interaction term in a linear form, that is, the Jauch field behaves as a "field". We shall study this case as to an example of β -decay interaction, since the extension to the general case is quite straightforward. Let us pay attention to the neutrino in the β -interaction H_{β} . When the neutrino has the transformation property of C or D type under space reflection, the interaction term can be assumed in somewhat a general form as first suggested by Sakata⁹⁾, that is,

$$\begin{aligned} H_{\beta} &= g(\tilde{\Psi}_p^* \Omega_{\alpha} \Psi_N)(\tilde{\psi}_e^* \Omega_{\alpha} \psi_{\nu}) + g^L(\tilde{\Psi}_p^* \Omega_{\alpha} \Psi_N)(\tilde{\psi}_e^* \Omega_{\alpha} \psi_{\nu}^*) + \text{h. c.} \\ &= F(g\psi_{\nu} + g^L\psi_{\nu}^*) + \text{h. c.}, \end{aligned} \quad (4.12)$$

where we have put for brevity

$$F = (\tilde{\Psi}_p^* \Omega_{\alpha} \Psi_N)(\tilde{\psi}_e^* \Omega_{\alpha}.$$

If $g^L=0$, the neutrino in H_{β} becomes Dirac particle, while if $g=g^L$ one of the two Majorana fields is eliminated and the β -decay neutrino becomes a pure Majorana particle. This is just the idea of Sakata's elimination theory.⁹⁾ In the usual literatures the β -interaction is assumed in the simplest form ($g^L=0$) but as far as there is no a priori reason for it we have rather to take the general form such as (4.12). From this standpoint it may be quite pointless to ask whether the neutrino is a Dirac or a Majorana particle. On the contrary we should study this problem by asking what value the ratio g_L/g of coupling constants amounts to.

Now, the interaction (4.12) can be written in terms of Jauch field $\phi_J(x; \rho)$ as follows :

$$\begin{aligned} H_{\beta} &= F/2\sqrt{1-\rho^2} \cdot [\{ (\sqrt{1+\rho} + \sqrt{1-\rho})g + (\sqrt{1-\rho} - \sqrt{1+\rho})g^L \} \phi_J(\rho) \\ &\quad + \{ (\sqrt{1-\rho} - \sqrt{1+\rho})g + (\sqrt{1+\rho} + \sqrt{1-\rho})g^L \} \phi_J^*(\rho)], \end{aligned} \quad (4.13)$$

or

$$= F/2\sqrt{1-\rho^2} \cdot [\{ (\sqrt{1+\rho} + \sqrt{1-\rho})g + (\sqrt{1-\rho} - \sqrt{1+\rho})g^L \} \phi_J(\rho)$$

$$+1/\rho \cdot \{(\sqrt{1-\rho}-\sqrt{1+\rho})g + (\sqrt{1+\rho}+\sqrt{1-\rho})g^L\} \psi_J(\rho) \\ -i\sqrt{1-\rho^2}/\rho \cdot \{(\sqrt{1-\rho}-\sqrt{1+\rho})g + (\sqrt{1+\rho}+\sqrt{1-\rho})g^L\} \psi_J(\rho). \quad (4.13)'$$

Therefore, H_β can be written in terms of $\mathcal{V}_J(x; \rho)$ only if there holds the relation between g and g^L

$$g^L/g = (\sqrt{1+\rho}-\sqrt{1-\rho})/(\sqrt{1+\rho}+\sqrt{1-\rho}) \equiv R(\rho). \quad (4.14)$$

In this case, H_β becomes, as in the elimination theory,

$$H_\beta = g_p(F\mathcal{V}_J(\rho)) + \text{h. c.}, \quad (4.15)^*$$

where

$$g_p = 2g/(\sqrt{1+\rho} + \sqrt{1-\rho}).$$

The function $R(\rho)$ runs from 0 to 1 as ρ increases from 0 to 1. As stated above the case $g^L/g=0$ corresponds to Dirac neutrino, the case $g^L/g=1$ to Majorana neutrino and the case $0 < g^L/g < 1$ to Jauch neutrino $\mathcal{V}_J(x; \rho)$, the ρ -value of which is given by $\rho = 2gg^L/(g^2 + g^{L2})$. In the last case the free part of neutrino Lagrangian can also be written in terms of $\mathcal{V}_J(x; \rho)$ only, and so the whole system can be described by $\mathcal{V}_J(x; \rho)$ only. Thus, we can say that the Dirac field with the interaction (4.12) is equivalent to the Jauch field with that of (4.15).

Generalizing the foregoing argument the Lagrangian of any Jauch field in interaction can always be reduced to the Lagrangian of Dirac field having, generally, all the possible interactions, coupling constants of which are ρ -dependent. (To this end we have only to insert (3.13) into the Jauch field Lagrangian.) In this sense, theories having different ρ -values respectively are different from each other because of different coupling constants. Thus we are led to the conclusion that at the present stage of the theory the problem of Jauch field in general should be considered from the view-point of the structure of interaction of Dirac field^{*)}.

Finally we should suggest, however, that in some cases the representation of Jauch field is preferable to that of Dirac field. As an example we shall again take the β -interaction. It is well known that for the probability of the double β -decay usual Dirac or

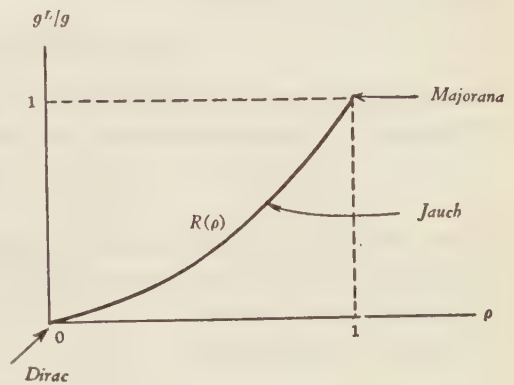


Fig. 1

Fig. 1 The relation between the ratio g^L/g and the ρ -values of Jauch field.

) If the neutrino has the transformation type A or B for space reflection, H_β is of the form $\sim F(g\psi_\nu + g^L\gamma_5\psi_\nu^) = F\mathcal{V}_J$. But in this case \mathcal{V}_J can not be a Jauch field, since independently of the values of g and g^L we have always $\{\mathcal{V}_J(x), \mathcal{V}_J(x')\} = 0$.

Majorana theory predicts 10^{24} years or 10^{16} years, respectively. If we adopt the interaction (4.15), however, we can get any arbitrary value between these two extreme values without raising any contradiction with the experimental results in the single β -decay, since the probability of neutrinoless double β -decay is proportional to ρ^2/g_p^2 . Suppose that the experimental value be found to be about 10^{20} years, say. In such a case, it would be very difficult, from the aesthetic view-point of Dirac field, to understand the reason why one of the coupling constant g' is extremely small as compared to another coupling constant g . From the view-point of Jauch field, however, we can avoid such an apparent asymmetry by suitably defining this field.

Within the framework of the present theory the above circumstance is merely a matter of convention, but we may expect that in some cases or rather in a future theory the concept of Jauch field may describe more clearly the physical reality than that of Dirac field. This point of course remains to be further investigated.

In conclusion we should like to express our gratitude to Prof. S. Sakata for helpful discussions. We are also indebted to Prof. J. M. Jauch and Dr. Y. Takahashi for their very profitable correspondence.

Appendix

This section will be devoted to the detailed discussion of quantization of the system described by the Lagrangian

$$L = -1/2 \cdot f_1 \bar{\phi}_1 A \phi_1 - 1/2 \cdot f_2 \bar{\phi}_2 A \phi_2. \quad (\text{A} \cdot 1)$$

Since ϕ_1 and ϕ_2 enter the above expression in a symmetrical way and they are mutually independent, it is sufficient to consider the system of ϕ_1 with the Lagrangian

$$L_1 = -1/2 \cdot f_1 \bar{\phi}_1 A \phi_1. \quad (\text{A} \cdot 2)$$

In the following the subscript 1 will be omitted for brevity. As it is necessary to find the interconnection of Jauch field operator with canonical variables, we shall first consider how to determine these variables.

Now, A is of the form

$$A = \gamma_4 (\gamma_\mu \partial_\mu + \kappa) \quad (\text{A} \cdot 3)$$

and so, when inserting (A.3) into (A.2) the coefficient of time derivative $\partial_4 \phi$ becomes a numerical multiple of unit matrix. Thus, if any component of ϕ is as usual taken as a canonical coordinate, the difficulty immediately arises that the corresponding momentum variable is no longer independent of the coordinate, leading, therefore, to inconsistency with the commutation relation. The difficulty is, however, avoided if it is possible to transform ϕ so as to bring the coefficient matrix of $\partial_4 \phi$ into the form

$$I_4 \rightarrow \begin{pmatrix} 0 & aI_2 \\ bI_2 & 0 \end{pmatrix}. \quad (\text{A} \cdot 4)$$

As such a transformation the following satisfies our condition, that is,

$$\phi = B\tilde{\zeta}, \quad (\text{A} \cdot 5)$$

with

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ iI_2 & -iI_2 \end{pmatrix}. \quad (\text{A} \cdot 6)$$

After this transformation terms of time derivative in (A·2) lead to

$$\begin{aligned} -\frac{1}{2} f \tilde{\phi} \partial_t \phi &= -\frac{i}{2} \tilde{\zeta} \tilde{B} B \dot{\zeta} = -\frac{i}{2} f \tilde{\zeta} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \dot{\zeta} \\ &= -\frac{if}{2} \left\{ \dot{\zeta}^1 \dot{\zeta}^3 + \dot{\zeta}^2 \dot{\zeta}^4 + \dot{\zeta}^3 \dot{\zeta}^1 + \dot{\zeta}^4 \dot{\zeta}^2 \right\}. \end{aligned} \quad (\text{A} \cdot 7)$$

In order to eliminate, in this expression, terms containing $\dot{\zeta}^3$ and $\dot{\zeta}^4$, the following 4-divergence is added to the original Lagrangian (A·2). Thus,

$$L' = L - 1/4 \cdot f (\partial / \partial x_\mu) \tilde{\phi} \gamma_\mu \phi. \quad (\text{A} \cdot 8)$$

In fact, since $-\frac{1}{4} f \partial (\tilde{\phi} \gamma_\mu \phi) / \partial x_\mu = -\frac{i}{2} f \left\{ \dot{\zeta}^1 \dot{\zeta}^3 + \dot{\zeta}^1 \dot{\zeta}^3 + \dot{\zeta}^2 \dot{\zeta}^4 + \dot{\zeta}^2 \dot{\zeta}^4 \right\}$, terms of time derivatives in L' , take the form

$$if/2 \cdot \left\{ \dot{\zeta}^3 \dot{\zeta}^1 + \dot{\zeta}^4 \dot{\zeta}^2 - \dot{\zeta}^1 \dot{\zeta}^3 - \dot{\zeta}^2 \dot{\zeta}^4 \right\}. \quad (\text{A} \cdot 9)$$

Hereupon, it is found that the usual procedure of quantization is consistently applied to variables $\tilde{\zeta}$'s in (A·8) and (A·9).

If $\tilde{\zeta}^1$ and $\tilde{\zeta}^2$ are regarded as canonical coordinates, the corresponding momenta are defined by

$$\begin{aligned} \pi^1 &= \partial L' / \partial \dot{\tilde{\zeta}}^1 = if \dot{\zeta}^3, \\ \pi^2 &= \partial L' / \partial \dot{\tilde{\zeta}}^2 = if \dot{\zeta}^4, \end{aligned} \quad (\text{A} \cdot 10)$$

respectively. Between these canonical variables anticommutation relation can be assumed as usual:

$$\left. \begin{aligned} \{\tilde{\zeta}^i(x), \pi^j(x')\} &= i \delta_{ij} \delta^3(x - x'), \\ \{\tilde{\zeta}^i(x), \tilde{\zeta}^j(x')\} &= \{\pi^i(x), \pi^j(x')\} = 0, \quad i, j = 1, 2. \end{aligned} \right\} \quad (\text{A} \cdot 11)$$

Then, (A·10) and (A·11) give the relation

$$\{\tilde{\zeta}^\mu(x), \tilde{\zeta}^\nu(x')\} = 1/f \cdot (\tilde{B} B)_{\mu\nu} \delta^3(x - x'), \quad (\text{A} \cdot 12)$$

or returning to the original representation ϕ

$$\{\phi(x), \tilde{\phi}(x')\} = 1/f \cdot \delta^4(x - x'). \quad (\text{A} \cdot 13)$$

From (A·5) and (A·10) $\phi(x)$ can be explicitly written in terms of canonical variables in the following way :

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & -iI_2/f \\ iI_2 & -I_2/f \end{pmatrix} \begin{pmatrix} \tilde{\zeta}^1(x) \\ \tilde{\zeta}^2(x) \\ \pi^1(x) \\ \pi^2(x) \end{pmatrix}, \quad (\text{A} \cdot 14)$$

which by a canonical transformation is further brought into a simple form

$$\phi(x) = \frac{1}{\sqrt{2f}} \begin{pmatrix} I_2 & -iI_2 \\ iI_2 & -I_2 \end{pmatrix} \begin{pmatrix} \tilde{\zeta}^1(x) \\ \tilde{\zeta}^2(x) \\ \pi^1(x) \\ \pi^2(x) \end{pmatrix}, \quad (\text{A} \cdot 15)$$

where the inequality $f \geq 0$, as proved in the text, was used.

The discussion developed above is essentially equivalent to the general theory of quantization of Schwinger.⁵⁾ The difference merely lies in the point that for convenience of consideration the 4-divergence (A·8) has been introduced in our case.

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Connection of the Strong Coupling Theory with the Weak Coupling Theory in the Bound Meson Problem

— The Symmetrical Scalar Theory —

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The bound meson problem is studied in the p -representation of meson oscillators. (\mathbf{p} is canonical momentum). The notion of this paper may be applied to other problems than the symmetrical scalar one. The domain of \mathbf{p} is divided into the inner and outer regions. It is proved by application of canonical transformations that such a process is natural. In the inner region the precession of the \mathbf{r} -spin is dominant, which takes place as bound mesons are virtually emitted or absorbed. On the other hand, in the outer region the coupling between the radial mode of meson with the \mathbf{r} -spin is primarily important. Since all relevant quantities appear as functions of $V|\mathbf{p}|$ (V is coupling constant), we can speak about the characteristics of regions independently of V , if we choose $1/V$ as the unit of length of \mathbf{p} . Then in this scale, we see that the wave functions of low-lying states are damped when the magnitude of $|\mathbf{p}|$ is larger than V . By way of these considerations, we can understand the characteristics of the problem over all ranges of V in a unified fashion. In a weak coupling case, the inner region is the only one to be taken into account, while the most noticeable features of a strong coupling case come from the outer region, to which the major part of a wave function belongs in this case. It is noticeable that the inner region is, however, essential to give finer details of a strong coupling case, which are of higher order in $1/V^2$.

With these qualitative prospects in mind, a new approach is tried to improve the results worked out by Sawada. However, our ability to treat a complicated form of operators is at present so limited that our results are not yet satisfactory. It is discussed what may be the key to the future improvement.

§ 1. Introduction

It is the purpose of this paper to find a unified stand-point from which the bound meson problem may be well investigated over all ranges of coupling constant. As is well known, the intermediate coupling theory is available in the intermediate region of coupling strength.¹⁾ But a result of that theory, e.g., the level of the ground state, is not yet represented in terms of analytical expressions of the coupling constant; and, in its current

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form, it seems hard to carry out the renormalization program. So it is desirable to fill gaps of such kinds, and one remarkable step has been made by Sawada²⁾ to this direction, who has obtained an interesting result by means of a simple type of canonical transformation. We shall make an approach of similar kind and study a possibility to improve it.

We shall here treat the symmetrical scalar theory as a simple example of such problems that their solution is made complicated by virtue of spin variables. If we confine our attention to bound mesons, the Hamiltonian is given by¹⁾

$$H=H_0+H' \quad (1)$$

with

$$H_0=(\mathbf{p}^2+\mathbf{q}^2)/2 \quad (\text{the bound meson Hamiltonian}),$$

and

$$H'=V(\boldsymbol{\tau} \cdot \mathbf{q}) \quad (\text{the interaction Hamiltonian}),$$

where $\boldsymbol{\tau}$ is the nucleonic isotopic spin, V is the effective coupling constant, and the oscillator's frequency is taken as unity; (as to the unit of energy, see reference 1 a, especially p. 609); \mathbf{q} and \mathbf{p} denote position- and momentum-operator of the oscillator, respectively.

It is known that the level shift of the ground state is $-(3/2)V^2$ in the weak limit, while it is $-(1/2)V^2-1+O(1/V^2)$ in the strong limit.³⁾ The numerical results are available at several points in the intermediate coupling region.⁴⁾ Sawada's method gives results which are in good agreement with correct values in the region $V<1$. But in the case $V>2$, it is not successful; it predicts that the level shift in the strong limit is $-(1/2)V^2+1$. Moreover, the sign of the isotopic spin-orbit coupling term is reversed when the coupling constant becomes larger than a certain value; for example, among the first excited states, the state $I=1/2$ is lower than the one $I=3/2$ in a strong coupling case. Nevertheless, it is instructive for us to investigate on what the conspicuous results of Sawada's method are based; hence we shall begin with re-examination of Sawada's result in next section.

By means of the analysis given in Sec. 2, we see how different the features of the system are in the strong and weak limits. These features are further analyzed by means of canonical transformations in Secs. 3 and 4. In Sec. 3, the transformation function

$$U_R=\exp\{i/2 \cdot \tan^{-1}[2V(\boldsymbol{\tau} \cdot \mathbf{p})]\} \quad (2)$$

will be introduced to treat the weak region of coupling constant. This transformation is of the type of half-arctangent, which is often found useful in the treatment of spin variables. Actually the interpretation of the transformation (2) can be given by taking account of some rotation of the $\boldsymbol{\tau}$ -spin. Simultaneously we can verify that there exists some regional characteristics in the domain of \mathbf{p} , which can be represented independently of V by taking $1/V$ as the unit of length of \mathbf{p} . In Sec. 3, the transformation function

$$U_T=\exp\{iV(\boldsymbol{\tau} \cdot \mathbf{p})\} \quad (3)$$

will be introduced to treat the most noticeable feature of the strong-coupling cases. The interpretation of the transformation (3) can be given in analogy to the Bloch-Nordsieck

transformation, if we take account of the fact that the angular modes of the system are apparently ineffective in the strong coupling case. But, through closer examination, this form of transformation function fails to reproduce finer details of the strong coupling theory.

In the course of analysis made in Secs. 2—4, qualitative aspects of our problems can be clarified. With these results in mind, the transformation function of general form

$$U = \exp \{ i [\text{an odd function of } V(\tau \cdot p)] \} = \exp \left\{ i \frac{(\tau \cdot p)}{|p|} f(|p|) \right\} \quad (4)$$

will be investigated in Sec. 5 to get some improvement to the Sawada's results. In view of the unsatisfactory development of the operator calculus, one is forced to use a function of the form (4); but it will turn out that the improvement to be achieved is rather minor and our results are by no means quantitatively parallel to our qualitative prospects. In Sec. 6, it will be discussed what kind of calculation should become possible before we improve our results.

§ 2. On Sawada's approximation

We shall re-examine Sawada's method and try to analyze the background for the success of this method. In this method we use the transformation function

$$U_S = \exp \{ i \lambda (\tau \cdot p) \}, \quad (5)$$

where λ is a variational parameter. Actually λ can be determined by the condition

$$(\lambda - V) + 2e^{-\lambda^2} [\lambda + (2\lambda^2 - 1)V] = 0. \quad (6)$$

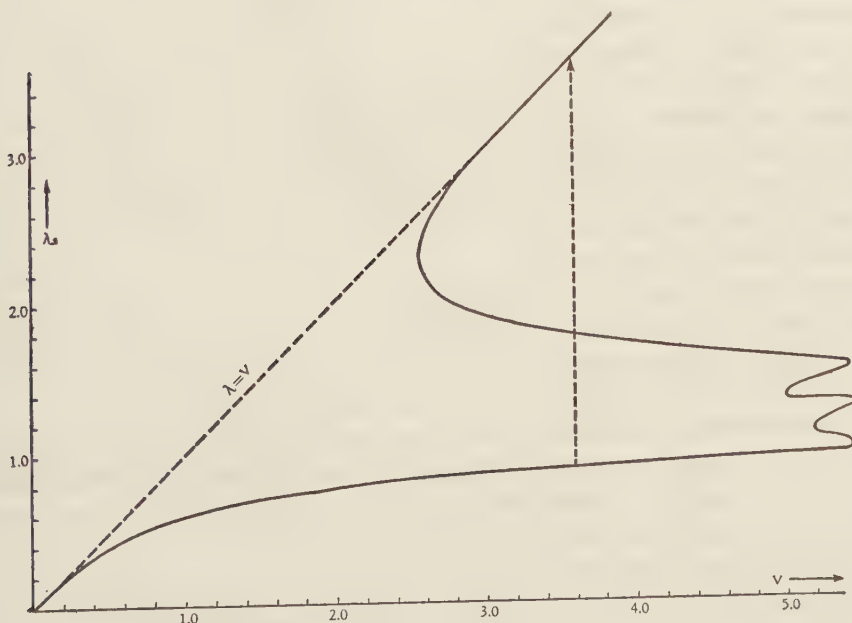


Fig. 1.

We shall hereafter denote by λ_s such λ that satisfies this condition. This condition gives the lowest approximative level of the ground state, $E_0 + (3/2)$, where $3/2$ is the zero-point energy of meson oscillators and E_0 is defined by

$$E_0 = {}_0\langle U_s^{-1} H U_s \rangle_0 - 3/2 = (1 + (\lambda^2/2) - V\lambda) - (1 + 2V\lambda)e^{-\lambda^2}. \quad (7)$$

(${}_0\langle \dots \rangle_0$ means to take the vacuum expectation value with respect to meson oscillators.) Simultaneously, the matrix elements for the ground state to emit or absorb a single meson vanish by virtue of this condition. (Cf. Eq. (11) below.)

The relation between λ_s and V is illustrated in Fig. 1. The value of E_0 is plotted in Fig. 2 as a function of V . In the intermediate region of V , λ_s and E_0 are not single-valued functions of V ; it is clear on the physical grounds that such branch of λ_s must be selected that gives the lowest E_0 . Then a certain range of λ_s is useless, since λ_s jumps from the lower region into the upper one when V becomes larger than a certain value*. This jump of parameter reminds us of a phase-transition of a condensing system, and it may be taken as an evidence that the nature of the problem is different in two regions. But we must be careful before we take this jump for something of real meaning from the physical view-point; we must examine thoroughly what is the physical meaning of Sawada's transformation in different regions of λ_s .

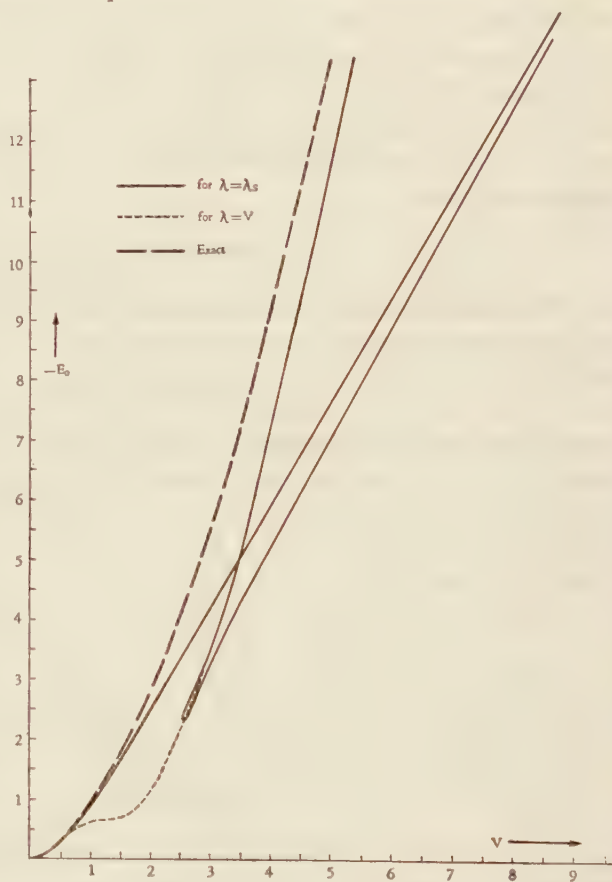


Fig. 2.

First we examine the situation in the weak-coupling region. It is instructive for us to compare the results for $\lambda=V$ with those obtained for $\lambda=\lambda_s$. In every respect it is better to use λ_s , when λ_s is much smaller than V ; this becomes clear if we analyze the transformed Hamiltonian in both cases.

The transformed Hamiltonian is given by

* This jump is shown by an arrow in Fig. 1.

$$U^{-1}HU = H_0 + H_{\text{pot}} + H_{\text{sp-orb}} + H'_{\text{rad}} + H'_{\text{aug}}, \quad (8)$$

where

$$H_{\text{pot}} = \lambda^2/2 - \lambda V + \sin^2(\lambda|\mathbf{p}|)/\mathbf{p}^2 - V \cdot \sin(2\lambda|\mathbf{p}|)/|\mathbf{p}|, \quad (8a)$$

$$H_{\text{sp-orb}} = [V \cdot \sin(2\lambda|\mathbf{p}|)/|\mathbf{p}| - \sin^2(\lambda|\mathbf{p}|)/\mathbf{p}^2][(\mathbf{p} \times \mathbf{q}) \cdot \boldsymbol{\tau}], \quad (8b)$$

$$H'_{\text{rad}} = \frac{1}{2}(V - \lambda)[(\boldsymbol{\tau} \cdot \mathbf{p})/\mathbf{p}^2 \cdot (\mathbf{p} \cdot \mathbf{q}) + \text{conj.}], \quad (8c)$$

$$H'_{\text{aug}} = \left[\frac{1}{2}V \cos(2\lambda|\mathbf{p}|) - \frac{1}{4} \frac{\sin(2\lambda|\mathbf{p}|)}{|\mathbf{p}|} \right] \left[(\boldsymbol{\tau} \cdot \mathbf{q}) - \frac{(\boldsymbol{\tau} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{q})}{\mathbf{p}^2} \right] + \text{conj.} \quad (8d)$$

The ordering procedure of these terms can be performed according to the formula

$$\begin{aligned} F(\mathbf{p}) &= \frac{1}{(2\pi)^3} \iint d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x})} e^{-i(\mathbf{x} \cdot \mathbf{y})} F(\mathbf{y}) \\ &= \sum_{m,n} \frac{1}{(2\pi)^3} \frac{(-1)^m}{m!} \frac{1}{n!} \left(\frac{1}{\sqrt{2}} \right)^{m+n} \iint d^3x d^3y e^{-i\mathbf{x} \cdot \mathbf{y}} e^{-i(\mathbf{x} \cdot \mathbf{y})} F(\mathbf{y}) (\hat{\boldsymbol{\epsilon}}^* \cdot \mathbf{x})^m (\hat{\boldsymbol{\epsilon}} \cdot \mathbf{x})^n, \end{aligned} \quad (9)$$

here we introduce the creation- and annihilation-operators, $\hat{\boldsymbol{\epsilon}}^*$ and $\hat{\boldsymbol{\epsilon}}$, and put

$$\mathbf{p} = (\hat{\boldsymbol{\epsilon}} - \hat{\boldsymbol{\epsilon}}^*)/\sqrt{2}i, \quad \mathbf{q} = (\hat{\boldsymbol{\epsilon}} + \hat{\boldsymbol{\epsilon}}^*)/\sqrt{2}.$$

After some manipulations we have the ordered interaction terms,

$$\begin{aligned} H'_{\text{aug}} + H'_{\text{rad}} &= I_1(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\epsilon}}^*) + I_2(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\epsilon}}^*)(\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}) + \dots \\ &+ (\text{terms of other types, e.g. } I_3'(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\epsilon}}^*)(\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}) \text{ etc.}) \\ &+ \text{conj.}, \end{aligned} \quad (10)$$

where I 's are given by

$$I_1 = \sqrt{2}/3 \cdot \left\{ \frac{1}{2}(V - \lambda) + e^{-\lambda^2}[(1 - 2\lambda^2)V - \lambda] \right\}, \quad (10a)$$

$$I_3' = \sqrt{2}/5 \cdot \left\{ (V - \lambda) + e^{-\lambda^2} \left[\frac{3}{2}(V - \lambda) + \lambda^3 + 2V\lambda^2(\lambda^2 - 3) \right] \right\}, \quad (10b)$$

$$I_3 = \sqrt{2}/15 \cdot \left\{ \frac{3}{2}(V - \lambda) - e^{-\lambda^2} \left[\frac{3}{2}(V - \lambda) + \lambda^3 + 2V\lambda^2(\lambda^2 - 3) \right] \right\}. \quad (10c)$$

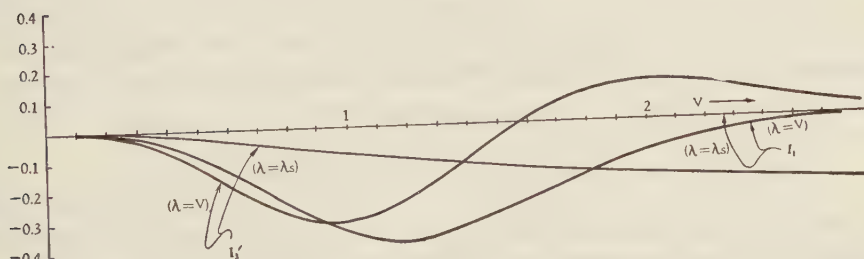


Fig. 3.

I 's are plotted in Fig. 3 in both cases of $\lambda = V$ and $\lambda = \lambda_s$. The matrix element for the creation (or annihilation) of single meson vanishes when we put $\lambda = \lambda_s$, as the result of the equation,

$$I_1 = -\frac{1}{6\sqrt{2}} \frac{\partial E_0}{\partial \lambda}. \quad (11)$$

With respect to matrix elements for creation or annihilation of more than one mesons, the results with $\lambda = \lambda_s$ are much better than those with $\lambda = V$ in the region $V \lesssim 1$.

If there were no involved effect caused by the τ -spin, the problem must have been solved by means of the transformation function $U = \exp[iV(\boldsymbol{\tau} \cdot \mathbf{p})]$ in analogy to the neutral scalar theory. As for the τ -spin, it is only τ_z that is diagonal in the usual representation of τ -matrices, but off-diagonal τ_1 and τ_2 appear in the Hamiltonian with non-vanishing coefficients as charged mesons are virtually emitted or absorbed. Then these off-diagonal terms change the eigenstates of τ -matrices. This fact can be represented by saying that the τ -spin is put into a precession when mesons are virtually emitted or absorbed. It is noticeable that λ_s is smaller than V ; the difference between λ_s and V must be responsible for the property of τ -spin. The effect of a component in some direction of τ -spin will be reduced smaller, if the precession is taken into account and the fluctuation is averaged out; and we may suppose that the generating function will be reduced from $V(\boldsymbol{\tau} \cdot \mathbf{p})$ to $\lambda_s(\boldsymbol{\tau} \cdot \mathbf{p})$. If this conjecture is true, the striking success of Sawada's method, as demonstrated above, shows that the precession of τ -spin is primarily important in the cases of weaker coupling strength.

On the other hand, the situation is quite different in the region of stronger coupling strength. When λ_s of the weak region is smoothly extrapolated beyond the jump-point, it gives poor results, (cf. Fig. 3). The best value for λ_s is nearly equal to V in the strong-coupling cases. If we put $\lambda = V$, all matrix elements for the emission or absorption of an odd number of mesons can be made exponentially small, as can be seen in Eqs. (10, a, b). And the level of the ground state can be made as large as $-(1/2)V^2$ only when we put $\lambda = V$. The transformation function will assume the form $\exp[iV(\boldsymbol{\tau} \cdot \mathbf{p})]$, if the τ -spin can be treated as if it were a c -number. Though we cannot directly prove here that the τ -spin behaves in the strong-coupling limit in such a way, it is remarkable that the degrees of freedom of τ -spin is apparently reduced to one in the strong-coupling theory. Thus we can expect results of correct order, at least to the first order of approximation, by putting $\lambda = V$. The effect of the transformation can be understood by considering a translation of $|\mathbf{q}|$ by $V(\boldsymbol{\tau} \cdot \mathbf{p})/|\mathbf{p}|$, which is primarily important in the strong limit.

However, the level of the ground state is not correctly given at the next order of expansion into powers of $1/V^2$. We find, putting $\lambda = V$,

$$E_0 = {}_0\langle H_{\text{pot}} \rangle_0 = -V^2/2 + 1 - e^{-V^2}(1 + 2V^2) \sim -V^2/2 + 1 \quad (12)$$

in the strong limit. The correct value is known as $-(1/2)V^2 - 1$. This failure is due to the fact that in our formulation H_{pot} gives rise to a peak of the effective potential in the neighbourhood of $|\mathbf{p}| \simeq 1/V$; the shape of H_{pot} is plotted in Fig. 5a, putting $\lambda = V$. In the outer region of p -space ($|\mathbf{p}| \gg 1/V$), H_{pot} is oscillating, except for the constant $-V^2/2$, and gives no contribution on the average, but the effect of the first and highest peak cannot be cancelled by virtue of the valleys around it, and it may give a contribution

+1 to the level shift. This peak of effective potential gives no serious effect in weak-coupling cases, since it does not overlap with the wave function of the ground state, because we have $1 \ll 1/V$ and the wave function is small in the region $|\mathbf{p}| > 1$; this is *a fortiori* true when we make use of λ_s , which is smaller than V . (Cf. Fig. 5a.)

§ 3. Elimination of the angular part of the interaction

We shall show in this section that we can eliminate the angular variables from the interaction Hamiltonian by means of a certain canonical transformation; simultaneously we can verify the conjecture made in the previous section concerning the qualitative difference between strong- and weak-coupling cases.

We begin with giving some general results of the transformation, assuming that the generating function of the transformation is an odd function of $(\boldsymbol{\tau} \cdot \mathbf{p})$. The validity of this assumption will be discussed later (Secs. 5 and 6). We put the transformation function in the form

$$U = \exp \{ i f [(\boldsymbol{\tau} \cdot \mathbf{p})] \}, \quad (13)$$

and

$$\cos [f(|\mathbf{p}|)] \equiv C, \quad (1/|\mathbf{p}|) \sin [f(|\mathbf{p}|)] \equiv S, \quad (14)$$

or

$$U = C + i(\boldsymbol{\tau} \cdot \mathbf{p})S. \quad (15)$$

By the definition we have the identity

$$C^2 + \mathbf{p}^2 S^2 = 1. \quad (16)$$

The variable \mathbf{q} after the transformation is given by

$$U^{-1} \mathbf{q} U = \mathbf{q} - \boldsymbol{\tau} C S + 2\mathbf{p} (\boldsymbol{\tau} \cdot \mathbf{p}) [\partial C / \partial (\mathbf{p}^2) \cdot \mathbf{S} - \partial S / \partial (\mathbf{p}^2) \cdot C] + [\mathbf{p} \times \boldsymbol{\tau}] S^2. \quad (17)$$

Consequently we have

$$\begin{aligned} U^{-1} \mathbf{q}^2 U = & \mathbf{q}^2 + [2\mathbf{p}^2 (\partial C / \partial (\mathbf{p}^2) \cdot \mathbf{S} - \partial S / \partial (\mathbf{p}^2) \cdot C) - C S]^2 + 2S^2 - 2S^2 (\mathbf{p} \times \mathbf{q}) \cdot \boldsymbol{\tau} \\ & - [C S (\boldsymbol{\tau} \cdot \mathbf{q}) + \text{conj.}] + 2[(\partial C / \partial (\mathbf{p}^2) \cdot \mathbf{S} - \partial S / \partial (\mathbf{p}^2) \cdot C) (\boldsymbol{\tau} \cdot \mathbf{p}) (\mathbf{p} \cdot \mathbf{q}) + \text{conj.}]. \end{aligned} \quad (18)$$

The results of transformation for $(\boldsymbol{\tau} \cdot \mathbf{q})$ is given by

$$\begin{aligned} U^{-1} (\boldsymbol{\tau} \cdot \mathbf{q}) U = & \{ (C^2 - \tfrac{1}{2}) (\boldsymbol{\tau} \cdot \mathbf{q}) + \text{conj.} \} + \{ S^2 (\boldsymbol{\tau} \cdot \mathbf{p}) (\mathbf{p} \cdot \mathbf{q}) + \text{conj.} \} \\ & + 2C S (\boldsymbol{\tau} \cdot [\mathbf{p} \times \mathbf{q}]) + 2(\partial C / \partial (\mathbf{p}^2) \cdot \mathbf{S} - \partial S / \partial (\mathbf{p}^2) \cdot C) \mathbf{p}^2 - 3C S. \end{aligned} \quad (19)$$

We notice the relation

$$\mathbf{p}^2 (\partial C / \partial (\mathbf{p}^2) \cdot \mathbf{S} - \partial S / \partial (\mathbf{p}^2) \cdot C) = \tfrac{1}{2} [S C - d f / d |\mathbf{p}|], \quad (20)$$

where we use the identity (16).

By means of Eqs. (18)–(20) and (16), the transformed Hamiltonian is

$$U^{-1} H U = H_0 + H_{\text{pot}} + H_{\text{sp-orb}} + H'_{\text{rad}} + H'_{\text{ang}}. \quad (21)$$

H_{pot} is given by

$$H_{\text{pot}} = \tfrac{1}{2} (d f / d |\mathbf{p}|)^2 - V (d f / d |\mathbf{p}|) + S^2 - 2V C S, \quad (22)$$

and can be regarded as the additional term to the potential of meson oscillators after the transformation; $H_{\text{sp-orb}}$ is given by

$$H_{\text{sp-orb}} = S(2VC - S)(\boldsymbol{\tau} \cdot [\mathbf{p} \times \mathbf{q}]) = (\boldsymbol{\tau} \cdot [\mathbf{p} \times \mathbf{q}])S(2VC - S), \quad (23)$$

(where it is remarked that this term is commutable with \mathbf{p}^2); and it gives rise to a separation between a state with parallel isotopic spin and charge vectors and a corresponding state with anti-parallel ones: H_{pot} and $H_{\text{sp-orb}}$ as well as H_0 are even operators by which the occupation number of mesons is changed by an even number, if any change takes place. On the other hand, H'_{rad} and H'_{aug} are odd terms by which the occupation number of mesons is necessarily changed by an odd number. They are given by

$$H'_{\text{rad}} = \frac{1}{2} \left[V - \frac{df}{d|\mathbf{p}|} \right] \frac{(\boldsymbol{\tau} \cdot \mathbf{p})}{p^2} (\mathbf{p} \cdot \mathbf{q}) + \text{conj.}, \quad (24)$$

and

$$\begin{aligned} H'_{\text{aug}} &= \left[V \left(C^2 - \frac{1}{2} \right) - \frac{1}{2} CS \right] \left[(\boldsymbol{\tau} \cdot \mathbf{q}) - \frac{(\boldsymbol{\tau} \cdot \mathbf{p})}{p^2} (\mathbf{p} \cdot \mathbf{q}) \right] + \text{conj.} \\ &= \frac{1}{2} \left[V \cos[2f] - \frac{\sin[2f]}{2|\mathbf{p}|} \right] \left[(\boldsymbol{\tau} \cdot \mathbf{q}) - \frac{(\boldsymbol{\tau} \cdot \mathbf{p})}{p^2} (\mathbf{p} \cdot \mathbf{q}) \right] + \text{conj.} \end{aligned} \quad (25)$$

respectively. Here H'_{rad} is independent of the angular modes of meson oscillators, while H'_{aug} depends on them.

In general both types of interaction term, H'_{rad} and H'_{aug} , appear after the transformation. Only when we put the transformation function in the form

$$U_R = \exp \{ i/2 \cdot \tan^{-1} [2V(\boldsymbol{\tau} \cdot \mathbf{p})] \}, \quad 2)$$

the angle-dependent term H'_{aug} can identically be canceled; (see (25)). For the moment, we recall the fact that a transformation function of half-arctangent-type has often proved useful in a problem in which $1/2$ -spin variables appear; the treatment of a free Dirac particle with non-vanishing momentum or the reduction of Ps - ps coupling in the meson theory are the examples, and in every case the physical interpretation of results has been given by considering some rotation of spin vector. Here the same type of transformation is powerful in treating the precession of $\boldsymbol{\tau}$ -spin. Accordingly, we name this type of transformation as the spin-rotation-type or rotation-type in short, and put the index 'R' to relevant quantities.

The results of the transformation generated by U_R are, according to Eqs. (22)–(24),

$$H_{\text{pot}}^R = \frac{1}{4p^2} \left(1 - \frac{1}{\sqrt{1+4V^2p^2}} \right)^2 + \frac{V}{2} \frac{1}{(1+4V^2p^2)^2} - \frac{2V^2}{\sqrt{1+4V^2p^2}}, \quad (29)$$

$$H_{\text{sp-orb}}^R = \frac{2V^2}{1 + \sqrt{1+4V^2p^2}} ([\mathbf{p} \times \mathbf{q}] \cdot \boldsymbol{\tau}), \quad (30)$$

$$H_{\text{rad}}^R = \frac{2V^3p^2}{1+4V^2p^2} \frac{(\boldsymbol{\tau} \cdot \mathbf{p})}{p^2} (\mathbf{p} \cdot \mathbf{q}) + \text{conj.} \quad (31)$$

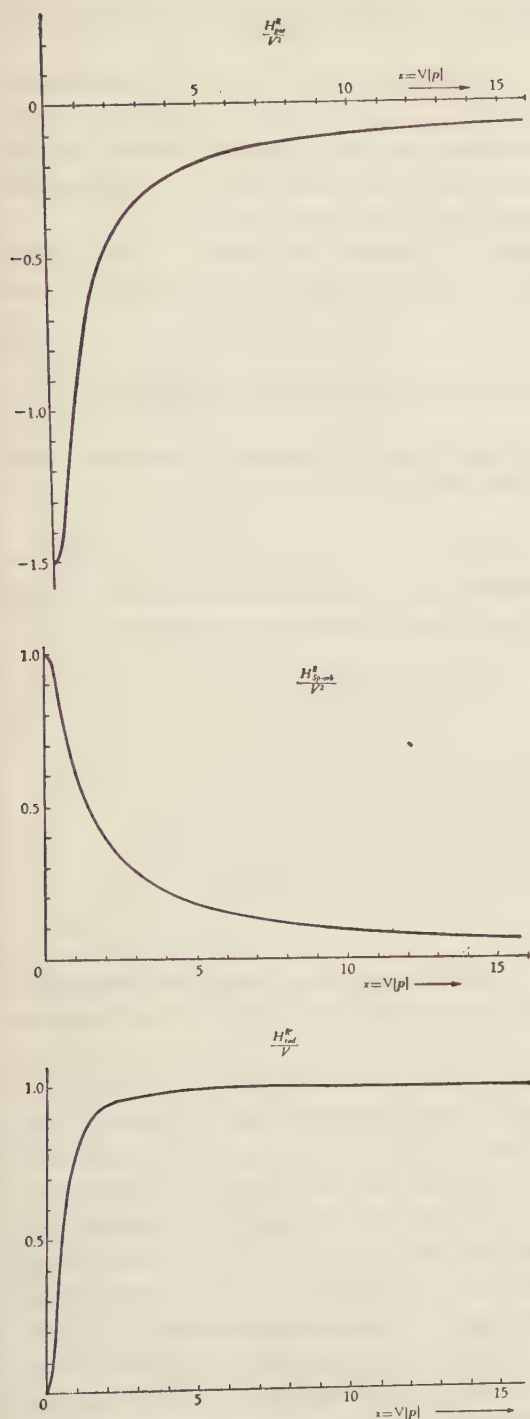


Fig. 4.

They are plotted in Fig. 4 as functions of $|p|$ except for factors $(\tau \cdot p)(p \cdot q)/p^2$ or $(\tau \cdot [p \times q])$.

The H^R_{sp-orb} term is larger in the neighbourhood of the origin of p -space, but is damped in the region $|p| \gg 1/2V$ with the asymptotic form $V/|p|$. It should be remembered that H^R_{sp-orb} gives rise to a splitting of correct sign between states with common value of $[p \times q]^2$, namely the states with parallel charge and isotopic spin vectors are made lower.

On the other hand, the remaining interaction term H^I_{rad} vanishes at the origin and is small in the region $|p| < 1/2V$, but is equal to V in the region far from the origin. ($|p| \gg 1/2V$).

The wave functions of low-lying states have an exponential factor $\exp(-\frac{1}{2}p^2)$ and in the region $|p| > 1$ their values are very small. In a weak-coupling case, we may safely neglect H^I_{rad} , because H^I_{rad} is small in region $|p| < 1$ since $1 < 1/2V$ and its matrix elements between low-lying states are very small. The isotopic spin-orbit coupling operator can readily be made diagonal in a given charge state, and the effective Hamiltonian $H^R_0 + H^R_{int} + H^R_{sp-orb}$ is free from any term which causes a change of meson occupation number by an odd number. Thus we can successfully eliminate the interaction Hamiltonian by virtue of the transformation U_R . In the strong-coupling case, on the contrary, we have $1 \gg 1/2V$, and the major part of wave function is perturbed by the interaction term of strength V ; while H^R_{int} and H^R_{sp-orb} are of secondary importance, for they affect only a small portion of wave function. Since the interaction term is averaged over angle and is correlated only with the radial mode, we have verified that, in the strong-

coupling limit, it is important to treat the interaction of the radial mode, first disregarding angular modes.

$(1/2)p^2 + H''_{\text{int}}$ is the effective potential of the meson oscillator after the transformation. H''_{int} has the value $-(3/2)V^2$ at the origin but it is damped in the region $|p| > 1/2V$ with the asymptotic form $-V/|p|$. We can expect a level shift of about $-(3/2)V^2$ for the ground state in a weak-coupling case. This is because we have $1/2V \ll 1$ in such a case and the ground state wave function is extended only in the region $|p| \lesssim 1$.

If the coupling constant V becomes larger, it is necessary to perform a second transformation to eliminate the interaction of radial mode, which will be considered in next section.

§ 4. Elimination of the radial mode in the interaction Hamiltonian

The radial mode can identically be eliminated from the interaction Hamiltonian, only when we put the transformation function in the form

$$U_T = \exp \{ iV(\tau \cdot p) \}. \quad (3)$$

This transformation corresponds to some translation in the q -space, and accordingly, we put the index 'T' to relevant quantities. The transformed Hamiltonian is given by

$$H_{\text{int}}^T = -\frac{1}{2}V^2 + \left\{ \frac{\sin^2(V|p|)}{p^2} - V \frac{\sin(2V|p|)}{|p|} \right\}, \quad (32)$$

$$H_{\text{spin-orb}}^T = -(\tau \cdot [p \times q]) \left\{ \frac{\sin^2(V|p|)}{p^2} - V \frac{\sin(2V|p|)}{|p|} \right\}, \quad (33)$$

$$H_{\text{int}}^T = \frac{1}{2} \left\{ V \cos(2V|p|) - \frac{\sin(2V|p|)}{2|p|} \right\} \left\{ (\tau \cdot q) - \frac{(\tau \cdot p)}{p^2} (p \cdot q) \right\} + \text{conj.} \quad (34)$$

These functions are illustrated in Fig. 5. In this case the remaining interaction Hamiltonian depends on angular variables; hence the transformation U_T has a character complementary to U_R of the preceding section.

The effective coupling strength in H_{int}^T oscillates rapidly in the region $|p| > 1/V$. In a strong-coupling case, H_{int}^T has small matrix elements between low-lying states; they are actually proportional to $\exp(-V^2)$ (see Eqs. (10, a—c)). This is because the major part of the wave function lies in the region $|p| < 1/V$, and are slowly varying. Consequently, we can consider the interaction Hamiltonian is approximately eliminated in strong-coupling cases by means of U_T . The effective potential also exhibits a characteristic feature of the strong-coupling theory. H_{int}^T oscillates rapidly about the value $-(1/2)V^2$ in the region $|p| > 1/V$, and to the first approximation, the level of the ground state may be $-(1/2)V^2$.

As mentioned earlier, the transformation function U_T is derived by assuming that the system is perturbed only by the radial mode of meson oscillators with constant strength V . Then we proceed in analogy to the Bloch-Nordsieck transformation. However, as is shown

in the preceding section, this assumption is justifiable only in the region $|p| \gg 1/V$. In the inner region $|p| \lesssim 1/V$, the components corresponding to angular modes can never be neglected.

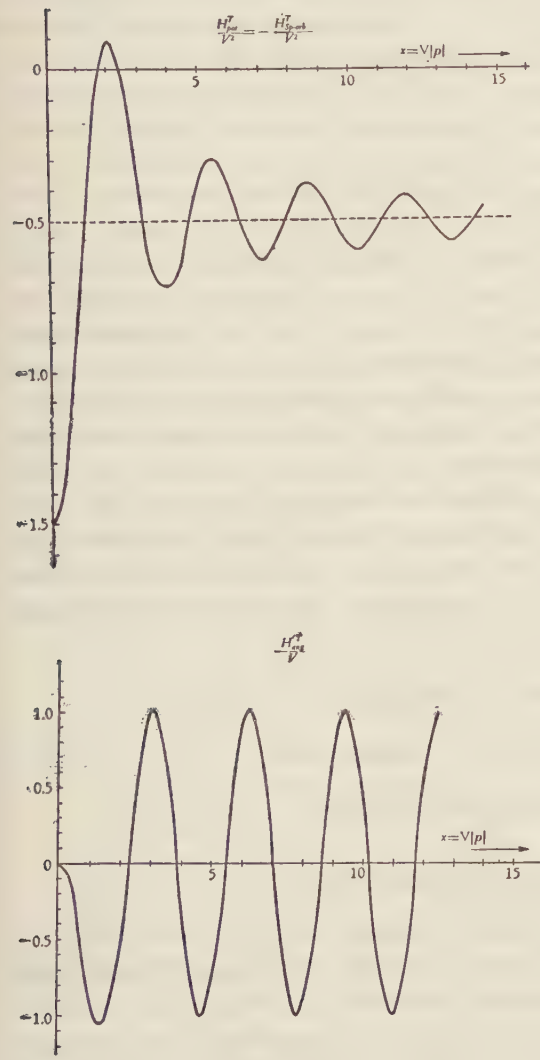


Fig. 5.

The behavior of transformed Hamiltonian is in fact unsatisfactory in the inner region. The effective potential has a high peak near $|p|=2/V$, whose effect of wrong sign cannot be canceled by virtue of neighbour potential valleys; and it makes the level shift a little wrong, i.e., $-(1/2)V^2+1$, in contradiction to the correct value $-(1/2)V^2-1$. The shape of H^T_{sp-orb} has some close relationship to that of H^T_{l-orb} and $H^T_{s_l-orb}$ has a deep valley near $|p|=2/V$. This valley makes the expectation value of $H^T_{s_l-orb}$ negative in a strong coupling case, which is a wrong sign; and among the first low-lying excited states, for example, the state with $I=1/2$ will be brought lower than the state with $I=3/2$, if the coupling constant becomes larger than a certain limiting value (about 0.81).^{*} We must make use of a generating function of such a form as discussed in the preceding section if we want to get rid of these defects in the inner region.

Summing up our analysis made so far, we must have a generating function of the form, which is similar to $(1/2)\tan^{-1}(2V|p|)$ in the inner region ($|p| < 1/V$), but approaches $V|p|$ asymptotically in the outer region ($|p| \gg 1/V$); the deviation from $V|p|$ must be largest in the region $|p| \sim 1/V$, since the situation is most involved there as is seen above.

^{*} We have

$$\langle 1|H_{sp-orb}^T|1\rangle = -[I(I+1) - 2 - 3/4] [e^{-V^2}(1 + 4V^2 - 4V^4) - 1],$$

where $\langle 1|\cdots|1\rangle$ denotes the expectation value taken with respect to any one of the first excited states of meson oscillators. I is the quantum number of total isotopic spin. The second factor vanishes at $V=0.81$ and is positive (negative) for V smaller (larger) than the critical value.

It is noticeable that V and $|\mathbf{p}|$ always appear in combination, so we can speak of regional character of p -space, independently of V , if we use $1/V$ for the unit of length of \mathbf{p} ; we put $x=V|\mathbf{p}|$ hereafter. The division of two regions at $x=1$ is somewhat arbitrary, and there may be some intermediate region. In fact, as will be discussed in § 6, our approach is to some extent powerless in the region $x \sim 1$. It requires a computational technique more powerful than is available at present to improve this point. But it seems tolerable for us to divide the p -space into two regions at $x=1$, as far as qualitative arguments are concerned. The curve of $f(x)$ plotted vs. x is somewhat similar to that of Sawada's parameter λ_s plotted vs. V , if the latter curve is made smooth in the intermediate region. This fact is more meaningful than a superficial similarity. The inner region $x < 1$ gives features common in weak coupling cases to our system, while the outer region $x > 1$ is essential for the features of strong coupling cases. As the wave functions extends approximately up to $x=V$, the magnitude of the coupling constant determines the character of regions which are covered by wave functions. On the basis of this fact we must expect some close relationship existing between the regional character of p -space, which is represented by $f(x)$, and the character of coupling strength, which is represented by λ_s ; where λ_s is the effective coupling constant in which some effects of higher order perturbation theory are taken into account in a kind of averaged form.

Incidentally, it may be natural for us to define the strong-coupling case as the one in which the weak-region branch of λ_s gives poorer results than $\lambda=V$ does, or in other words, a case in which the precession of τ -spin gives only a secondary effect. Then we have $V \sim 3.5$ for the lower limit of the strong coupling region. Thus we have the following scheme in the symmetrical scalar problem

$$\begin{array}{ll}
 V < 1/2 & \text{the weak coupling region,} \\
 1/2 < V < 3.5 & \text{the intermediate coupling region,} \\
 3.5 < V & \text{the strong coupling region.}
 \end{array} \tag{35}$$

§ 5. Variational calculation

We have thus far exclusively used a generating function which is an odd function of the argument $(\tau \cdot \mathbf{p})$. We are forced to consider this type of generating function, since otherwise we cannot perform ordering of operators in the transformed Hamiltonian by known techniques of operator calculus. This is concluded in the following way: An exponential function of operators can be ordered, at present, if and only if the argument is a linear or bilinear form of operators. Thus we can perform ordering of the transformed Hamiltonian, applying Fourier transformation to the terms to be ordered (cf. Eq. (9)), if the argument of the transformation function is a linear or bilinear form of operators. But no bilinear form is useful, since we want to eliminate the interaction term. Hence we should consider a generating function, which is an odd function of the argument

$$\lambda(\tau \cdot \mathbf{p}) + \mu(\tau \cdot \mathbf{q}), \tag{36}$$

where λ and μ are arbitrary numerical constants. If we perform a transformation by

$$R = \exp[-i(\tan^{-1}\mu/\lambda)(\mathbf{p}^2 + \mathbf{q}^2)/2], \quad (37)$$

the total Hamiltonian in the new representation is put into the form

$$R^{-1}HR = H_0 + \frac{\lambda V}{\sqrt{\lambda^2 + \mu^2}}(\boldsymbol{\tau} \cdot \mathbf{q}) + \frac{\mu V}{\sqrt{\lambda^2 + \mu^2}}(\boldsymbol{\tau} \cdot \mathbf{p}), \quad (38)$$

and at the same time the generating function is put into the form

$$R^{-1}UR = \exp\{if[\sqrt{\lambda^2 + \mu^2}(\boldsymbol{\tau} \cdot \mathbf{p})]\}. \quad (39)$$

Then it is evident that we must put $\mu=0$ in the new representation, because the last term on the right hand side of Eq. (38) remains unchanged by the transformation generated by (39). Consequently, we are led to consider a transformation function of the form used in this paper.

Let us now try to minimize effects of interaction terms of any type by virtue of a suitable transformation. Some remark was given in preceding section about the shape of generating function. But it is impossible to invent a transformation function which can identically eliminate both radial and angular modes from the interaction Hamiltonian, as far as the generating function is restricted to the type considered here. Then we must require that our transformation should satisfy the condition, 'The radial and angular interaction terms should give no remarkable effect to the lowest state.'. This condition may be represented by the following two equations

$$\int_0^\infty \left(\begin{array}{c} \text{effective coupling strength in } H'_{\text{rad}} \\ \text{considered as a function of } |\mathbf{p}| \end{array} \right) \exp(-|\mathbf{p}|^2) \cdot \mathbf{p}^2 d|\mathbf{p}| \rightarrow \min. \quad (40)$$

$$\int_0^\infty \left(\begin{array}{c} \text{effective coupling strength in } H'_{\text{ang}} \\ \text{considered as a function of } |\mathbf{p}| \end{array} \right) \exp(-|\mathbf{p}|^2) \cdot \mathbf{p}^2 d|\mathbf{p}| \rightarrow \min. \quad (41)$$

If these minimum values are actually very small, we may safely neglect effects of H'_{rad} and H'_{ang} -terms in the transformed Hamiltonian, at least when we are concerned with low-lying states. Then we have only to use the effective Hamiltonian

$$H_{\text{eff}} = H_0 + H_{\text{pot}} + H_{\text{sp-orb}}.$$

In each charge state, the spin-orbit coupling operator in $H_{\text{sp-orb}}$ can be made diagonal, and we have an effective Hamiltonian representing oscillators in p -space with modified potential for each state.

At this point, some closer examination of H_{pot} may be useful. It consists of two parts, as is shown in Eq. (22)

$$H_{\text{pot}} = H_{\text{pot}}^1 + H_{\text{pot}}^2$$

with

$$H_{\text{pot}}^1 = \left[\frac{1}{2} (df/d|\mathbf{p}|)^2 - V(df/d|\mathbf{p}|) \right], \quad \text{and} \quad H_{\text{pot}}^2 = S^2 - 2VCS.$$

Only H'_{pot} gives remarkable effect in the outer region of p -space, and its physical meaning is understandable, if we take $df/d|p|$ for the effective coupling strength of our system and adopt an analogy to the Bloch-Nordsieck transformation. On the other hand, we may regard the effect of rotation of τ -spin as the cause of H'_{pot} . There is another support to a conjecture that H'_{pot} may be closely related to the precession of τ -spin; the effective strength of $H'_{\text{sp-orb}}$, which represents characteristically a result of τ -spin's precession, is of the just same form but of the reverse sign as this term. We may naturally expect that if the terms $S^2 - 2VCS$ are correctly given, it will then be possible in the cases of strong coupling to get a level shift which is not far from the correct value $-(1/2)V^2 - 1$, and to obtain simultaneously correct spin-orbit splitting of levels.

From a practical point of view, we will here introduce a variational approach, and assume a rather simple functional form for the generating function, in which some parameters are varied as to satisfy above conditions. The form here adopted is

$$f(x) = \frac{1}{2} \tan^{-1} \left[\frac{2x}{1 + \lambda'x^2} \right] + \frac{x^3}{\mu' + x^2}, \quad (42)$$

or

$$f(x) = x - \frac{3x^3}{4 + \lambda x^2 + \mu x^4}, \quad (43)$$

with two parameters to be varied. This form is in accordance with the general remark given in the preceding section, as it approaches to $f(x) = x$ as $x \rightarrow \infty$, and to $f(x) = (1/2)\tan^{-1}(2x)$ as $x \rightarrow 0$. The numerical results are given in Table 1. Calculations are made at $V = 0.2, 1, 5$, which are representing the weak, intermediate and strong-coupling region respectively.

Table 1.

V	0.2			1.0	5.0	
$0\langle H_{\text{pot}} \rangle_0$	-1.41×0.04			-0.935×1	-0.478×25 = ($-0.5 \times 25 + 0.55$)	-0.477×25
$1\langle H_{\text{sp-orb}} \rangle_1$	0.93×0.04			0.315×1	$-0.373 =$ -0.0149×25	
$1\langle H'_{\text{rad}} \rangle_0$ (Eq. (40))	0.	0.14×0.2	0.18×0.2	0.390×1	-0.0559×5	
$1\langle H'_{\text{anc}} \rangle_0$ (Eq. (41))	-0.05×0.2	-0.01×0.2	0.00×0.2	-0.345×1	0.0030×5	
parameter						
λ		15.	7.2	7.2	4	4
μ	1	1	0.5

It is true that some improvement to Sawada's results is obtained, but it is insufficient in the strong coupling cases, and we are far from reproducing finer details of the strong coupling theory.

We have here computed expectation values of various terms not using the eigenfunctions of the effective Hamiltonian but using the eigenfunctions of the unperturbed harmonic oscillators, in the hope that the former may well be approximated by the latter. Some further improvement on our results is possible, if we make use of the more exact wave functions for each state, which are eigenfunctions of the effective Hamiltonian, $H_0 + H_{i, \text{int}} + H_{\text{sp-orb}}$. However, it seems improbable that we can expect much in this way, because there is some serious difficulty in carrying out our approach* in not-weak-coupling cases perfectly in a line with the qualitative prospect thus far obtained. This difficulty arises from the limited form of our generating function. And the region $x \sim 1$ can never be well treated.

§ 6. Discussions

We cannot quantitatively obtain a result, which parallels the qualitative prospect thus far discussed, so long as the generating function is restricted to some odd functions of $(\tau \cdot p)$ alone. The general form of the generating function G can be obtained from the solution of the operator equation

$$-i[H_0, G] = H' - (1/3)[[H', G]G] - (1/45)[\cdots [H'G]G]G] - \cdots \\ - \{2^{2n} B_n / (2n)\} [\cdots \overbrace{[H'G] \cdots G}^{2n \text{ brackets}}], \quad (44)$$

where B_n 's are Bernoulli numbers; this equation is obtainable by means of the method investigated by one of the authors⁽⁵⁾ (S.T.); the n -th coefficient is that of x^{2n} in the expansion of $x \cot x$. When the generating function G is expanded into powers of V and is put in the form

$$G = VG^{(1)} + V^2 G^{(2)} + \cdots, \quad (45)$$

the first term should be the solution of the equation

$$-i[H_0, G^{(1)}] = (\tau \cdot q). \quad (46)$$

This equation can readily be solved and the result is

$$G^{(1)} = (\tau \cdot p).$$

Then $G^{(3)}$ should be the solution of the equation

$$i[H_0, G^{(3)}] = (1/3)[[(\tau \cdot q), G^{(1)}]G^{(1)}] \\ = (1/3) \{ (p^2 + q^2)(\tau \cdot q) + \text{conj.} \} \\ + (1/3) \{ (p^2 - q^2)(\tau \cdot q) - [(p \cdot q) + (q \cdot p)](\tau \cdot p) + \text{conj.} \}. \quad (47)$$

* Alone with the condition that H'_{int} should give a minimum effect or that $H'_{\text{sp-orb}}$ should give a maximum effect, we are led to the answer that $f(x)$ is $(1/2)\tan^{-1}(2x)$; while alone with the condition that the H'_{int} should give a minimum effect, we are led to the answer that $f(x)$ is x . Actually we are forced to determine the best value of parameters by the condition that the approximative level of ground state (computed with the unperturbed wave function) should become as low as possible.

If only the first member of the right hand side is retained we obtain a part of $G^{(3)}$, which is given by

$$G^{(3)'} = -(1/3) \{ (p^2 + q^2) (\tau \cdot p) + \text{conj.} \}.$$

If we further make an approximation to replace q^2 by p^2 , considering that the same expectation value is obtainable when applied to the ground state, then we have

$$G^{(3)'} \sim -(4/3) (\tau \cdot p)^3$$

which gives the second term of the expansion of $(1/2) \tan^{-1}(2V(\tau \cdot p))$. However neglect of the second member on the right hand side of Eq. (47) cannot be permissible. There is no more justification, if any, than that its inclusion causes so much complication of the generating function G that we can hardly give the ordered form of the transformed Hamiltonian explicitly. In fact the exact solution to Eq. (47) is given by

$$G^{(3)} = -(2/3) (q^2 (\tau \cdot p) + \text{conj.}) + (1/3) [(p \cdot q) + (q \cdot p)] (\tau \cdot q) + \text{conj.}.$$

Particularly, omission of such a term $G^{(3)''}$,

$$G^{(3)''} \sim [(p \cdot q) + (q \cdot p)] (\tau \cdot q)$$

causes a serious effect on the transformed Hamiltonian. Since $i[H, G^{(3)''}]$ is just the term of the type of H'_{rad} , inclusion of $G^{(3)''}$ or a term of similar nature yields an effect that, to some extent, H'_{rad} is reduced smaller, while H'_{int} remains nearly unchanged. Thus we can expect that the results of our transformation will be characteristically affected, if we include a function of the form $G^{(3)''}$ into the generating function, though we will have some difficulty in performing the ordering procedure of the transformed Hamiltonian.

In analogy to the treatment of rotation of τ -spin in more usual cases, we must have two types of τ -operator in the argument of the generating function of our transformation. This means that we must include both $(\tau \cdot p)$ and $(\tau \cdot q)$ into the generating function. But as was discussed in the previous section, the linear combination

$$\lambda(\tau \cdot p) + \mu(\tau \cdot q)$$

is useless; and, as suggested above, the part containing $(\tau \cdot q)$ must appear in the form

$$(\tau \cdot q) \{ [(p \cdot q) + (q \cdot p)] + \dots \}.$$

The transformation function which is given in terms of $(\tau \cdot q)(p \cdot q)$ or similar forms may be effective in the intermediate region of p -space. The properly inner ($|p| \sim 1/V$) or outer ($|p| \sim 1/V$) regions of p -space can be manipulated by generating functions of the type which are similar to $f(x) = (1/2) \tan^{-1}(2x)$ or $f(x) = x$. Indeed, it is in the region $1/V \lesssim |p| \lesssim 3/V$ that we are annoyed by a high peak of H_{pot} . If a transformation function given in terms of $(\tau \cdot q)(p \cdot q)$ or similar forms were successfully applied, H'_{rad} in the intermediate region of p -space could be reduced smaller and then the transformations of rotation-type, which has been discussed in Sec. 3, would have a wider range of applicability in p -space, so that H_{pot} after such transformations would become far less oscillating and assume a well-behaved form which we expect on the basis of qualitative analysis made

in this paper. If the intermediate region of p -space could successfully be dealt, then $H_{\text{sp-orb}}$ would correctly be given and the results of the strong-coupling theory might well be reproduced in its finer detail.

In conclusion, we have made analysis of the bound-meson problem of the symmetrical scalar theory, and pointed out the regional character in p -space. When the knowledge about these regional character are combined with the fact that the wave functions of low-lying states are not extended beyond a certain limit, we can predict how the character of our problem is changed as the coupling constant is varied from the weak-coupling region into the strong-coupling region. Because of our limited possibility in handling a complicated form of operators, the success achieved is only partial as regards to its quantitative aspects. However, the direction has been discussed in which the future improvement might be explored.

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Letters to the Editor

Some Remarks on the Applicability of the Field Theory from the Stand-point of the Distribution Analysis

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The divergence difficulties are the serious defect of the field theory, but so far as quantum electrodynamics is concerned, we can overcome these difficulties by the renormalization method, and can get the results which are in good agreement with experiments. In spite of this success, the renormalization method contains a mathematically inadmissible process, i.e., 'the subtraction of infinite quantities'. As we can obtain some results of physical significance through the mathematically inadmissible process, we may consider that if we reformulate the theory with mathematical rigour, we shall be able to arrive at the same results without such difficulties as divergences. Such reformulation will be made using the concept of distributions of Schwartz¹⁾, because singular functions in the field theory, which are important but strange, can be treated rigorously only in connection with this concept²⁾. In the following, we will call these singular functions 'A-type functions.'

At first, we define various spaces as follows³⁾:

- i) The space (b) :
The space of complex-valued functions of n real variables, each of which is indefinitely differentiable and has a compact carrier.
- ii) The space (b') (The space of distributions):
The adjoint space of (b) .
- iii) The space (\mathfrak{B}) :
The space of complex-valued functions, each of which is indefinitely differentiable and is 'of rapid decay at infinity.'
- iv) The space (\mathfrak{B}') (The space of temperate distributions):
The adjoint space of (\mathfrak{B}) .
- v) The space (\mathfrak{h}) (The Hilbert space):

The space of complex-valued functions, each of which is square integrable. (\mathfrak{h}) is self-adjoint. These spaces satisfy the relation

$$(b) \subset (\mathfrak{B}) \subset (\mathfrak{h}) \subset (\mathfrak{B}') \subset (b'). \quad (1)$$

All A-type functions belong to (\mathfrak{B}') . In (\mathfrak{B}') , we can define easily the Fourier transformation, and the Fourier transform of each function of (\mathfrak{B}') belongs to (\mathfrak{B}') again⁴⁾, so in this space we can treat the problem equally well both in the p -representation and in the x -representation. Therefore, in the field theory it is not necessary to use (b') , and we have only to use (\mathfrak{B}') .

In the Schrodinger representation, the space of state vectors is defined as follows:

$$(\mathfrak{F}) = (\mathfrak{h}_0) \oplus (\mathfrak{h}_1) \oplus \cdots, \quad (2)$$

where (\mathfrak{h}_n) , $n=0, 1, \cdots$, is the Hilbert space (\cdot) of $3n$ variables (coordinates of n particles), and \oplus denotes the direct sum. The Schrödinger equation is usually given by

$$i(\partial/\partial t)\Psi = H\Psi \quad \text{in } (\mathfrak{F}). \quad (3)$$

In these theories, H must be an Hermitian operator, though it is not so in many cases and $H\Psi$ cannot be contained in (\mathfrak{F}) in general, because A-type functions are not contained in (\mathfrak{h}) . Corresponding to (2), therefore, we define a new space (\mathfrak{E}') as follows:

$$(\mathfrak{E}') = (\mathfrak{B}_0') \oplus (\mathfrak{B}_1') \oplus \cdots, \quad (4)$$

and denote its adjoint by (\mathfrak{E}) . We can show that

$$(\mathfrak{E}) \subset (\mathfrak{F}) \subset (\mathfrak{E}'), \quad (5)$$

and that (\mathfrak{E}) is everywhere dense in (\mathfrak{F}) , and (\mathfrak{F}) in (\mathfrak{E}') ⁵⁾.

As A-type functions in the field theory belong to (\mathfrak{B}') , domains and ranges of all operators are contained in (\mathfrak{E}') . We cannot define a multiplication of any two distributions, and so, when we deal with a product of two operators, it must be considered whether it is admissible or not. In view of these facts, we propose that the Hamiltonian H must be modified so as to contain *admissible products only*, and that *the precise meaning of the Schrödinger equation (3) should be as follows:*

$$i(\partial\Psi/\partial t) = H\Psi \quad \text{in } (\mathfrak{E}'), \quad (\text{not in } (\mathfrak{F})!), \quad \Psi \in (\mathfrak{E}). \quad (3')$$

In the following, we will call H 'quasi-Hermitian' if, for any vectors Φ_1 and Φ_2 in the domain of H , there exists an 'inner product' $(\Phi_1, H\Phi_2)$ satisfying the condition

$$(\Phi_1, H\Phi_2) = (H\Phi_1, \Phi_2).$$

H is not Hermitian in general, but it must be 'quasi-Hermitian,' and if so, the condition $|\Psi(t)| = \text{const.}$ is satisfied.

Now we must study the existence of solutions of the Schrödinger equation. We can, however, give no proof as yet and can only surmise that the existence of them is very probable at least in the case of quantum electrodynamics. The Schrödinger equation of quantum electrodynamics in the interaction representation is

$$i(\partial\Psi/\partial t) = H\Psi \text{ in } (\mathfrak{S}'), H(t) = -\int j_\mu(x) A_\mu(x) dx^3. \quad (6)$$

As H is 'quasi-Hermitian,' we can show that, if there exists a solution of (6) which satisfies an initial condition, it is unique.

Since a product $H(t_1) \cdot H(t_2)$ has no mathematical meaning, the perturbation expansion is not admissible. Therefore, we regularize H , and make the perturbation expansion admissible⁶⁾. Let the regularized Hamiltonian be H_j , then we can assume without loss of generality⁷⁾

$$\lim_{j \rightarrow \infty} H_j = H, j = 1, 2, \dots.$$

Now consider the equation

$$i(\partial\Psi_j/\partial t) = H_j\Psi_j \text{ in } (\mathfrak{S}) \quad |\Psi(t_0)| = 1. \quad (7)$$

We can easily show that there is one and only one solution of (7), which can be given by the following series:

$$\begin{aligned} \Psi_j(t) = & \Psi(t_0) + (-i) \int_{t_0}^t dt_1 H_j(t_1) \Psi(t_0) \\ & + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_j(t_1) H_j(t_2) \Psi(t_0) + \dots \end{aligned} \quad (8)$$

Further, we can also show that in such a case we can apply Dyson's renormalization method with mathematical rigour⁸⁾, and that, when $j \rightarrow \infty$, the limits of the results of these renormalizations are the same as those given by Dyson. As Dyson's results are in good agreement with experiments, we may consider that, in the case of quantum electrodynamics, there exists a solution of (6), which can be given by $\lim_{j \rightarrow \infty} \Psi_j$, but it cannot be expanded in

powers of e (coupling constant).

Hitherto many authors have adopted equations like (7), as is seen in 'the non-local interaction theory'⁹⁾, but almost all of those authors denied the equation (6) and introduced such equations as (7) by some physical reasonings. From our standpoint the equation (6) is not denied, but as we know no method to solve it, we have introduced (7) as a mathematical procedure.

Now we may conclude as follows: At least in the case of quantum electrodynamics, the field theory is acceptable in the above-stated sense, and the divergence difficulties arise from the use of inadmissible method of computation. If only we can find an admissible approximation method, there will be more difficulties. Also in other cases, it is necessary to discover an admissible approximation method, and if there remains any difficulty further, or if it is shown that (3') has no solution, then we must seek after its modification from some physical standpoint.

- 1) L. Schwartz, *Théorie des distributions*, I, II, Hermann, Paris (1950-51).
- 2) T. Takahashi, *Prog. Theor. Phys.* **11** (1954), 1.
- 3) Cf. ref. 1) Chap. III §§ 1, 3; Chap. VII §§ 3, 4.
- 4) Cf. ref. 1) Chap. VII § 6.
- 5) Cf. ref. 1) Chap. III § 3; Chap. VII §§ 3, 4.
- 6) K. Gotō, *Prog. Theor. Phys.* **13** (1955), 112.
- 7) Cf. ref. 1) Chap. VI § 4.
- 8) F. J. Dyson, *Phys. Rev.* **75** (1949), 486, 1736.
- 9) P. Kristensen and C. Møller, *Dan. Mat. Fys. Medd.* **27** (1952), no. 7.

The Character of the Roton State in Liquid Helium

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One of the authors¹⁾ has computed an energy spectrum for the elementary excitations in liquid

helium, using a trial wave function of the form

$$\psi = \sum_i f(R_i) \varphi, \quad (1)$$

where φ is the ground state wave function and the sum runs over all the atoms in the liquid. For an excitation of momentum $\hbar K$, $f(R) = \exp iK \cdot R$ and the resulting spectrum is

$$E(K) = \hbar^2 K^2 / 2mS(K), \quad (2)$$

where $S(K)$ is the liquid structure factor and is known from experiment²⁾. This spectrum is correct for phonons ($K < 1\text{\AA}^{-1}$) and has the qualitatively correct feature of exhibiting a minimum at 2\AA^{-1} (roton region). Landau and subsequent workers³⁾ have shown that the specific heat and second sound velocity data require the value $\Delta/k = 9.6^\circ\text{K}$ where Δ is the minimum roton energy and k is Boltzmann's constant. This number is probably accurate⁴⁾ to 0.2°K . Formula (1) locates the minimum correctly but gives $\Delta/k = 19^\circ\text{K}$.

To improve the theoretical result an improved roton trial function was sought. Mathematically the next order of complication

$$\psi = \sum_{i,j} f(R_i, R_j) \varphi \quad (3)$$

suggests itself. From the fact that the momentum is $\hbar K$ we know $f(R_1, R_2)$ must be of the form $g(R_1 - R_2) \exp iK \cdot R_1$. The resulting variational equations to determine the best $g(R)$ turn out to be very complicated. This general attack was abandoned and we tried instead to find a physically reasonable mathematical form for $g(R)$ containing only a few parameters to be minimized.

A number of different physical arguments suggested about the same function but perhaps the clearest argument is the observation¹⁾ that the function (1) does not satisfy the conservation of current, and that in fact the current in (1) must be returned by a back flow. This flow, far from the roton, is similar to that from a dipole source, corresponding to a velocity potential $mS(R)$ where S is of the form

$$S(R) = a(K \cdot R)/R^3 \quad (4)$$

with a certain constant a . This suggests the function

$$\psi = \sum_i \exp(iK \cdot R_i + i \sum_{j \neq i} S(R_{ij})) \varphi. \quad (5)$$

For very small R of course S should differ from (4), but no atom comes closer than the 2.7\AA range of repulsion, and we propose to omit the term $i=j$ in the sum. Furthermore we shall leave a as an adjustable parameter to compensate for imperfections in the

form of S . This function is also very difficult to handle, but the variation in $\sum_j S(R_{ij})$ over the configurations allowed by φ is small, and we may use instead the function obtained by expanding the exponent,

$$\psi = \sum_i (\exp iK \cdot R_i) (1 + i \sum_j S(R_{ij})) \varphi. \quad (6)$$

This is the trial function which we finally used.

Some of the resulting integrals involve the coordinates of three atoms, and it was necessary to approximate such a three particle correlation function by a product of three two particle correlation function. Careful inspection and elaborate check calculations of the various terms which contribute to the answer shows that this approximation introduces a negligible error. We hope to publish a more detailed account of these considerations soon. Choosing a to minimize the energy we obtained the new value⁵⁾ $\Delta/k = 11.7^\circ\text{K}$. The position of the minimum, $K_0 = 2\text{\AA}^{-1}$, is substantially unchanged by this improvement in the wave function, and phonon energies (K very small) are also unchanged.

The roton state represented by the function (5) can be described roughly classically as a vortex ring of such small radius that only one atom can pass through the center⁶⁾. Outside the ring there is a slow drift of atoms returning for another passage through the ring. There are at least three ways that the classical picture is modified. (1) The momentum of atoms passing through the center cannot be made smaller because the wave function must return to its original value when, after one moves through, another stands in its old place. The wave length must be the atomic spacing. (2) The ring does not drift forward as a large smoke ring, because as it is as small as possible there is no force tending to shrink it; which force in a classical ring is balanced as a consequence of the forward drift. (3) The location of the ring is not definable. In typical quantum mechanical fashion the lowest energy state corresponds to superposition of amplitude to find the ring anywhere in the liquid. The energy is less than the kinetic energy $\hbar^2 K_0^2 / 2m$ of one atom with momentum $\hbar K_0$ because there is a correlated motion of many atoms moving together so the effective inertia is higher (the energy Δ/k corresponds to 2.5 atoms moving together at total momentum $\hbar K_0$).

The decided improvement in the value of Δ/k supports our belief that the roton wave function is close to (6) or (5), and that the above physical picture is roughly correct.⁷⁾

- 1) R. P. Feynman, *Phys. Rev.* **94** (1954), 262.
- 2) L. Goldstein and J. Reekie, *Phys. Rev.* **98** (1955), 857. We have chosen to normalize $S(K)$ slightly differently. With our normalization $S(K)=1.26$ at $K=2A^{-1}$, slightly less than the value in this reference.
- 3) De Klerk, Hudson and Pellam, *Phys. Rev.* **93** (1954), 28.
- 4) J. R. Pellam, (Private communication).
- 5) The value of α expected for the conservation of current argument leading to (4) is $1/4\pi\rho$, where ρ is the number density of atoms. The value of α which minimize the energy is very close to this.
- 6) "Progress in Low Temperature Physics," North Holland Publishing Co. (1955), Chap. II, R. P. Feynman.
- 7) We are indebted to C. G. Kuper for sending us a preprint of a paper in which he obtains substantially the same result for A/k with a wave function of form (3). A detailed comparison of our wave functions has not yet been made.

* Presently visiting Yukawa Hall, Kyoto University. This author wishes to express his gratitude for the kind hospitality he experienced during his visit to Japan.

The Cloudy Crystal Ball Model for 14 Mev Neutron Reactions

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As is well known the cloudy crystal ball model proposed by Feshbach, Porter and Weisskopf¹⁾ has

successfully accounted for low energy neutron experiments. Therefore, it seems worthwhile investigating whether this model still holds for the nucleon scattering at a higher energy region, and consequently seeing the energy dependence of the parameters of the optical potential. In this respect, Wood and Saxon²⁾ has actually carried out the analysis of the 20 Mev proton scattering data. There has, however, been no analysis, based upon the cloudy crystal ball model, for the neutron scattering at the intermediate energy.

Recently, Graves and Davis³⁾ have carried out an experiment on the nonelastic neutron scattering at 14 Mev and a comprehensive accumulation of data for 14 Mev neutron experiments has been tabulated in their paper. In this paper they especially note the fact that the ratio of the scattering cross section to that of the compound nucleus formation shows a characteristic diffraction-like-pattern if this quantity is plotted against $A^{1/3}$, and this behavior can hardly be fitted to the flat curve expected from the continuum theory.

In view of such situations, we have attempted to account for these observed behavior of the 14 Mev neutron scattering by the use of the cloudy crystal ball model. The preliminary result of our calculations will be reported here. At such a high energy as 14 Mev we may safely expect that the so-called compound elastic scattering is negligibly small. We have carried out the exact calculation by the partial wave method, using the square well optical potential. The parameters of the potential used are as follows:

$$\begin{cases} V=40 \text{ Mev} \\ W=8 \text{ Mev} \end{cases}, \begin{cases} V=40 \text{ Mev} \\ W=12 \text{ Mev} \end{cases}, \begin{cases} V=30 \text{ Mev} \\ W=6.5 \text{ Mev} \end{cases}$$

On the other hand the estimation of W by means of the semi-classical Fermi gas model yields the value of $W=10 \text{ Mev}$.⁴⁾

The results of the calculations together with the experimental data are plotted in Fig. 1, 2 and 3, in which σ_t represents the total cross section, σ_c the cross section for compound nucleus formation and σ_{sc} the scattering cross section. The calculated curve with $V=30 \text{ Mev}$ is not plotted since it was found that σ_{sc}/σ_c curve could not be fitted to the observed one by any choice of the nuclear radius R within a reasonable range, although the choice of the nuclear radius is somewhat arbitrary. Among the several formulae thus far proposed for R in terms of A , the mass number of the nucleus, we have tentatively adopted the one: $R=a+bA^{1/3}$. If we choose the value $a=0.5 \times 10^{-13} \text{ cm}$, $b=1.35 \times 10^{-13} \text{ cm}$

the curve σ_t vs $A^{1/3}$ is well reproduced by the calculated one with $W=12$ Mev, while, in the positions of the maximum and minimum of the σ_{sc}/σ_o vs $A^{1/3}$ curve, a slight discrepancy is observed. In order to remove the discrepancy about these points we have to choose the values, $a=0.78 \times 10^{-13}$ cm and $b=1.20 \times 10^{-13}$ cm. It is clear from the figures that the general trend of the cross sections as well as the ratios σ_{sc}/σ_o can qualitatively be accounted for by the cloudy crystal ball calculation. The existence of the second minimum in large $A^{1/3}$ region seems to be apparent in our calculated curve, σ_{sc}/σ_o vs $A^{1/3}$, but there is no sufficient experimental data to test our result about this point.

Quantitatively, however, the calculated values of σ_{sc}/σ_o are too high, in comparison with the observed data, especially in the region of $A^{1/3}$ near the first minimum. On taking still larger W our values of σ_{sc}/σ_o are found to decrease, but the general trend of the curve becomes flatter than the one obtained with the smaller W , in contradiction to the experiment. On account of the fact that the curve given by the continuum theory shows no minimum or maximum and is quite flat, we might say that the stronger the absorption of the incident wave in the nuclear matter becomes, the flatter the curve of σ_{sc}/σ_o against $A^{1/3}$. Thus, it would be hopeless to reproduce the experimental curve quantitatively so far as we adhere to the square well potential. This difficulty might probably be removed by adopting the optical potential with tail, just as has first been suggested by Feshbach, Porter and Weisskopf¹⁾ and as has been actually worked by Woods and Saxon²⁾ in their analysis of 20 Mev proton scattering. Then, because of the reduced reflection of the incident wave

at the smoothed-out nuclear boundary, we may expect the smaller σ_{sc} and correspondingly larger σ_o and thus reduced ratio σ_{sc}/σ_o without changing the depth of the imaginary potential W . In fact, allowing for the states of affairs that the use of the sharp edge optical potential is insufficient for accounting for the neutron scattering data in the energy region so far investigated it would be highly desirable to carry out an analysis with a rounded optical potential also for the 14 Mev neutron scattering.

The authors wish to express their thanks to Prof. T. Muto for his encouragement throughout the work.

- 1) Feshbach, Porter and Weisskopf, Phys. Rev. **96** (1954), 448.
- 2) R. D. Woods and D. S. Saxon, Phys. Rev. **95** (1954), 577.
- 3) E. R. Graves and R. W. Davis, Phys. Rev. **97** (1955), 1205.
- 4) Morrison, Muirhead and Rosser, Phil. Mag. **44** (1953), 1326.
Hayakawa, Kawai and Kikuchi, Prog. Theor. Phys. (to be published).

(caption of figures)

Fig. 1.

Total cross sections σ_t and cross sections for compound nucleus formation σ_o vs $A^{1/3}$. Points are experimental values, taken from Barshall et. al. for σ_t and Graves and Davis for σ_o , respectively. Solid curve represents the calculated value with the parameter $V=40$ Mev, $W=12$ Mev, and $R=(1.35A^{1/3})$

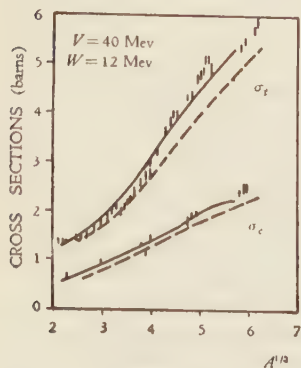


Fig. 1

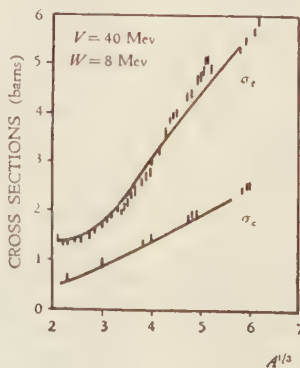


Fig. 2

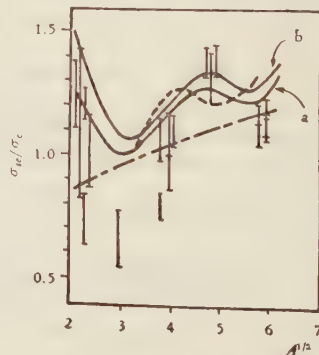


Fig. 3

$+0.35) \times 10^{-13} \text{cm}$ and the broken curve the result with the same V and W and $R = (1.20A^{1/3} + 0.5) \times 10^{-13} \text{cm}$.

Fig. 2.

Total and compound nucleus formation cross sections $vs A^{1/3}$. Points are experimental values taken from the same sources as in Fig. 1. Solid curve represents the calculated value with the parameter $V=40 \text{ Mev}$, $W=8 \text{ Mev}$ and $R = (1.35A^{1/3} + 0.5) \times 10^{-13} \text{cm}$.

Fig. 3.

The ratio $\sigma_{so}/\sigma_{\theta}$ $vs A^{1/3}$, points are the experimental

values taken from Graves and Davis. Solid curve a represents the calculated value with the parameter $V=40 \text{ Mev}$, $W=8 \text{ Mev}$, $R = (1.20A^{1/3} + 0.78) \times 10^{-13} \text{cm}$, and b curve expresses the calculated value with the parameter $V=40 \text{ Mev}$, $W=12 \text{ Mev}$ and $R = (1.20A^{1/3} + 0.78) \times 10^{-13} \text{cm}$. The broken curve represents the calculated value with the parameter $V=40 \text{ Mev}$, $W=13 \text{ Mev}$ and $R = (1.35A^{1/3} + 0.5) \times 10^{-13} \text{cm}$. The dott-broken curve represents the calculated value based on the continuum theory, with the nuclear radius $R = (120A^{1/3} + 0.78) \times 10^{-13} \text{cm}$.

Errata

Potential in Quantum Field Theory

Tsutomu IMAMURA

Progress of Theoretical Physics, 13, (1955), 183.

There were two mistake in our previous paper.

i) Eqs. (22), (23) and (24) must be replaced by

$$\langle pr, p'r' | R' | qs, q's' \rangle = \langle pr, p'r' | L | qs, q's' \rangle + \sum \langle pr, p'r' | L | lv, lv' \rangle (1/a_0) \langle lv, lv' | R' | qs, q's' \rangle, \quad (22)$$

$$\langle pr, p'r' | R' | qs, q's' \rangle = 2Krr' s s' (pH_0(p) ; p'H_0(q) + H_0(q') - H_0(p) ; qH_0(q)) \quad (23)$$

$$\text{and } \langle pr, pr' | L | lv, lv' \rangle = 2Lrr' v v' (pH_0(p), p'H_0(q) + H_0(q') - H_0(p) ; lH_0(l)). \quad (24)$$

where $H_0(q) + H_0(q')$ is the initial energy which corresponds to E in Brueckner-Watson's potential. Thus L is not a pure operator, hence the attempt to exclude E -dependence from a potential does not succeed. Therefore the discussions concerning this E -dependence in §3 must be removed. On account of the relations, $a=a_0$ (for $p^0 + p'^0 = E$) and $\langle pr, p'r' | R | qs, q's' \rangle = \langle pr, pr' | R' | qs, p's' \rangle$ (for $H_0(p) + H_0(p') = H_0(q) + H_0(q')$), the procedure by which we can get the transition matrix elements for scattering problems from eq. (22) and the procedure from eq. (10) are just the same if we replace L by v . Thus we can take L as the effective potential for scattering problems.

ii) The following terms must be added in the brackets in the right hand side of eq. (25),

$$u_{+\gamma} v (l) u_{-\gamma'} v' (l') (P/p^0 + p'^0 - H_0(l) + H_0(l')) (2/l^0 - H_0(l) + i\delta) u_{+\delta''} (l) u_{-\delta'v'} (l') \\ + u_{-\gamma} v (l) u_{+\gamma'} v' (l') (P/p^0 + p'^0 + H_0(l) - H_0(l')) (2/l^0 + H_0(l) - i\delta) u_{-\delta''} (l) u_{+\delta'v'} (l').$$

The author wishes to thank Dr. J. Iwadare for his kindness for drawing the author's attention to the above mentioned points.

Relativistic Wave Equations with Maximum Spin Two*

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The relativistic wave equations with maximum spin 2 in the canonical form are analyzed under the assumption that each matrix involved is a direct sum of two mutually commutable Duffin-Kemmer operators. Five independent sets of tensor equations are established by algebraic procedures and each set of equations is further reduced to several simple sets of equations, each corresponding to definite values of mass and spin. Physical consequences of the theory are discussed. Two mass values are possible, one being twice as large as the other, and possible values of spin are 2, 1 and 0. The field with spin 2 is of Fierz-Pauli type. But among the fields with spin 1 we find some ones governed by the equations different from the usual ones of the vector meson. The total energy of the whole field is not positive definite, the contributions of the parts with lower and higher mass values being positive and negative respectively.

§ 1. Introduction

The first order relativistic wave equations in the canonical form

$$(\partial^\rho \alpha_\rho + \kappa)\psi = 0 \quad (1)$$

have long been the subject of wide-spread investigations in the hope that they might describe the elementary particles of higher spin. Most of these works have been based on the assumption that each matrix α_ρ was the direct sum of s independent Dirac operators in the case of maximum spin $s/2$, with a few exceptions among which we note particularly the Fierz-Pauli equation of spin $3/2$ in Dirac form due to K. K. Gupta¹⁾ and Bhabha's equation for a particle having two mass states.²⁾ The investigations so far developed have been concerned mainly with the general group-theoretical aspect of the theory.³⁾ But on account of the complicated nature of the algebra and representation of the matrix, the explicit solution of the general particle wave equation has not yet been given. Especially for the case of maximum spin $3/2$ the irreducible representations of the matrices have been found by Hönl and Boerner⁴⁾ and by Madhava Rao and others⁵⁾ independently, and some physical consequences have been discussed. However for the case of maximum spin two we have no satisfactory theory. In 1941 Tonnelat investigated the field equations satisfied by the field quantities which are the direct product of four independent solutions of the Dirac equations⁶⁾. However this fusion theory of spin two can not be regarded, strictly speaking, to be based on the canonical form (1) alone. Recently Green has taken up the problem

* The essential ideas and main results of the present work were presented at the annual meeting of the Physical Society of Japan held at Osaka, October 31, 1954.

of deriving tensor equations from eq. (1) on the assumption⁷⁾

$$\alpha_p = \beta_p \times E + E \times \beta_p' \quad (2)$$

where β_p 's are the Duffin-Kemmer matrices, though his basic procedures are essentially the same as those already used by Tonnelat. In his algebraic treatment we find a certain deficiency originated from laying too much stress on the role played by the commutation relations. In the present paper we shall present a satisfactory solution of the same problem on the basis of the linearly independent elements of the Duffin-Kemmer algebra constructed by one of the authors⁸⁾.

§ 2. Preliminary treatment of the theory. The case of spin 0 and 1

As a guide for proceeding to the case of maximum spin 2, we shall exhibit how one can construct a theory of spin 0 and 1 on the basis of the Dirac algebra defined by the commutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} E. \quad (3)$$

As is well known, the linearly independent basis of the Dirac algebra is formed by the 16 elements

$$\gamma_A = \{E, \gamma_\rho, \sigma_{\mu\nu} \gamma_5 = (i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \gamma_5, i\gamma_\rho \gamma_5, \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4\}, \quad (4)$$

for each of which we have $\gamma_A^2 = E$. (The reason for taking $\sigma_{\mu\nu} \gamma_5$ instead of $\sigma_{\mu\nu}$ will be seen in eq. (11).) The Duffin-Kemmer wave equation reads

$$(\partial^\rho \beta_\rho + \kappa) \psi = 0 \quad (5)^*$$

with a 16-component column matrix ψ . Here we assume that β_ρ is constructed from two mutually independent Dirac operators γ_ρ and γ_ρ' as

$$\beta_\rho = (1/2)(\gamma_\rho \times E + E \times \gamma_\rho'). \quad (6)$$

Corresponding to this representation of β_ρ the operand ψ must be specified by a pair of indices μ and ν as $\psi_{\mu\nu}$, γ_ρ operating on the index μ and γ_ρ' on the ν . Then for the operation $\beta_\rho \psi$ we have

$$\begin{aligned} (\beta_\rho \psi)_{\mu\nu} &= (1/2)[(\gamma_\rho)_\mu^\lambda (E)_{\nu}^\sigma + (E)_\mu^\lambda (\gamma_\rho')_{\nu}^\sigma] \psi_{\lambda\sigma} \\ &= (1/2)[(\gamma_\rho)_\mu^\lambda \psi_{\lambda\nu} + \psi_{\mu\sigma} (\gamma_\rho')_{\nu}^\sigma], \end{aligned}$$

since $(E)_{\mu\nu} = \delta_{\mu\nu}$. If we regard each $\psi_{\mu\nu}$ as an element of a 4-dimensional square matrix \mathcal{P} , situated in the μ -th row and ν -th column, the last equation establishes the correspondence

$$\beta_\rho \psi \longleftrightarrow (1/2)(\gamma_\rho \mathcal{P} + \mathcal{P} \gamma_\rho'^T),$$

where the superscript T denotes the transposed matrix. Thus, in place of the original equation (5) we have

⁸⁾ Throughout in this paper we shall use the natural unit and put $x_4 = x^4 = it$. We adopt the Euclidian metric and summation convention over repeated indices is also observed.

$$(1/2)\partial^p(\gamma_p\mathcal{V} + \mathcal{V}\gamma_p'^T) + \kappa\mathcal{V} = 0.$$

Then on multiplication from right by a non-singular matrix A such that $\gamma_p'^T = A\gamma_p A^{-1}$ the last equation gives

$$(1/2)\partial^p\{\mathcal{V}A, \gamma_p\} + \kappa\mathcal{V}A = 0, \quad (7)$$

where $\{A, B\} \equiv AB + BA$.

Under any Lorentz transformation $x_p' = a_p^\lambda x_\lambda$, the meson wave function ψ transforms into $\psi' = S\psi$, where S is defined by $S^{-1}\beta_p S = a_p^\lambda \beta_\lambda$. For our present case the transformation operator S must be expressible as

$$S = S_D(\gamma) \times S_D(\gamma')$$

in terms of the transformation operators $S_D(\gamma)$ and $S_D(\gamma')$ in Dirac space defined respectively by

$$S_D(\gamma)^{-1}\gamma_p S_D(\gamma) = a_p^\lambda \gamma_\lambda \quad \text{and} \quad S_D(\gamma')^{-1}\gamma_p' S_D(\gamma') = a_p^\lambda \gamma_\lambda'. \quad (8)$$

From these relations we find that

$$S_D(\gamma).A^{-1}S_D(\gamma')^T.A = c.E,$$

with a numerical factor c which is certainly unity for proper transformations but may be either $+1$ or -1 for improper ones. The reason is that the matrix S_D is defined by eq. (8) only to within a constant factor which is limited to the four roots of unity $\pm 1, \pm i$ by the additional requirement that $\det(S_D)$ shall be unity. Thus there exist the alternatives of $c=1$ and $c=\det(a)$. Then for the transformed matrix \mathcal{V}' corresponding to ψ' one gets

$$\mathcal{V}'A = S_D(\gamma)\mathcal{V}S_D(\gamma')^T.A = c.S_D(\gamma)\mathcal{V}AS_D(\gamma)^{-1}.$$

A quantity $\Phi = \mathcal{V}A$ transforms as $\Phi' = c.S_D\Phi S_D^{-1}$, so that 16 wave functions

$$\varphi_A = \text{hs}(\gamma_A\Phi), \quad (9)$$

where hs is the half spur operator, transform as

$$\varphi'_A = c.\text{hs}(S_D^{-1}\gamma_A S_D\Phi).$$

Thus each φ_A transforms as γ_A does and according to the choice of c we have a pair of dual theories.

The wave equations for φ_A are given at once by multiplying eq. (7) by γ_A and applying the half spur operator;

$$\partial^p \text{hs}[(1/2)\{\gamma_p, \gamma_A\}\Phi] + \kappa \text{hs}(\gamma_A\Phi) = 0. \quad (10)$$

The 16 linearly independent elements γ_A of the Dirac algebra are subdivided into three closed subsets with respect to the operation of anticommutation with γ_p ;

$$\left. \begin{aligned} (1/2)\{\gamma_p, \gamma_3\} &= 0, & (1/2)\{\gamma_p, E\} &= \gamma_p, \\ (1/2)\{\gamma_p, \gamma_\mu\} &= \delta_{p\mu}E, \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} (1/2) \{ \gamma_\rho, i \gamma_\mu \gamma_5 \} &= \sigma_{\rho\mu} \gamma_5, \\ (1/2) \{ \gamma_\rho, \sigma_{\mu\nu} \gamma_5 \} &= \partial_{\rho\mu} i \gamma_\nu \gamma_5 - \partial_{\rho\nu} i \gamma_\mu \gamma_5. \end{aligned} \right\} \quad (11)$$

According to these relationships we get the well-known meson wave equations:

$$\left. \begin{aligned} \kappa \varphi &= 0, & \partial^\mu \chi_\rho + \kappa \chi_\rho &= 0, & \partial^\mu \varphi_{\rho\mu} + \kappa \varphi_\mu &= 0, \\ \partial_\rho \chi + \kappa \chi_\rho &= 0, & \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu + \kappa \varphi_{\mu\nu} &= 0, \end{aligned} \right\} \quad (12)$$

where

$$\left. \begin{aligned} \varphi &= \text{hs}(\gamma_5 \psi), & \chi &= \text{hs}(\psi), & \varphi_\mu &= \text{hs}(i \gamma_\mu \gamma_5 \psi), \\ \chi_\rho &= \text{hs}(\gamma_\rho \psi), & \varphi_{\mu\nu} &= \text{hs}(\sigma_{\mu\nu} \gamma_5 \psi). \end{aligned} \right\} \quad (13)$$

Here we have derived wave equations on the basis of the anticommutators, but we may as well use commutators $[A, B] \equiv AB - BA$. By multiplying eq. (7) by γ_5 from the left we get

$$(1/2) \partial^\rho [\gamma_5 \psi, \gamma_\rho] + \kappa \gamma_5 \psi = 0, \quad (7')$$

which then yields

$$\partial^\rho \text{hs}((1/2) [\gamma_\rho, \gamma_A] \gamma_5 \psi) + \kappa \text{hs}(\gamma_A \gamma_5 \psi) = 0. \quad (10')$$

In the present case the linearly independent elements γ_A are classified into three subsets as

$$\left. \begin{aligned} (1/2) [\gamma_\rho, E] &= 0, & (1/2) [\gamma_\rho, \gamma_\mu \gamma_5] &= \partial_{\rho\mu} \gamma_5, \\ (1/2) [\gamma_\rho, \gamma_5] &= \gamma_\rho \gamma_5, \\ i(1/2) [\gamma_\rho, \gamma_\mu] &= \sigma_{\rho\mu}, \\ (1/2) [\gamma_\rho, \sigma_{\mu\nu}] &= i(\partial_{\rho\mu} \gamma_\nu - \partial_{\rho\nu} \gamma_\mu), \end{aligned} \right\} \quad (11')$$

which together with eq. (10') afford the same wave equations (12) with the definitions (13).

On the assumption that β_ρ 's are all Hermitean the canonical conjugate of ψ is given by $\bar{\psi} = \psi^\dagger \gamma_4$, where ψ^\dagger is the Hermitean conjugate of ψ and $\gamma_4 = 2i\beta_4^2 - E$ satisfies the well-known relations

$$\eta_\rho \beta_\lambda + \beta_\lambda \eta_\rho = 0 \quad \text{for } \rho \neq \lambda \quad \text{and} \quad \eta_\rho \beta_\rho = \beta_\rho \eta_\rho = \beta_\rho. \quad (14)$$

The Lagrangean density for the original equation (5) is then

$$L = -\bar{\psi} (\partial^\rho \beta_\rho + \kappa) \psi. \quad (15)$$

Now since γ_4 is represented as $\gamma_1 \times \gamma_4'$, the canonical conjugate to ψ is $\bar{\psi} = \gamma_4 \psi^\dagger \gamma_1$, (γ_ρ and γ_ρ' are now Hermitean and hence A is unitary) and eq. (15) is rewritten as

$$L = -\text{sp} [\bar{\psi} (1/2) \partial^\rho \{ \psi, \gamma_\rho \} + \kappa \bar{\psi} \psi]. \quad (16)$$

This can further be rewritten with the aid of the identity (see Appendix)

$$\text{sp}(XY) = \text{hs}(\gamma^A X) \cdot \text{hs}(\gamma_A Y) \quad (17)$$

valid for any two 4×4 matrices X and Y , as

$$L = -\text{hs}(\gamma^A \bar{\Psi}) \cdot \text{hs}[\partial^\rho (1/2) \{\gamma_\rho, \gamma_A\} \Psi] - \kappa \text{hs}(\gamma^A \bar{\Psi}) \cdot \text{hs}(\gamma_A \Psi). \quad (18)$$

Now we shall proceed to rewrite this expression explicitly in terms of the wave functions φ_A . We shall mean by φ^{*A} the conjugate complex of φ^A , that is φ^{A*} , multiplied by $\sigma(A) = (-1)^n$, where n is the number of 4's among the superscript A :

$$\varphi^{*A} = \sigma(A) \varphi^{A*}. \quad (19)$$

Moreover the γ^A multiplied by γ_4 from both sides gives again the same γ^A with positive or negative signs, which fact can be expressed as

$$\gamma_4 \gamma^A \gamma_4 = \sigma(A) d(\gamma^A) \gamma^A, \quad (20)$$

with $d(\gamma_A) = +1$ for $\gamma_A = E, i\gamma_4\gamma_5$ and -1 for others. Now γ_ρ 's being all Hermitean, we have $\gamma_A^+ = \gamma_A$ so that from $\varphi^A = \text{hs}(\gamma^A \Psi)$ we get

$$\begin{aligned} \varphi_A^* &= \text{hs}(\gamma^{A+} \Psi^+) = \text{hs}(\gamma_4 \gamma^A \gamma_4 \bar{\Psi}) \\ &= \sigma(A) d(\gamma^A) \text{hs}(\gamma^A \bar{\Psi}). \end{aligned}$$

Hence,

$$\text{hs}(\gamma^A \bar{\Psi}) = d(\gamma^A) \sigma(A) \varphi^{A*} = d(\gamma^A) \varphi^{*A}. \quad (21)$$

The Lagrangean density (18) is now rewritten according to eqs. (11), (13) and (21) as

$$\begin{aligned} L &= \kappa \varphi^* \varphi - \mathcal{L}^* (\partial^\nu \mathcal{L}_\rho + \kappa \mathcal{L}) + \mathcal{L}^{*\nu} (\partial_\nu \mathcal{L} + \kappa \mathcal{L}_\rho) \\ &\quad - \varphi^{*\mu} (\partial^\rho \varphi_{\rho\mu} + \kappa \varphi_\mu) + \varphi^{*\mu\nu} (\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu + \kappa \varphi_{\mu\nu}). \end{aligned} \quad (22)$$

Finally we add the energy-momentum tensor and the charge-current density respectively rewritten in our formalism as

$$T_{\mu\nu} = \text{sp}[\bar{\Psi} \partial_\mu \{\gamma_\nu, \Psi\}] \quad (23)$$

and

$$J_\mu = i \text{sp}[\bar{\Psi} \{\gamma_\mu, \Psi\}]. \quad (24)$$

§ 3. Passage to the case of maximum spin two

Corresponding to eqs. (5) and (6) we have in the case of maximum spin two

$$(\partial^\rho \alpha_\rho + \kappa) \psi = 0 \quad (1)$$

and

$$\alpha_\rho = \beta_\rho \times E + E \times \beta'_\rho, \quad (2)$$

where β_ρ and β'_ρ are two independent Duffin-Kemmer matrices defined by the commutation relations

$$\beta_{\lambda}\beta_{\mu}\beta_{\nu} + \beta_{\nu}\beta_{\mu}\beta_{\lambda} = \delta_{\lambda\mu}\beta_{\nu} + \delta_{\nu\mu}\beta_{\lambda}. \quad (25)$$

We may put without loss of generality $\beta'_p = \beta_p^T$, and eq. (1) passes over into

$$\partial^p \{ \mathcal{P}, \beta_p \} + \kappa \mathcal{P} = 0, \quad (26)$$

where \mathcal{P} is a 16×16 matrix whose elements are the components of the original column matrix ψ . The canonical conjugate to \mathcal{P} is $\mathcal{P}^* = \gamma_4 \mathcal{P}^* \gamma_4$, and the Lagrangean density, the energy-momentum tensor and the charge-current density are given respectively by

$$L = -\text{sp} [\bar{\mathcal{P}} (\partial^p \{ \mathcal{P}, \beta_p \} + \kappa \mathcal{P})], \quad (27)$$

$$T_{\mu\nu} = \text{sp} [\bar{\mathcal{P}} \partial_{\mu} \{ \beta_{\nu}, \mathcal{P} \}] \quad (28)$$

and

$$J_{\mu} = i \text{sp} [\bar{\mathcal{P}} \{ \beta_{\mu}, \mathcal{P} \}]. \quad (29)$$

We are now to manipulate the β -matrices as is seen from eq. (26). Therefore, in order to be able to proceed quite analogously to the case of the theory of spin 0 and 1, we should have the linearly independent bases of the four-dimensional Duffin-Kemmer algebra. This algebra is decomposable into three subalgebras, being spanned respectively by 1, 25 and 100 linearly independent elements. In the preceding paper by one of the authors⁽¹⁾ it was shown that each of them can be constructed from β_p , so as to have a pair of sets of indices and to transform as a tensor of certain rank for any linear transformations of β -space. We shall first list these linearly independent elements and relate their basic properties. For the definitions and the detailed discussions of them one may refer to the previous paper mentioned above.

The 1×1 trivial subalgebra consists of a single element E_0 for which

$$E_0^2 = E_0, \quad E_0 \beta_p = \beta_p E_0 = 0 \quad \text{and} \quad \text{sp}(E_0) = 1. \quad (30)$$

The 5×5 subalgebra consists of 25 elements

$$\left(\begin{array}{c|c} P & P\beta_{\nu} \\ \hline \beta_{\mu}P & \beta_{\mu}P\beta_{\nu} \end{array} \right)$$

which are constructed on the basis of an idempotent P for which

$$\left. \begin{aligned} P^2 &= P, \quad P\beta_p P = 0, \quad \text{sp}(P) = 1 \\ \beta_{\mu}\beta_{\nu}P &= P\beta_{\mu}\beta_{\nu} = \delta_{\mu\nu}P. \end{aligned} \right\} \quad (31)$$

The 100 elements constituting the 10×10 subalgebra are given by

$$\left(\begin{array}{c|c} P_{\mu;\nu} & P_{\mu;\nu}\beta_p \\ \hline \beta_{\lambda}P_{\mu;\nu} & \beta_{\lambda}P_{\mu;\nu}\beta_p \end{array} \right)$$

where for the 16 basic elements $P_{\mu;\nu}$ we have the following set of relations;

$$P_{\mu;\rho}P_{\lambda;\nu} = \delta_{\rho\lambda}P_{\mu;\nu}, \quad P_{\mu;\nu}\beta_p P_{\lambda;\sigma} = 0, \quad \text{sp}(P_{\mu;\nu}) = \delta_{\mu\nu},$$

$$P_{\mu;\nu}\beta_\rho + P_{\mu;\rho}\beta_\nu = 0, \quad \beta_\lambda P_{\mu;\nu} + \beta_\mu P_{\lambda;\nu} = 0, \quad (32)$$

and

$$P_{\mu;\nu}\beta_\lambda\beta_\rho = \delta_{\lambda\rho}P_{\mu;\nu} - \delta_{\nu\rho}P_{\mu;\lambda}, \quad \beta_\lambda\beta_\rho P_{\mu;\nu} = \delta_{\lambda\rho}P_{\mu;\nu} - \delta_{\lambda\mu}P_{\rho;\nu}.$$

We have seen in eq. (11) that the Dirac algebra, while being irreducible itself, splits up into the three closed subsets with regard to the operation of anticommutation with γ_ρ and accordingly that the three independent sets of field equations have been obtained from eq. (7). In order to treat eq. (26) in the same way, we shall now show that the linearly independent elements of each irreducible part can further be subdivided into several closed subsets with respect to the operations of anticommutation with β_ρ . For E_0 we immediately see

$$\{E_0, \beta_\rho\} = 0. \quad (33)$$

For the elements belonging to the 5×5 subalgebra, we get

$$\left. \begin{aligned} \{P, \beta_\rho\} &= P\beta_\rho + \beta_\rho P, \\ \{P\beta_\mu, \beta_\rho\} &= \delta_{\mu\rho}P + \beta_\rho P\beta_\mu, \\ \{\beta_\mu P, \beta_\rho\} &= \delta_{\mu\rho}P + \beta_\mu P\beta_\rho, \\ \{\beta_\mu P\beta_\nu, \beta_\rho\} &= \delta_{\nu\rho}\beta_\mu P + \delta_{\mu\rho}P\beta_\nu, \end{aligned} \right\} \quad (34)$$

and

so that a closed subset containing P is obtained by adding the 2nd and the 3rd ones in eqs. (34) and further by symmetrizing the 4th equation with respect to the indices μ and ν . The elements $P\beta_\mu - \beta_\mu P$ together with $\beta_\mu P\beta_\nu - \beta_\nu P\beta_\mu$ resulting from antisymmetrization form another closed subset. These facts can be put into a single set of relations

$$\left. \begin{aligned} \{K^{(\varepsilon)}, \beta_\rho\} &= (1 + \varepsilon)L_\rho^{(\varepsilon)}, \\ \{L_\mu^{(\varepsilon)}, \beta_\rho\} &= M_{\rho\mu}^{(\varepsilon)} + \delta_{\rho\mu}K^{(\varepsilon)}, \\ \{M_{\mu\nu}^{(\varepsilon)}, \beta_\rho\} &= \delta_{\rho\nu}L_\mu^{(\varepsilon)} + \varepsilon\delta_{\rho\mu}L_\nu^{(\varepsilon)} \end{aligned} \right\} \quad (35)$$

and

by introducing symbols

$$\left. \begin{aligned} K^{(\varepsilon)} &= (1/2)(1 + \varepsilon)P, \\ L_\mu^{(\varepsilon)} &= (1/2)(P\beta_\mu + \varepsilon\beta_\mu P), \\ M_{\mu\nu}^{(\varepsilon)} &= (1/2)(\beta_\mu P\beta_\nu + \varepsilon\beta_\nu P\beta_\mu), \end{aligned} \right\} \quad (36)$$

and

with $\varepsilon=1$ for the symmetrized set and $\varepsilon=-1$ for the antisymmetrized one.

For the 10×10 subalgebra the same arguments are equally valid. By introducing symbols

$$\left. \begin{aligned} U_{\mu\nu}^{(\varepsilon)} &= (1/2)(P_{\mu;\nu} + \varepsilon P_{\nu;\mu}), \\ V_{\mu[\nu\lambda]}^{(\varepsilon)} &= (1/2)(P_{\mu;\nu}\beta_\lambda + \varepsilon\beta_\lambda P_{\nu;\mu}), \\ W_{\mu[\rho\nu\lambda]}^{(\varepsilon)} &= (1/2)(\beta_\rho P_{\mu;\nu}\beta_\lambda + \varepsilon\beta_\lambda P_{\nu;\mu}\beta_\rho) \end{aligned} \right\} \quad (37)$$

and

with $\varepsilon=1$ for the symmetrized set and $=-1$ for the antisymmetrized set as before, we have

$$\left. \begin{aligned} \{U_{\mu\nu}^{(\varepsilon)}, \beta_p\} &= V_{\nu[\nu\rho]}^{(\varepsilon)} + \varepsilon V_{\nu[\mu\rho]}^{(\varepsilon)}, \\ \{V_{[\mu\nu\lambda]}^{(\varepsilon)}, \beta_p\} &= W_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} - \partial_{\rho\nu} U_{\mu\lambda}^{(\varepsilon)} + \partial_{\rho\lambda} U_{\mu\nu}^{(\varepsilon)} \\ \text{and } \{W_{[\mu\sigma][\nu\lambda]}^{(\varepsilon)}, \beta_p\} &= \partial_{\rho\sigma} V_{\mu[\nu\lambda]}^{(\varepsilon)} - \partial_{\rho\mu} V_{\sigma[\nu\lambda]}^{(\varepsilon)} \\ &\quad + \varepsilon (\partial_{\rho\lambda} V_{\nu[\mu\sigma]}^{(\varepsilon)} - \partial_{\rho\nu} V_{\lambda[\mu\sigma]}^{(\varepsilon)}). \end{aligned} \right\} \quad (38)$$

Thus, together with E_ν , the linearly independent elements are classified into five independent sets with respect to the anticommutation with β_p ; from the 5×5 subalgebra we have the two sets given by eqs. (36) which we shall hereafter call "the lower class" and also from the 10×10 subalgebra the two sets given by eqs. (37) which we shall term "the higher class". Incidentally it must be noted that the following elements obtained from the higher class by contracting their suffixes;

$$\left. \begin{aligned} \partial^{\mu\nu} U_{\mu\nu}^{(\varepsilon)} &= U^{(\varepsilon)}, \quad \partial^{\mu\nu} V_{\mu[\nu\lambda]}^{(\varepsilon)} = V_{\lambda}^{(\varepsilon)} \\ \text{and } \partial^{\mu\nu} W_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} - U_{\rho\lambda}^{(\varepsilon)} &= W_{\rho\lambda}^{(\varepsilon)} \end{aligned} \right\} \quad (39)$$

have the same anticommutators with β_p as those of the lower class.

The generic one of the linearly independent elements so far discussed may be expressed as $P_{A;B}$, which satisfies the fundamental relations corresponding to eqs. (30), (31) and (32)

$$P_{A;R} P_{C;D} = \delta_{RC} P_{A;D} \quad \text{and} \quad \text{sp}(P_{A;B}) = \delta_{AB}. \quad (40)$$

For the present case of Hermitean β_p , we have

$$P_{A;B}^+ = P_{B;A}. \quad (41)$$

The procedures of symmetrization in eqs. (36) and (37) are expressed simply as

$$P_{AB}^{(\varepsilon)} = (1/2) (P_{A;B} + \varepsilon P_{B;A}). \quad (42)$$

Now the field quantities defined by

$$\psi_{AB}^{(\varepsilon)} = \text{sp}(P_{AB}^{(\varepsilon)} \mathcal{F}) \quad (43)$$

satisfy the field equations

$$\partial^\rho \text{sp}(\{\beta_\rho, P_{AB}^{(\varepsilon)}\} \mathcal{F}) + \kappa \text{sp}(P_{AB}^{(\varepsilon)} \mathcal{F}) = 0. \quad (44)$$

We introduce as before the sign function $\sigma(AB) = (-1)^n$ where n is the number of 4's among the indices of $\psi_{AB}^{(\varepsilon)}$ and adopt the convention

$$\varphi^{*(\varepsilon)AB} = \sigma(AB) (\psi_{AB}^{(\varepsilon)})^*, \quad (45)$$

where $(\psi_{AB}^{(\varepsilon)})^*$ is the complex conjugate of $\psi_{AB}^{(\varepsilon)}$. Moreover it is easily found that

$$\eta_A P_{AB}^{(\varepsilon)} \eta_A = \sigma(AB) d(P_{A;B}) P_{AB}^{(\varepsilon)} \quad (46)$$

with sign functions $d(P_{A;B})$ equal to unity except those for the elements with odd numbers of suffixes which are equal to -1 . Incidentally we may define this sign function by

$d(P_{A;B})P_{A;B} = \gamma_5 P_{A;B} \gamma_5$ with $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ anticommutable with each β_μ . Hence according to eqs. (41), (43) and (46)

$$\begin{aligned} (\phi_{AB}^{(\varepsilon)})^* &= \text{sp}(P_{AB}^{(\varepsilon)+} \bar{\psi}^+) = \text{sp}(\gamma_4 P_{BA}^{(\varepsilon)} \gamma_4 \bar{\psi}) \\ &= \sigma(AB) d(P_{A;B}) \text{sp}(P_{BA}^{(\varepsilon)} \bar{\psi}), \end{aligned}$$

and further

$$\text{sp}(P^{(\varepsilon)BA} \bar{\psi}) = d(P_{A;B}) \phi^{*(\varepsilon)AB}. \quad (47)$$

According to eqs. (40) we have (see Appendix)

$$\text{sp}(XY) = \sum_{\varepsilon=\pm 1} \text{sp}(P^{(\varepsilon)BA} X) \cdot \text{sp}(P_{AB}^{(\varepsilon)} Y), \quad (48)$$

and the Lagrangean density (27) reads

$$L = - \sum_{\varepsilon=\pm 1} \sum_{A,B} d(P_{A;B}) \phi^{*(\varepsilon)AB} [\partial^\mu \text{sp}(\{\beta_\mu, P_{AB}^{(\varepsilon)}\} \bar{\psi}) + \kappa \phi_{AB}^{(\varepsilon)}]. \quad (49)$$

The energy-momentum tensor (28) and the charge current density (29) can be rewritten in the similar way.

Now we shall write down explicitly the field equations (44) in terms of the field quantities (43) which are enumerated as follows;

$$\left. \begin{aligned} \psi_0 &= \text{sp}(E_0 \bar{\psi}); \quad \varphi^{(\varepsilon)} = \text{sp}(K^{(\varepsilon)} \bar{\psi}), \quad \varphi_\mu^{(\varepsilon)} = \text{sp}(L_\mu^{(\varepsilon)} \bar{\psi}), \quad \varphi_{\mu\nu} = \text{sp}(M_{\mu\nu}^{(\varepsilon)} \bar{\psi}) \\ \text{and} \quad \psi_{\mu\nu}^{(\varepsilon)} &= \text{sp}(U_{\mu\nu}^{(\varepsilon)} \bar{\psi}), \quad \psi_{[\mu\nu\lambda]}^{(\varepsilon)} = \text{sp}(V_{[\mu\nu\lambda]}^{(\varepsilon)} \bar{\psi}), \quad \psi_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} = \text{sp}(W_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} \bar{\psi}). \end{aligned} \right\} \quad (50)$$

On the basis of the fact that the linearly independent elements are subdivided into five independent sets with respect to the anticommutation with β_μ , we get the following five independent sets of field equations. First we have according to the anticommutator (33) a trivial scalar equation

$$\kappa \psi_0 = 0 \quad (51)$$

and according to the anticommutators (35) and (38) we have the field equations of the lower class

$$(1 + \varepsilon) \partial^\rho \varphi_\rho^{(\varepsilon)} + \kappa \varphi^{(\varepsilon)} = 0, \quad (52a)$$

$$\partial^\rho \varphi_{\rho\lambda}^{(\varepsilon)} + \partial_\lambda \varphi^{(\varepsilon)} + \kappa \varphi_\lambda^{(\varepsilon)} = 0, \quad (52b)$$

$$\partial_\lambda \varphi_\mu^{(\varepsilon)} + \varepsilon \partial_\rho \varphi_\lambda^{(\varepsilon)} + \kappa \varphi_{\lambda\rho}^{(\varepsilon)} = 0 \quad (52c)$$

and the field equations of the higher class

$$\partial^\rho (\psi_{\mu[\nu\rho]}^{(\varepsilon)} + \varepsilon \psi_{\nu[\mu\rho]}^{(\varepsilon)}) + \kappa \psi_{\mu\nu}^{(\varepsilon)} = 0, \quad (53a)$$

$$\partial^\rho \psi_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} + \partial_\lambda \psi_{\mu\nu}^{(\varepsilon)} - \partial_\nu \psi_{\mu\lambda}^{(\varepsilon)} + \kappa \psi_{\mu[\nu\lambda]}^{(\varepsilon)} = 0, \quad (53b)$$

$$\partial_\rho \psi_{[\mu\nu\lambda]}^{(\varepsilon)} - \partial_\mu \psi_{[\nu\lambda\rho]}^{(\varepsilon)} + \varepsilon (\partial_\lambda \psi_{\nu[\mu\rho]}^{(\varepsilon)} - \partial_\nu \psi_{\lambda[\mu\rho]}^{(\varepsilon)}) + \kappa \psi_{[\mu\rho][\nu\lambda]}^{(\varepsilon)} = 0. \quad (53c)$$

The Lagrangean density (49) splits into two parts;

$$L = L^{(1)} + L^{(2)}.$$

The $L^{(1)}$ for the fields of the lower class is

$$\begin{aligned} L^{(1)} = & - \sum_{\varepsilon = \pm 1} [\varphi^{*(\varepsilon)} \{ (1 + \varepsilon) \partial^\rho \varphi_\rho^{(\varepsilon)} + \kappa \varphi^{(\varepsilon)} \} \\ & - 2 \varphi^{*(\varepsilon) \lambda} \{ \partial^\rho \varphi_{\rho \lambda}^{(\varepsilon)} + (1/2) (1 + \varepsilon) \partial_\lambda \varphi^{(\varepsilon)} + \kappa \varphi_\lambda^{(\varepsilon)} \} \\ & + \varphi^{*(\varepsilon) \lambda \rho} \{ \partial_\lambda \varphi_\rho^{(\varepsilon)} + \varepsilon \partial_\rho \varphi_\lambda^{(\varepsilon)} + \kappa \varphi_{\lambda \rho}^{(\varepsilon)} \}] \end{aligned} \quad (54)$$

and the $L^{(2)}$ for those of the higher class is

$$\begin{aligned} L^{(2)} = & - \sum_{\varepsilon = \pm 1} [\psi^{*(\varepsilon) \mu \nu} \{ \partial^\rho \psi_{[\mu, \nu \rho]}^{(\varepsilon)} + \varepsilon \psi_{\nu [\mu \rho]}^{(\varepsilon)} + \kappa \psi_{\mu \nu}^{(\varepsilon)} \} \\ & - \psi^{*(\varepsilon) \mu \nu \lambda} \{ \partial^\rho \psi_{[\mu \rho] \nu \lambda}^{(\varepsilon)} + \partial_\lambda \psi_{\mu \nu}^{(\varepsilon)} - \partial_\nu \psi_{\mu \lambda}^{(\varepsilon)} + \kappa \psi_{\mu [\nu \lambda]}^{(\varepsilon)} \} \\ & + (1/4) \psi^{*(\varepsilon) \mu \nu \lambda} \{ \partial_\rho \psi_{\mu \nu \lambda}^{(\varepsilon)} - \partial_\mu \psi_{\nu \lambda}^{(\varepsilon)} \\ & + \varepsilon (\partial_\lambda \psi_{\nu [\mu \rho]}^{(\varepsilon)} - \partial_\nu \psi_{\lambda [\mu \rho]}^{(\varepsilon)} + \kappa \psi_{\mu \rho [\nu \lambda]}^{(\varepsilon)} \}]. \end{aligned} \quad (54')$$

Likewise the energy momentum tensor and the charge current density are given respectively by

$$\left. \begin{aligned} T_{\mu \nu}^{(1)} = & \sum_{\varepsilon = \pm 1} [(1 + \varepsilon) (\varphi^{*(\varepsilon)} \partial_\mu \varphi_\nu^{(\varepsilon)} - \varphi_\mu^{*(\varepsilon)} \partial_\nu \varphi^{(\varepsilon)}) \\ & + 2 (\varphi^{*(\varepsilon) \lambda} \partial_\mu \varphi_{\nu \lambda}^{(\varepsilon)} - \varphi_{\nu \lambda}^{*(\varepsilon)} \partial_\mu \varphi^{(\varepsilon) \lambda})], \\ J_\mu^{(1)} = & i \sum_{\varepsilon = \pm 1} [(1 + \varepsilon) (\varphi^{*(\varepsilon)} \varphi_\mu^{(\varepsilon)} - \varphi_\mu^{*(\varepsilon)} \varphi^{(\varepsilon)}) \\ & + 2 (\varphi^{*(\varepsilon) \rho} \varphi_{\mu \rho}^{(\varepsilon)} - \varphi_{\mu \rho}^{*(\varepsilon)} \varphi^{(\varepsilon) \rho})] \end{aligned} \right\} \quad (55)$$

for the fields of the lower class and

$$\left. \begin{aligned} T_{\mu \nu}^{(2)} = & \sum_{\varepsilon = \pm 1} [2 (\psi_{[\nu \lambda]}^{*(\varepsilon)} \partial_\mu \psi_{\rho \lambda}^{(\varepsilon)} - \psi_{\rho \lambda}^{*(\varepsilon)} \partial_\mu \psi_{[\nu \lambda]}^{(\varepsilon)}) \\ & + (\psi^{*(\varepsilon) \rho \lambda \sigma} \partial_\mu \psi_{[\nu \rho] \lambda \sigma}^{(\varepsilon)} - \psi_{[\nu \rho] \lambda \sigma}^{*(\varepsilon)} \partial_\mu \psi^{(\varepsilon) \rho \lambda \sigma})], \\ J_\mu^{(2)} = & i \sum_{\varepsilon = \pm 1} [2 (\psi_{[\mu \lambda]}^{*(\varepsilon)} \psi_{\rho \lambda}^{(\varepsilon)} - \psi_{\rho \lambda}^{*(\varepsilon)} \psi_{[\mu \lambda]}^{(\varepsilon)}) \\ & + (\psi^{*(\varepsilon) \rho \lambda \sigma} \psi_{[\mu \rho] \lambda \sigma}^{(\varepsilon)} - \psi_{[\mu \rho] \lambda \sigma}^{*(\varepsilon)} \psi^{(\varepsilon) \rho \lambda \sigma})] \end{aligned} \right\} \quad (56)$$

for those of the higher class.

§ 4. Reduction of the field equations

Having derived the independent sets of field equations above, our object in this section is to reduce each of these sets into several simple field equations so that we may ascribe to each of them definite values of spin and mass. We shall first consider the case of the lower class and then proceed to the analysis of the more complicated fields of the higher class. Hereafter the superscript (ε) in the wave functions will be omitted for the sake of simplicity.

i) The field equations of the lower class

For the antisymmetric set eqs. (52) are just Proca's set of field equations for the vector meson. For the symmetric case of eq. (52) a scalar field with mass $\kappa/2$ and a

vector field with mass κ are obtained in the following way. By taking divergence of eqs. (52b) and (52c) it can easily be seen that

$$\{\square - (\kappa/2)^2\} \varphi = 0. \quad (57)$$

Then for the field quantities χ 's defined by

$$\left. \begin{aligned} \chi_\rho &= \varphi_\rho + (2/\kappa) \partial_\rho \varphi \\ \chi_{\lambda\rho} &= -(1/\kappa) (\partial_\lambda \chi_\rho + \partial_\rho \chi_\lambda) \end{aligned} \right\} \quad (58)$$

we get the vector equations

$$\partial^\rho \chi_\rho = 0, \quad \partial^\rho \chi_{\rho\lambda} + \kappa \chi_\lambda = 0. \quad (59)$$

ii) The field equations of the higher class

For the case of this class the symmetric and the antisymmetric set of field equations can be dealt with in completely parallel way.

a) Subtraction of the contracted quantities

In view of the fact that the contracted elements $U^{(e)}$, $V_\mu^{(e)}$ and $W_{\mu\nu}^{(e)}$ defined by eq. (39) have the same anticommutators with β_ρ as those of the lower class, we can show that the quantities

$$\delta^{\mu\nu} \psi_{\mu\nu} = \varphi, \quad \delta^{\mu\nu} \psi_{[\nu\lambda]} = \varphi_\lambda \quad \text{and} \quad \delta^{\mu\nu} \psi_{[\mu\rho][\nu\lambda]} - \psi_{\rho\lambda} = \varphi_{\rho\lambda} \quad (60)$$

satisfy the same equations as those of the lower class. This can also be confirmed by contracting the suffixes μ and ν in eqs. (53). First of all we shall subtract these quantities from the original ψ 's. With the aid of eqs. (52) it can be shown that the quantities defined by

$$\left. \begin{aligned} \varphi'_{\mu\nu} &= (1/3) (\partial_{\mu\nu} \varphi - \varphi_{\mu\nu}), \\ \varphi'_{\mu[\nu\lambda]} &= (1/3) (\partial_{\mu\nu} \varphi_\lambda - \partial_{\mu\lambda} \varphi_\nu), \\ \text{and} \quad \varphi'_{[\mu\rho][\nu\lambda]} &= (1/3) (\partial_{\mu\nu} \varphi_{\rho\lambda} - \partial_{\mu\lambda} \varphi_{\rho\nu} + \partial_{\rho\lambda} \varphi_{\mu\nu} - \partial_{\rho\nu} \varphi_{\mu\lambda}) \end{aligned} \right\} \quad (61)$$

satisfy eqs. (53). Therefore, for the quantities

$$\left. \begin{aligned} \psi'_{\mu\nu} &= \psi_{\mu\nu} - \varphi'_{\mu\nu}, \\ \psi'_{\mu[\nu\lambda]} &= \psi_{\mu[\nu\lambda]} - \varphi'_{\mu[\nu\lambda]}, \\ \text{and} \quad \psi'_{[\mu\rho][\nu\lambda]} &= \psi_{[\mu\rho][\nu\lambda]} - \varphi'_{[\mu\rho][\nu\lambda]} \end{aligned} \right\} \quad (62)$$

satisfying eqs. (53) we have

$$\delta^{\mu\nu} \psi'_{\mu\nu} = 0, \quad \delta^{\mu\nu} \psi'_{\mu[\nu\lambda]} = 0 \quad \text{and} \quad \delta^{\mu\nu} \psi'_{[\mu\rho][\nu\lambda]} = \psi'_{\rho\lambda}, \quad (63)$$

where we have made use of $\delta^{\mu\nu} \varphi_{\mu\nu} = \varphi$ resulting from eqs. (52). It must be noted that the 1st and the 3rd ones in eqs. (63) follow from the 2nd by means of eqs. (53). In this way the field quantities of the higher class are split into φ 's satisfying the field equations of the lower class and ψ 's satisfying those of the higher class with the additional conditions (63). In what follows we shall omit the prime of ψ 's and investigate eqs. (53) with

the conditions (63).

b) *Further reduction of the field equations of the higher class*

Operating ∂^μ on eqs. (53a and b) we get

$$\partial^\rho (\partial^\mu \zeta'_{\mu[\nu\rho]}) + \kappa (\partial^\mu \zeta'_{\mu\nu}) = 0$$

and

$$\partial_\lambda (\partial^\mu \zeta'_{\mu\nu}) - \partial_\nu (\partial^\mu \zeta'_{\mu\lambda}) + \kappa (\partial^\mu \zeta'_{\mu[\nu\lambda]}) = 0.$$

These are just the equations for a vector field, since by introducing symbols

$$\hat{\xi}_\mu = (1/\kappa) \partial^\rho \zeta'_{\rho\mu} \quad \text{and} \quad \hat{\xi}_{[\mu\nu]} = (1/\kappa) \partial^\rho \zeta'_{\rho[\mu\nu]} \quad (64)$$

these are rewritten respectively as

$$\partial^\mu \hat{\xi}_{[\nu\mu]} + \kappa \hat{\xi}_\nu = 0 \quad \text{and} \quad \partial_\nu \hat{\xi}_\mu - \partial_\mu \hat{\xi}_\nu + \kappa \hat{\xi}_{[\mu\nu]} = 0. \quad (65)$$

So as to subtract this $\hat{\xi}_\mu$ from ζ' s, we construct $\hat{\xi}_{\mu\nu}$ as

$$\hat{\xi}_{\mu\nu} = (1/\kappa) (\partial_\mu \hat{\xi}_\nu - \varepsilon \partial_\nu \hat{\xi}_\mu), \quad (66)$$

for which

$$\partial^\mu \nu \hat{\xi}_{\mu\nu} = 0 \quad \text{and} \quad \partial^\mu \hat{\xi}_{\mu\nu} = \kappa \hat{\xi}_\nu. \quad (67)$$

Then for a quantity $\eta_{\mu\nu}$ defined by

$$\eta_{\mu\nu} = \zeta'_{\mu\nu} - \hat{\xi}_{\mu\nu} \quad (68)$$

we have from eqs. (63), (64) and (67)

$$\partial^\mu \nu \eta_{\mu\nu} = 0 \quad \text{and} \quad \partial^\mu \eta_{\mu\nu} = 0. \quad (69)$$

We can deduce from eqs. (53b and c)

$$\begin{aligned} & \square \partial^\rho (\zeta'_{\mu[\nu\rho]} + \varepsilon \zeta'_{\nu[\mu\rho]}) - \partial^\lambda \partial^\rho (\partial_\mu \zeta'_{\lambda[\nu\rho]} + \varepsilon \partial_\nu \zeta'_{\lambda[\mu\rho]}) \\ & - \kappa \zeta'_{\mu\nu} + \kappa \partial_\nu \partial^\mu \zeta'_{\mu\rho} - \kappa \partial^\mu \partial_\nu \zeta'_{\mu\rho} = 0. \end{aligned}$$

Adding to this the one with indices μ and ν interchanged and multiplied by ε we have according to eq. (53a)

$$\begin{aligned} & \square \zeta'_{\mu\nu} - (\kappa/2)^2 \zeta'_{\mu\nu} + (1/2\kappa) \partial^\lambda \partial^\rho (\partial_\mu \zeta'_{\rho[\nu\lambda]} + \varepsilon \partial_\nu \zeta'_{\rho[\mu\lambda]}) \\ & - (1/4) \partial^\rho (\partial_\nu \zeta'_{\mu\rho} + \varepsilon \partial_\mu \zeta'_{\nu\rho}) = 0. \end{aligned}$$

By means of eqs. (64–69) this leads to the Klein-Gordon equation for $\eta_{\mu\nu}$:

$$\{\square - (\kappa/2)^2\} \eta_{\mu\nu} = 0. \quad (70)$$

$\eta_{\mu\nu}$ is completely determined by eqs. (69) and (70). Now that the reduction of $\zeta'_{\mu\nu}$ has been finished we shall proceed to investigate the parts of $\zeta'_{\mu[\nu\lambda]}$ and $\zeta'_{[\mu\rho][\nu\lambda]}$ left after the separation of the two fields $\hat{\xi}_\mu$ and $\eta_{\mu\nu}$. First we consider

$$\eta_{\mu[\nu\lambda]} = (2/\kappa) (\partial_\nu \eta_{\mu\lambda} - \partial_\lambda \eta_{\mu\nu}),$$

$$\eta_{[\mu\rho][\nu\lambda]} = (1/\kappa) \{ \partial_\mu \eta_{\rho[\nu\lambda]} - \partial_\rho \eta_{\mu[\nu\lambda]} + \varepsilon (\partial_\nu \eta_{\lambda[\mu\rho]} - \partial_\lambda \eta_{\nu[\mu\rho]}) \}.$$

It is easy to prove the following relations ;

$$\begin{aligned}\partial^\mu \eta_{\mu[\nu\lambda]} &= 0, & \partial^\rho \eta_{\mu[\nu\rho]} &= -(\kappa/2) \eta_{\mu\nu}, & \partial^{\mu\nu} \eta_{\mu[\nu\lambda]} &= 0, \\ \partial^\rho \eta_{[\mu\rho][\nu\lambda]} &= (\kappa/2) \eta_{\mu[\nu\lambda]} & \text{and} & & \partial^{\mu\nu} \eta_{[\mu\rho][\nu\lambda]} &= \eta_{\rho\lambda}.\end{aligned}$$

It must be noted here that these η 's satisfy the original field equations (53) by itself. In the same way we introduce $\xi_{\mu[\nu\lambda]}$ defined by

$$\xi_{\mu[\nu\lambda]} = (1/\kappa) (\partial_\nu \xi_{\mu\lambda} - \partial_\lambda \xi_{\mu\nu})$$

for which we have

$$\partial^\mu \xi_{\mu[\nu\lambda]} = \kappa \xi_{[\nu\lambda]}, \quad \partial^\rho \xi_{\mu[\nu\rho]} = -\partial_\mu \xi_\nu \quad \text{and} \quad \partial^{\mu\nu} \xi_{\mu[\nu\lambda]} = \xi_\lambda$$

and from these relations we see

$$\partial^\rho (\xi_{\mu[\nu\rho]} + \varepsilon \xi_{\nu[\mu\rho]}) + \kappa \xi_{\mu\nu} = 0.$$

$\xi_{[\mu\rho][\nu\lambda]}$ having the same form as $\eta_{[\mu\rho][\nu\lambda]}$ vanishes identically. Hence ξ 's again satisfy eqs. (53) by itself. Therefore we put

$$\begin{aligned}\theta_{\mu[\nu\lambda]} &= \psi'_{\mu[\nu\lambda]} - \eta_{\mu[\nu\lambda]} - \xi_{\mu[\nu\lambda]}, \\ \theta_{[\mu\rho][\nu\lambda]} &= \psi_{[\mu\rho][\nu\lambda]} - \eta_{[\mu\rho][\nu\lambda]} - \xi_{[\mu\rho][\nu\lambda]}\end{aligned}\tag{71}$$

and analyze these θ 's. From eqs. (53) together with the properties of ξ 's and η 's we get

$$\left. \begin{aligned}\partial^\rho (\theta_{\mu[\nu\rho]} + \varepsilon \theta_{\nu[\mu\rho]}) &= 0, \\ \partial^\rho \theta_{[\mu\rho][\nu\lambda]} + \kappa \theta_{\mu[\nu\lambda]} &= 0, \\ \partial_\rho \theta_{\mu[\nu\lambda]} - \partial_\mu \theta_{\rho[\nu\lambda]} + \varepsilon (\partial_\nu \theta_{\lambda[\mu\rho]} - \partial_\lambda \theta_{\nu[\mu\rho]}) + \kappa \theta_{[\mu\rho][\nu\lambda]} &= 0,\end{aligned}\right\}\tag{72}$$

the second of which yields at once

$$\partial^\mu \theta_{\mu[\nu\lambda]} = 0.\tag{73}$$

Then on operating ∂^λ to the result obtained by eliminating $\theta_{[\mu\rho][\nu\lambda]}$ from the 2nd and the 3rd ones in eqs. (72) and using the 1st we get

$$\partial^\rho \theta_{\mu[\nu\rho]} = 0.\tag{74}$$

Also we have from eqs. (63) and the 1st of eqs. (71)

$$\partial^{\mu\nu} \theta_{\mu[\nu\lambda]} = -\xi_\lambda\tag{75}$$

which shows that θ 's still depend on ξ 's. It is possible to subtract ξ 's from θ 's with the similar procedures. The quantities

$$\begin{aligned}\xi'_{\mu[\nu\lambda]} &= (1/2) (\xi_{\mu[\nu\lambda]} - \partial_{\mu\nu} \xi_\lambda + \partial_{\mu\lambda} \xi_\nu), \\ \xi'_{[\mu\rho][\nu\lambda]} &= (1/2) (\partial_{\mu\nu} \xi_{\rho\lambda} - \partial_{\rho\nu} \xi_{\mu\lambda} + \partial_{\rho\lambda} \xi_{\mu\nu} - \partial_{\mu\lambda} \xi_{\rho\nu})\end{aligned}$$

are shown to satisfy a set of equations ;

$$\partial^{\mu\nu}\hat{\zeta}'_{[\mu\nu\lambda]} = -\hat{\zeta}_\lambda, \quad \partial^\mu\hat{\zeta}'_{[\mu\nu\lambda]} = \partial^\rho\hat{\zeta}'_{[\mu\nu\rho]} = 0$$

and

$$\partial^\rho\hat{\zeta}'_{[\mu\rho\nu\lambda]} = -\kappa\hat{\zeta}'_{[\mu\nu\lambda]}.$$

Hence $\hat{\zeta}'$'s satisfy all the equations for θ 's. Therefore we can attain our aim by introducing

$$\zeta_{[\mu\nu\lambda]} = \theta_{[\mu\nu\lambda]} - \hat{\zeta}'_{[\mu\nu\lambda]} \quad (76)$$

and

$$\zeta_{[\mu\rho\nu\lambda]} = \theta_{[\mu\rho\nu\lambda]} - \hat{\zeta}'_{[\mu\rho\nu\lambda]}$$

satisfying all the equations for θ 's except eq. (75) which is now replaced by

$$\partial^{\mu\nu}\zeta_{[\mu\nu\lambda]} = 0. \quad (77)$$

Finally we can easily derive the Klein-Gordon equation for $\zeta_{[\mu\nu\lambda]}$:

$$(\square - \kappa^2)\zeta_{[\mu\nu\lambda]} = 0. \quad (78)$$

§ 5. Physical consequences of the theory

We have seen in the preceding section that the sets of field equations (52) and (53), which are mutually independent in algebraic sense, are further reduced to the several sets of simple equations which might be called "analytically irreducible". We shall first evaluate spin values of the various fields, recapitulating our results obtained.

The field equations of the lower class present no difficulty. The antisymmetric set of eqs. (52) is nothing but Proca's set of equations with mass κ , and the symmetric set is reduced to the scalar field φ with mass $\kappa/2$ and the vector field χ_μ with mass κ which has not been so far contemplated in the usual formulation of the vector theory. The quantities extracted from the higher class by eq. (60) satisfy the equations of the same form as the lower class, so that we meet with the double occurrence of the fields of the lower class. From the higher class, after the subtraction of contracted quantities, we have first the ordinary vector fields $\hat{\zeta}_\mu$ with mass κ which are governed by eqs. (65) irrespective of the distinction of symmetry. Eqs. (69) restrict the number of independent components of $\eta_{\mu\nu}$ to 5 for the symmetric case and to 3 for the antisymmetric one. Thus the symmetric $\eta_{\mu\nu}$ is of spin 2, which appears in the Fierz-Pauli theory. To the antisymmetric one we must ascribe spin value 1 but it is of the form foreign to that of the ordinary vector field. The field quantities $\zeta_{[\mu\nu\lambda]}$ satisfy eqs. (73), (74), (77) and (78) so that its mass is κ and the number of independent components is 6. But we conjecture that these may further split up into two independent parts, each containing odd number of components, thus yielding two fields of integer spin. We hope the complete solution will be found in the future investigation.

The total energy and the total charge can be evaluated in terms of the reduced field quantities according to eqs. (55) and (56). For the total energy we have the results;

$$E = E^{(1)} + E^{(2)}$$

with

$$E^{(1)} = -(4/\kappa) \sum_{\varepsilon=\pm 1} \int dV \{ 2(1+\varepsilon) \partial_4 \varphi^{*(\varepsilon)} \partial_4 \varphi^{(\varepsilon)} - \partial_4 \chi^{*(\varepsilon)\rho} \partial_4 \chi_\rho^{(\varepsilon)} \},$$

$$E^{(2)} = - (2/\kappa) \sum_{\varepsilon=\pm 1} \int dV \{ 8 \partial_{\lambda} \eta^{*(\varepsilon)\mu\nu} \partial_{\lambda} \eta_{\mu\nu}^{(\varepsilon)} - 2 \partial_{\lambda} \xi^{*(\varepsilon)\rho} \partial_{\lambda} \xi_{\rho}^{(\varepsilon)} - \partial_{\lambda} \xi^{*(\varepsilon)\mu[\nu\lambda]} \partial_{\lambda} \xi_{\mu[\nu\lambda]}^{(\varepsilon)} \} + (1/3) E^{(1)} \quad (79)$$

where the 2nd term on the right hand side of the last equation is the contribution arising from the quantities of the lower class separated from the higher class. The total energy can not be positive definite. However the contributions from the field with mass $\kappa/2$ is positive definite whereas those of the fields with mass κ is negative definite. The total charge is obtained as

$$J_{\mu} = J_{\mu}^{(1)} + J_{\mu}^{(2)}$$

with

$$\begin{aligned} J_{\mu}^{(1)} &= -4(i/\kappa) \sum_{\varepsilon=\pm 1} \int dV \{ 2(1+\varepsilon) \varphi^{*(\varepsilon)} \partial_{\mu} \varphi^{(\varepsilon)} - \chi^{*(\varepsilon)\rho} \partial_{\mu} \chi_{\rho}^{(\varepsilon)} \}, \\ J_{\mu}^{(2)} &= -2(i/\kappa) \sum_{\varepsilon=\pm 1} \int dV \{ \eta^{*(\varepsilon)\rho\lambda} \partial_{\mu} \eta_{\rho\lambda}^{(\varepsilon)} - \xi^{*(\varepsilon)\rho} \partial_{\mu} \xi_{\rho}^{(\varepsilon)} - \xi^{*(\varepsilon)\rho[\lambda\sigma]} \partial_{\mu} \xi_{\rho[\lambda\sigma]}^{(\varepsilon)} \} + (1/3) J_{\mu}^{(1)}, \end{aligned} \quad (80)$$

so that the contributions of individual fields are not positive definite as was expected.

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Appendix

For the Dirac algebra we have, as is well known,

$$\text{sp}(E) = 4 \quad \text{and} \quad \text{sp}(\gamma_A) = 0 \quad \text{for} \quad \gamma_A \neq E.$$

Any 4×4 matrix X may be expressed in terms of the linearly independent basis of the Dirac algebra as

$$X = \sum_A x_A \gamma_A \equiv x^A \gamma_A,$$

where x_A are numerical coefficients. Then according to the relationship $\text{sp}(\gamma_A \gamma_B) = 4 \delta_{AB}$, we have $\text{sp}(\gamma_B X) = 4 x_B$, so that X is expressible as

$$X = (1/4) \text{sp}(\gamma^A X) \cdot \gamma_A.$$

Now by multiplying this by another arbitrary 4×4 matrix Y and operating sp we get the desired result

$$\text{sp}(XY) = (1/4) \text{sp}(\gamma^A X) \cdot \text{sp}(\gamma_A Y) = \text{hs}(\gamma^A X) \cdot \text{hs}(\gamma_A Y).$$

Then, an arbitrary element X belonging to the Duffin-Kemmer algebra is expanded in the form

$$X = x^{A;B} P_{A;B}$$

with numerical coefficients $x_{A;B}$. According to eq. (40) we have

$$\text{sp}(P_{C;D} X) = x^{A;B} \cdot \text{sp}(P_{C;D} P_{A;B}) = x_{D;C}.$$

Therefore we have

$$X = \text{sp}(P^{B;A} X) \cdot P_{A;B}$$

and further

$$\text{sp}(XY) = \text{sp}(P^{B;A} X) \cdot \text{sp}(P_{A;B} Y) = \sum_{\varepsilon=\pm 1} \text{sp}(P^{(\varepsilon)BA} X) \cdot \text{sp}(P_{AB}^{(\varepsilon)} Y).$$

In the above equations the summation runs through all ones of the linearly independent elements, each one appearing only once. Therefore we must be careful of the following two points when the abstract symbols A and B are treated concretely as tensor indices. First, when we sum over an antisymmetric pair of indices $[\mu\nu]$ with μ and ν running through the four numbers 1, 2, 3 and 4, the factor 1/2 must always be attached. Another point concerns with the case where the linearly independent element has odd number of indices. Since when expressed in the form $P_{A;B}$ it is natural to write P as $P_{0;0}$, $P_i^{\beta\mu}$ and $\beta_\mu P$ are regarded respectively as $P_{0;\mu}$ and $P_{\mu;0}$. On symmetrizing the indices we get $P_{\mu\alpha}^{(\varepsilon)}$ and $P_{\mu\alpha}^{(\varepsilon)}$ as $P_{A;B}^{(\varepsilon)}$. But when we consider wave functions the suffix zero is irrelevant so the summation, if taken only over μ , must be doubled. The same caution is applied to the case of $V_{\mu[\nu\lambda]}^{(\varepsilon)}$ in the higher class. These two rules must be observed when we pass from eq. (49) to eqs (54) and (54').

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Note added in proof: The field $\zeta_{\mu[\nu\lambda]}$ derived from the higher class are made up of two independent fields, one with spin 0 and the other with spin 2 for both the symmetric and the antisymmetric cases. The complete solution will soon be published.

The Vector Representation of Spinning Particle in the Quantum Theory, I*

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A quantum-mechanical non-relativistic spinning particle is represented equivalently with a non-linear vector field which defines a new kind of hydrodynamics of a spinning fluid. In this hydrodynamics, quantum effects are separated as non-linear terms which mean the occurrence of "internal potential" and "internal magnetic field".

Mathematically, the method is based upon the replacement of the original calculus in terms of a spinor ψ by the calculus in terms of tensors formed as the bilinear expressions in ψ^* and ψ . This replacement is ensured by setting up the identities which should hold among those tensor quantities.

§ 1. Introduction

In the usual formulation of quantum mechanics, wave function which is not observable itself plays the essential role. However, as was shown in previous papers¹⁾²⁾, an alternative *hydrodynamical* formulation is possible for quantum mechanics in case of spinless particle, in which formulation the wave function is replaced by real and gauge-independent quantities.

The second characteristic situation of the ordinary formulation of quantum mechanics, however, arises in the treatment of spinning particle; viz., the appearance of a *spinor* with its special double-valued transformation property. In quantum mechanics, wave function can be such a spinor, since originally wave function is not directly observable but rather it works as the intermediary deriving observable values through its bilinear expressions^{3)4a)}. On the other hand in *classical* theories which have to operate according to some realistic model**, spinor quantity with its double-valued transformation property cannot appear, at least as an essential thing***.

* This article (hereafter called **I**) and also the major part of the subsequent one (hereafter called **II**) which will appear in succession, are the translation of the paper which appeared in *Soryusiron-Kenkyu* 7, pp. 600-640, (January 1955) in Japanese, while a brief account of these works was reported in *Prog. Theor. Phys.* 12 (1954), 810.

Very recently articles by Bohm, Schiller, and Tiomno appeared (*Nuovo Cimento* 1, Suppl. (1955), 48, 67), in which they investigated, independently of us, the similar hydrodynamical method for the purpose of a "causal interpretation" of quantum theory in the case of a non-relativistic spinning particle. However, their method stands upon Euler angle representation of the spinor (cf. **II**), not directly concerned with the self-contained tensor formulation of the spinor equation.

** We may rather consider here this property as the definition of a theory being classical.

*** Proca⁴⁾ considered a classical spin theory from just the opposite viewpoint to ours.

It should thus be necessary to transform, beforehand, the spinor representation into some *tensor* representation equivalently, in order to reveal the *picture* in classical sense underlying the spinor representation of quantum mechanics, or to find the *correspondence principle* and also the *classical approximation* for a spinor equation. In the present and the subsequent papers we give the answer to those problems in the case of a single non-relativistic spinning particle*.

Now, as regards the general problem to replace a spinor by some tensor quantities, we may first recollect the work of Darwin⁵⁾, who tried to interpret the quantum mechanics of a spinning particle, which was just then formulated by Pauli⁶⁾, in terms of a certain vector wave. His method, however, was unsatisfactory in that the vector itself had arbitrariness.

More lately Kramers⁷⁾ could represent the spinor by a complex zero-vector, which method may be regarded as providing the geometrical representation of a spinor. Yet, he could not express the equation of motion in terms of that zero-vector in a closed form.

In the present paper we establish a vector formulation of spinor theory which is mathematically consistent and self-contained, and is at the same time physically fully interpreted and thus much more significant. Our formulation is gauge-independent, and defines just a particular kind of hydrodynamics of a fluid carrying classical "spin", where quantum effects are separated as the non-linear dynamical terms in the equations of motion. These terms are interpreted as the "internal potential" and "internal magnetic field" occurring inside the fluid.

The formulation thus provides an essentially new insight and image to the quantum mechanics of a spinning particle or to a spinor field. It would accordingly be expected to be useful also for various practical applications.

§ 2. General description

i) We consider the quantum mechanics of a single non-relativistic spinning particle. Its state is represented by a two-component spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, which changes according to the Schrödinger-Pauli equation :

$$\{(\hbar/i)\partial_0 + \underline{H}\} \psi = 0, \quad (\partial_0 \equiv \partial/\partial t) \quad (2.1)$$

where the Hamiltonian H is taken to be

$$\underline{H} = (1/2m) (\underline{p} - e\underline{A}/c)^2 + e\underline{A}_0 - (e/mc) \underline{H} \underline{s}, \quad (2.2)**$$

as we consider, for simplicity, in the lowest non-relativistic approximation. In (2.2),

⁴ Similar treatment of the Dirac equation has also been carried through in separate articles: T. Takabayasi, *Prog. Theor. Phys.* **13** (1955), 222; *Nuovo Cimento*. (in press)

It is to be remarked that our theory supplies a picture in classical sense to a quantum-mechanical spinning particle, and so stands in different lines from various existing theories^{(4), (11)} which attempt to give some model to the freedom of particle spin in the usual quantum-mechanical formulation.

** Underlined quantities mean quantum-mechanical q-numbers.

(\mathbf{A}, A_0) is the external electromagnetic potential, $\mathbf{H} = \text{curl } \mathbf{A}$ the magnetic field strength, and $\underline{s} = (\hbar/2)\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ being the Pauli matrices.

Now, in the case of spinless particle, the hydrodynamical description of the Schrödinger equation was easily derived by making use of de Broglie-Madelung substitution^{(1),(2)}

$$\psi = R e^{iS/\hbar}, \quad (R, S: \text{real}). \quad (2.3)$$

In the present case, however, the direct extension of (2.3),

$$\psi_\alpha = R_\alpha \exp(i\chi_\alpha), \quad (R_\alpha, \chi_\alpha: \text{real}; \alpha = 1, 2) \quad (2.4)$$

for the spinor wave function would not work suitably*, because then the quantities R_α and χ_α have no simple transformation properties and accordingly must not have any significant physical meanings.

In the present case we should start, as stated in § 1, upon a more general line of searching for the set of tensor quantities which represents the spinor ψ in a mathematically equivalent and physically meaningful way. When we can find such a set of tensors, which we shall call the "set of basic quantities", every physical relation originally expressed in the language of ψ will have to be expressed in terms of those basic quantities only.

Now, we may take as such basic quantity, the complex zero-vector of Kramers mentioned in § 1. This is defined by

$$\vec{\xi} = \tilde{\psi} \sigma \psi, \quad (2.5)$$

using the spinor $\tilde{\psi}$ contragradient to ψ ,

$$\tilde{\psi} = (\psi^1, \psi^2) \equiv (\psi_2, -\psi_1), \quad (2.6)$$

and satisfies

$$\vec{\xi}^2 = 0. \quad (2.7)$$

If we write

$$\vec{\xi} = \mathbf{M} + i\mathbf{N}, \quad (\mathbf{M}, \mathbf{N}: \text{real}) \quad (2.8)$$

(2.7) becomes

$$\mathbf{M}^2 = \mathbf{N}^2, \quad \mathbf{M} \cdot \mathbf{N} = 0. \quad (2.9)$$

Really, the spinor ψ can be represented either with $\vec{\xi}$ restricted by the condition (2.7) or with the set (\mathbf{M}, \mathbf{N}) restricted by (2.9). Given a $\vec{\xi}$, the corresponding ψ is determined aside from the arbitrariness of factor ± 1 .

In this method, however, $\vec{\xi}$ is not gauge-invariant, and moreover the equation of motion for $\vec{\xi}$ cannot conveniently be expressed in a closed form.

ii) We should then look for other basic quantities. We thus adopt, in place of the tensors bilinear in $\tilde{\psi}$ and ψ , the tensors bilinear in ψ^* and ψ , which may be called "density functions" and have more direct physical meanings.

* In reference 1) we have shown that the substitution (2.4) leads to the two-fluid model of spin.

Such covariant quantities* are generally produced, by using σ_k and/or the gradient operator $\partial_i \equiv \partial/\partial x_i$, as

$$\left. \begin{array}{ll} \text{tensor :} & \partial_i \partial_j \cdots \psi^* \partial_l \partial_m \cdots \psi, \\ \text{pseudotensor :} & \partial_i \partial_j \cdots \psi^* \sigma_k \partial_l \partial_m \cdots \psi. \end{array} \right\} \quad (2.10)$$

The simplest of them are**,

$$\text{scalar :} \quad P = \psi^* \psi \quad (\geq 0), \quad (2.11)$$

$$\text{axial vector :} \quad (\sigma) = \psi^* \sigma \psi, \quad (2.12)$$

$$\text{vector :} \quad \text{Im}(\psi^* \nabla \psi) = (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi) / 2i. \quad (2.13)$$

Of these, (2.13) is not gauge-invariant, so we may better replace it by the gauge-invariant vector :

$$j_k = \frac{\hbar}{2mi} (\psi^* D_k \psi - \text{c.c.}) = \frac{\hbar}{2mi} (\psi^* \partial_k \psi - \text{c.c.}) - \frac{e}{mc} P A_k, \quad (2.14)$$

where $D_k \equiv \partial_k - (ie/\hbar c) A_k$, and c.c. stands for 'conjugate complex'.

For our purpose it is often more convenient to replace the density functions, (2.12), (2.13), and (2.14), by the corresponding "particle functions", i.e.,

$$\vec{\Sigma} = (\sigma) / P, \quad \text{or} \quad \mathbf{S} = (\hbar/2) \vec{\Sigma} = \psi^* \underline{\mathbf{s}} \psi / \psi^* \psi, \quad (2.15)$$

$$\mathbf{p}(\mathbf{x}) = (\hbar/2i) (\psi^* \nabla \psi - \text{c.c.}) / \psi^* \psi, \quad (2.16)$$

and

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &= \mathbf{j} / P = (\mathbf{p} - e\mathbf{A}/c) / m \\ &= (\hbar/2mi) (\psi^* \nabla \psi - \text{c.c.}) / \psi^* \psi - (e/mc) \mathbf{A}. \end{aligned} \quad (2.17)$$

P is usually normalized: $\int P d\mathbf{x} = 1$, while (σ) or \mathbf{S} satisfies the identity :

$$(\sigma)^2 = P^2, \quad \text{i.e.,} \quad \vec{\Sigma}^2 = 1, \quad \text{or} \quad \boxed{\mathbf{S}^2 = \hbar^2/4.} \quad (\text{I})$$

Now the first important point for our method is that

(α) : we can adopt the set $(P, \mathbf{S}, \mathbf{v})$ as the basic set representing the spinor.

The four quantities, P and (σ_k) , which exhaust bilinear quantities involving no ∂ -operator, contain three independent quantities on account of (I), whereas the original spinor ψ contains four real quantities. The remaining variable, involved in ψ but not reflected by $(P, (\sigma))$, is obviously the sum of phases of both components of ψ , i.e., $\chi = \chi_1 + \chi_2$, where χ_α 's are those defined in (2.4). But this χ is neither a scalar nor a tensor component (see II) ; and, in order to reflect this variable by a tensor quantity bilinear in ψ^* and ψ ,

* In this paper by a tensor or a covariant is meant one in three-dimensional sense. A Latin index means a vector index running over 123. The transformation properties, including those for space inversion and time reversal, of our formulation will intimately be explained in II.

** Note that the real part of $\psi^* \nabla \psi$ is merely $\nabla P/2$.

we need a vector involving ∂ -operator, such as \mathbf{j} or \mathbf{v} . In so doing we arrive at our basic set, $(P, \mathbf{S}, \mathbf{v})$.

Then such introduction of \mathbf{v} for the representation of χ must be accompanied by two more constraints, which we find (see § 4) to be

$$\text{curl}_i \mathbf{v} = (4/m\hbar^2) \varepsilon_{lmn} \underline{S}_l \partial_j \underline{S}_m \partial_k \underline{S}_n - (e/mc) H_i, \quad (\text{II})$$

$$(i=j \times k)^*$$

where ε_{lmn} is the completely antisymmetric pseudotensor of third rank with $\varepsilon_{123}=1$. We also apply the summation convention for repeated indices**. The relation (II) may also be written

$$m(\partial_j v_k - \partial_k v_j) = S_l^{-1} (\partial_j \underline{S}_m \partial_l \underline{S}_n - \partial_k \underline{S}_m \partial_j \underline{S}_n) - (e/c) H_i, \quad (\text{II}')$$

$$(l=m \times n)$$

when we re-express the determinant term, $\varepsilon_{lmn} \underline{S}_l \partial_j \underline{S}_m \partial_k \underline{S}_n$, of (II) by the help of (I) [see § 4-v]. Eq. (II) contains just two independent relations, because the divergence of each side of it vanishes identically on account of (I) [see (4.25)].

Now, given the set of basic quantities $(P, \mathbf{S}, \mathbf{v})$ obeying the subsidiary conditions (I) and (II), the corresponding ψ is determined except for a common arbitrary additive constant, $\chi_0/2$, in the phases of ψ_1 and ψ_2 . This arbitrariness arises from the fact that \mathbf{v} was defined to involve the ∂ -operator on ψ^* or ψ . χ_0 is, however, a physically meaningless thing, as long as we deal with single particle problems. Thus, given $(P, \mathbf{S}, \mathbf{v})$, the corresponding quantum-mechanical state is specified just uniquely.

At this point we consider the physical meanings to be assigned to the basic quantities in the viewpoint of our formalism. First, corresponding to the fact that in ordinary quantum mechanics P and $\mathbf{j}=P\mathbf{v}$ are probability density and current, we imagine the hydrodynamical field having density distribution P and velocity distribution \mathbf{v} . Further, \mathbf{S} is regarded as the distribution of *classical spin* (i.e., intrinsic angular momentum) which the hydrodynamical field is assumed to carry. This is required in order that the hydrodynamical picture should reproduce the quantum-mechanical expectation value of spin component correctly, as $\int \psi^* s_z \psi d\mathbf{x} = \int P S_z d\mathbf{x}$ [see also Appendix B-b)]. The spin field \mathbf{S} is of constant magnitude $\hbar/2$ everywhere due to (I), and is related to the vorticity through (II).

It is the basic feature of our formulation that it represents the physical contents of the spinor wave function in splitting them into two kinds of vector fields, spin and velocity, and also a density field.

Our definitions of density and velocity functions, (2.11) and (2.17), are formally

* $i=j \times k$ signifies that (ijk) is an even permutation of (123) .

** This convention is explicitly indicated by underlining dummy indices, when it is desirable. The convention will also be applied for spinor indices α, β, \dots , or "4-vector" indices μ, ν, \dots , afterwards (see e.g., eq. (3.7) or (3.4)).

identical with those in the hydrodynamical formulation of a spinless particle, but in the latter case the hydrodynamical field was described by P and \mathbf{v} alone, with the "quasi-irrotationality condition"^{*)}:

$$\text{curl } \mathbf{v} = -e\mathbf{H}/2mc. \quad (2.18)$$

In the present case the subsidiary condition (2.18) is replaced by (II), and \mathbf{r} becomes a general rotational field (cf. II). S , being the intrinsic angular momentum, must certainly reflect internal degrees of freedom of the underlying fluid, but it is not necessary further to explain it by such a model where each element of the fluid rotates intrinsically (cf. II).

Anyway we consider a certain kind of hydrodynamics of a spinning fluid. The appropriateness of such notion will be made more remarkable when we study the dynamical aspect of the problem [iv].

iii) Now, to the statement that (P, S, \mathbf{r}) constitute the basic set, may be attributed further implications, that is, every higher tensor included in (2.10):

$$\partial_i \psi^* \partial_k \psi, \quad \psi^* \partial_i \partial_k \psi, \quad \psi^* \sigma_i \partial_k \psi, \quad \dots, \quad (2.19)$$

can be reduced to a rational expression* of (P, S, \mathbf{r}) and their space derivatives. By virtue of this circumstance, e.g., the equations of motion for P , S , and \mathbf{r} , calculated on the basis of the wave equation (2.1), can be led to rational simultaneous equations closed concerning (P, S, \mathbf{v}) [see § 3].

Thus an important mathematical basis for setting up our formalism consists in the systematic clarification of the identical relations holding among various bilinear quantities of (2.10); viz., the *identities between basic quantities only*—these constitute the subsidiary conditions (I) and (II) —, and the formulae for the *reductions of higher tensor quantities to basic ones*. The actual treatment of these problems will be carried out in § 4. The relations obtained furnish the mathematical apparatus which enables us to replace the original calculus in terms of a spinor by the tensor calculus of our formalism.

Such formulae and technique will be useful also for the treatments of a two-component spinor in general. The formulae obtained are concerned, except for (I), with the bilinear quantities involving differentiation, such as (4.10), (4.11), and (4.15), and do not seem to have been stated in the literature. On the other hand, it is also to be noted that *our formulation endows those mathematical identities with significant physical meanings*.

Our formulation is thus based upon the bilinear quantities in ψ^* and ψ , in place of ψ itself. Hence the linear property possessed by the original ψ -formalism (wave equation (2.1), etc.), i.e., the superposition principle, becomes implicit there, with the entrance of *non-linear terms, which generally just embodies the quantum effects in our formulation*. The linear and homogeneous character of the original formalism, however, remains in the fact that our formalism is homogeneous in P (though not so in \mathbf{v} or S).

We are now in a position to express every physical relation for a spinning particle in

* Speaking more precisely, the quantities (2.19) can be reduced to the form:
 $P \times \{\text{polynomial of } \log P, S, \mathbf{v}, \text{ and their space derivatives}\}.$

terms of our basic quantities (P, S, \mathbf{v}) , and thereby bring to light some new aspect implicitly involved in the original spinor.

Measurable in quantum mechanics are expectation values and probability distributions of various physical quantities. But they are of expressions in ψ of similar type with our basic quantities, and therefore can conveniently be expressed in terms of the latter. Moreover the results come out to be understandable by our picture for many important cases [Appendix B].

More important is the dynamical problem, which we will consider in the following.

iv) We can show (see § 3) that the physical contents of the wave equation (2.1) are cast equivalently into the following simultaneous equations of motion for (P, S, \mathbf{v}) , i.e.,

$$\partial_0 P + \text{div}(\mathbf{P}\mathbf{v}) = 0, \quad (\text{III})$$

$$m \frac{dv_i}{dt} = K_i + \frac{e}{mc} \partial_i H_k \cdot S_k - \partial_i \Pi^P - \frac{1}{mP} \partial_k (P \partial_i S_l \cdot \partial_k S_l), \quad (\text{IV})$$

$$\frac{d\mathbf{S}}{dt} = \frac{e}{mc} [\mathbf{S} \times \mathbf{H}] + \frac{1}{mP} [\mathbf{S} \times \partial_k (P \partial_k \mathbf{S})], \quad (\text{V})$$

where

$$d/dt \equiv \partial_0 + \mathbf{v} \cdot \nabla, \quad (\partial_0 \equiv \partial/\partial t)$$

$$\mathbf{K} = e\mathbf{E} + (e/c)[\mathbf{v} \times \mathbf{H}], \quad (2.20)$$

with

$$E_i = -\partial_0 A_i/c - \partial_i A_0 = \text{electric field strength},$$

and

$$\Pi^P = -(\hbar^2/4m) \{ \Delta P/P - (\nabla P)^2/2P^2 \} = -(\hbar^2/2m) \Delta(P^{1/2})/P^{1/2}. \quad (2.21)$$

These equations of motion (III), (IV), and (V), together with the subsidiary conditions (I) and (II), constitute the fundamental set of equations of our formalism. It is at once clear that

(β): it presents the real and gauge-independent tensor formalism for a quantum-mechanical spinning particle.

Moreover we can see that those equations of motion, (III), (IV), and (V), can be interpreted as the hydrodynamical equations of motion for our spinning fluid introduced before, with a supplementary assumption that the spin field \mathbf{S} be accompanied by the distribution of magnetic moment $(e/mc)\mathbf{S}$.

First, (III) means obviously the equation of continuity, i.e., the conservation of fluid.

Second, (IV) implies Euler's equation of flow, noting that d/dt means the substantial derivative. Indeed, in its right side, the first term K_i is the Lorentz force exerted on the element of the fluid by the external field, and the third term is the so-called "quantum force" which appeared also in the similar (but more simple) hydrodynamical formulation of a spinless particle^{(1),(9)}. But there appear, in addition, the second and the fourth terms

which couple the orbital motion to the spin field. The former means the classical reaction upon the orbital motion of the element of the fluid by the possible inhomogeneity of the magnetic field, through the associated magnetic moment $(e/mc)\mathbf{S}$, while the latter is regarded as the force due to a new stress ("spin stress") inside the fluid which arises from the inhomogeneity of spin distribution.

Finally, (V) means the equation of motion for the spin. In its right side, the first term is the classical torque exerted on the spin by the external magnetic field through the associated magnetic moment, while the second term implies that the spin is also subjected to the additional torque ("spin torque"), which is dependent on the gradient of the spin distribution itself, and tends to align neighboring spins parallel [see § 3-viii)].

In all, we can say that

(γ) : *our formalism describes just a particular kind of hydrodynamics of a spinning fluid.*

The realization of the above picture in classical sense should be considered not accidental, but to originate from the correspondence principle existing between quantum mechanics and classical theory. Indeed, *our method reveals a most precise and explicit form of such correspondence-theoretical relations in the case of a spinning particle.*

We know that the purely classical equations of motion for a classical spinning particle are given by

$$\left\{ \begin{array}{l} m d\mathbf{v}_i/dt = \mathbf{K}_i + \gamma \partial_i H_{\underline{k}} \cdot \underline{s}_{\underline{k}}, \\ \end{array} \right. \quad (2.22)$$

$$\left\{ \begin{array}{l} d\mathbf{s}/dt = \gamma [\mathbf{s} \times \mathbf{H}], \end{array} \right. \quad (2.23)$$

where \mathbf{r} , \mathbf{s} , and $\gamma \mathbf{s}$ denote the the velocity, spin, and associated spin magnetic moment of the particle, respectively (cf. II). The comparison of our equations of motion (IV) and (V) with (2.22) and (2.23) indicates that

(\hat{n}) : *in our formulation quantum effects are separated as non-linear, dynamical terms of the order \hbar^2 .*

Especially, spin stress or spin torque term means a sort of "spin-orbit coupling" or a sort of "spin-spin interaction," respectively. It is to be noted that we have obtained these terms in our hydrodynamical formalism, as quantum-mechanical effects, for the system with simple Hamiltonian (2.2) which does not contain the (well-known) spin-orbit coupling or spin-spin interaction term originally.

Now, these quantum terms may be provided with still clearer images by the introduction of the notions of "total quantum potential" Π and "internal magnetic field" \mathbf{H}^m . The former consists of the usual quantum potential Π' and the contribution from the spin field, such that

$$\Pi = \Pi' - |\nabla \mathbf{S}|^2/2m, \quad |\nabla \mathbf{S}|^2 \equiv \sum_{k,l} (\partial_{\underline{k}} \underline{s}_{\underline{l}})^2, \quad (2.24)$$

while the latter is defined by

$$\mathbf{H}^m = (c/eP) \partial_{\underline{k}} (P \partial_{\underline{k}} \mathbf{S}) = (c/e) (\mathbf{S} + \partial_{\underline{k}} P \cdot \partial_{\underline{k}} \mathbf{S}/P). \quad (2.25)$$

The "effective magnetic field",

$$\mathbf{H}^{\text{eff}} = \mathbf{H} + \mathbf{H}^m, \quad (2.26)$$

to act on the spin magnetic moment, is also used. We can then re-express [see § 3-iv)] (IV) and (V) in quite simple forms, i.e.,

$$mdv_i/dt = K_i - \partial_i \Pi + (e/mc) \partial_i H_k^{\text{eff}} \cdot S_k, \quad (\text{IV}')$$

$$dS/dt = (e/mc) [S \times H^{\text{eff}}]. \quad (\text{V}')$$

These results show that

(E) : the hydrodynamical motion representing the motion of a quantum-mechanical spinning particle is conducted in such a way that each element of the fluid moves like a classical spinning particle under the actions not only of external force but also of the internal potential Π and the internal magnetic field H^{in} .

The equation of motion (IV') may further be rewritten

$$mdv_i/dt = \{eE_i^{\text{eff}} + (e/c)[\mathbf{v} \times \mathbf{H}]_i\} + (e/mc) \partial_i H_k^{\text{eff}} \cdot S_k, \quad (\text{IV}'')$$

with
$$E_i^{\text{eff}} = -\partial_0 A_i/c - \partial_i A_0^{\text{eff}}, \quad A_0^{\text{eff}} = A_0 + \Pi/e. \quad (2 \cdot 27)$$

In the forms of (IV'') and (V'), our equations become formally identical with the purely classical equations of motion for a spinning particle, (2·22) and (2·23), with the substitution :

$$A_0 \rightarrow A_0^{\text{eff}}, \quad \mathbf{H} \rightarrow \mathbf{H}^{\text{eff}}, \quad (2 \cdot 28)$$

for the scalar potential and the magnetic field acting on spin. In other words,

(E') : the quantum effects can be viewed simply as renormalizing the scalar potential and magnetic field strength acting, by (2·28).

v) At any rate we have obtained the hydrodynamical formulation which describes a state and motion of a quantum-mechanical spinning particle, with (P, S, \mathbf{v}) under the subsidiary conditions (I) and (II), and with the equations of motion (III), (IV'), and (V').

It is possible to find out the Lagrangian which leads to those equations of motion together with the subsidiary conditions (see II). This also ensures the consistency of our formalism.

At this point we remark that in our formulation Planck's constant \hbar does not appear explicitly in the equations of motion (III), (IV) and (V), except at the quantum potential term Π^P . Accordingly, the "spin stress" and "spin torque" terms in (IV) and (V) may also be considered like classical effects, contrary to the viewpoint of (E). On the other hand, however, S itself is restricted by (I) (which may be called the "quantum condition" for the spin field), so that it has a constant magnitude $\hbar/2$. [In this connection \hbar also appears in (II), but disappears if we re-express it as (II').] Thus in our formulation Planck's constant plays two-fold roles, i.e., the coupling constant for the quantum potential and the magnitude of spin (cf. II).

Finally, our formulation for a quantum-mechanical spinning particle may also be understood by the picture of a sort of statistical ensemble of numerous classical motions of

a spinning particle, in place of hydrodynamical picture (Appendix A). In that picture, however, individual motions belonging to an ensemble are not independent of one another, but must be regarded as mutually interacting through the internal potential U and internal magnetic field \mathbf{H}^{in} . Neither such an ensemble picture nor the hydrodynamical one can be accepted as literally real and to permit at once the realistic interpretation of quantum mechanics. This situation is similar to that met with in case of simpler treatments of spinless particle^{1),9)}.

§ 3. Derivation of the hydrodynamical equations, energy-momentum tensor, etc.

i) For proving the equivalency of the original wave equation (2.1) with the set of our hydrodynamical equations of motion (III), (IV), and (V), we have various possible ways. In what follows, however, we limit ourselves to the explanation that (IV) and (V) can be deduced from (2.1).

We rewrite here (2.1) as

$$iD_0\psi = \{ -(\hbar^2/2m)\mathbf{D}^2 - (1/2)\gamma\mathbf{H}\boldsymbol{\sigma} \} \psi, \quad (3.1)$$

$$\text{with} \quad D_0 = \partial_0 + (ie/\hbar)A_0, \quad \gamma = e/mc. \quad (3.2)$$

The Lagrangian density, which reproduces (3.1) as its field equation, is given by

$$\mathcal{L}^0 = (i\hbar/2) (\psi^* D_0 \psi - \text{c.c.}) - (\hbar^2/2m) \mathbf{D}\psi^* \mathbf{D}\psi + \gamma \psi^* \mathbf{H} \boldsymbol{\sigma} \psi; \quad (3.3)$$

and this enable us to derive, in an ordinary fashion, the canonical energy-momentum tensor $T_{\mu\nu}^0$ for the ψ -field, satisfying

$$\partial_\mu T_{\mu\nu}^0 = \partial \mathcal{L}^0 / \partial (x_\nu). \quad (3.4)^*$$

For instance, the momentum and energy densities are given respectively by

$$\mathcal{G}_i = T_{0i}^0 = (\hbar/2i) (\psi^* \partial_i \psi - \text{c.c.}) = P p_i, \quad (3.5)$$

and

$$\mathcal{H} = -T_{00}^0 = (\hbar^2/2m) \mathbf{D}^* \psi^* \mathbf{D} \psi + e A_0 \psi^* \psi - \gamma \psi^* \mathbf{H} \boldsymbol{\sigma} \psi. \quad (3.6)$$

$T_{\mu\nu}^0$, however, is not gauge-invariant, nor symmetrical, so we replace it by the gauge-invariant energy-momentum tensor:

$$\theta_{\mu\nu} = - \left\{ \frac{\partial \mathcal{L}^0}{\partial (\partial_\mu \psi_\alpha)} D_\nu \psi_\alpha + \text{c.c.} \right\} + \partial_{\mu\nu} \mathcal{L}^0, \quad (3.7)$$

which is symmetric as regards its space components, such that

$$\theta_{ik} = (\hbar^2/2m) (D_i^* \psi^* D_k \psi + \text{c.c.}) + \partial_{ik} \mathcal{L}^0; \quad (3.8)$$

* Indices μ, ν run over 0 1 2 3; and $x_0 \equiv t$. $\partial \mathcal{L}^0 / \partial (x_\nu)$ means the derivative with respect to x_ν , for constant ψ and $\partial_\mu \psi$.

while

$$\theta_{0k} = -(i\hbar/2) (\psi^* D_k \psi - \text{c.c.}) = m j_k = m P v_k. \quad (3.9)$$

We can re-express the conservation equation (3.4) in terms of $\theta_{\mu\nu}$, and especially obtain

$$m \partial_0 j_i + \partial_k \theta_{ik} = \mathfrak{R}_i + \gamma P \partial_i H_k \cdot S_k, \quad (3.10)$$

as the momentum conservation relation, where $\mathfrak{R} = cPE + (e/c)[\mathbf{j} \times \mathbf{H}] = P\mathbf{K}$ is the Lorentz force density.

ii) At this point it is needed for our purpose to express θ_{ik} in terms of our basic quantities: The first term in the right side of (3.8) is reduced by (4.20). On the other hand, \mathcal{L}^0 may be written, for a permissible motion, as

$$\mathcal{L}^0 = -(\hbar^2/4m) (\psi^* \mathbf{D}^2 \psi + \mathbf{D}^* \psi^* \mathbf{D} \psi) + \text{c.c.},$$

which is further simplified, by (4.19), to

$$\mathcal{L}^0 = -(\hbar^2/4m) \Delta P. \quad (3.11)$$

We thus get

$$\theta_{ik} = m P v_i v_k + \pi_{ik} + \sigma_{ik}, \quad (3.12)$$

with

$$\left\{ \begin{array}{l} \pi_{ik} = (\hbar^2/4m) (\partial_i P \partial_k P / P - \partial_{ik} \Delta P), \\ \sigma_{ik} = (\hbar^2/4m) P \cdot \partial_i \sum_l \partial_k \sum_l = (1/m) P \cdot \partial_i S_l \partial_k S_l. \end{array} \right. \quad (3.13)$$

$$(3.14)$$

Here, the second term in π_{ik} , i.e. $\partial_{ik} \Delta P$, may be replaced by $\partial_i \partial_k P$, since the difference is divergenceless. Accordingly we may replace π_{ik} by

$$\pi'_{ik} = -(\hbar^2/4m) P \cdot \partial_i \partial_k (\log P), \quad (3.15)$$

and θ_{ik} by

$$\theta'_{ik} = m P v_i v_k + \pi'_{ik} + \sigma_{ik}. \quad (3.16)$$

By the way, this θ'_{ik} is the quantity which can be written as

$$\theta'_{ik} = (\hbar^2/4m) (-\psi^* D_i D_k \psi + D_i^* \psi^* D_k \psi) + \text{c.c.}, \quad (3.17)$$

in terms of ψ .

Anyway, inserting (3.12) or (3.16) into (3.10) we get

$$m \{ \partial_0 j_i + \partial_k (v_k j_i) \} = \mathfrak{R}_i + \gamma P \partial_i H_k \cdot S_k - \partial_k (\pi'_{ik} + \sigma_{ik}), \quad (3.18)$$

but this is equivalent with the Euler equation (IV), noting (III) and also the relation:

$$\partial_k \pi'_{ik} = P \cdot \partial_i \Pi^P \quad (3.19)$$

iii) As is naturally expected, $T_{\mu\nu}^0$ or $\theta_{\mu\nu}$, when expressed in terms of our basic quantities, can be taken as the energy-momentum tensor of our hydrodynamical field. This is justified because in the first place it satisfies its respective conservation law, and in the second the momentum and energy densities, (3.5) and (3.6), when integrated throughout over space, yield the quantum-mechanical expectation values of momentum and energy [for this point see also Appendix B].

Eq. (3.18) of course expresses the momentum conservation law for our hydrodynamical field, where (3.9) is the kinetic momentum density, and $\pi'_{ik} + \sigma_{ik}$ is the internal stress tensor.

The energy density of our hydrodynamical field is obtained, merely by reducing (3.6) to our basic quantities applying the formula (4.20) again, in the form :

$$\mathcal{H} = P \left\{ \frac{1}{2} m \mathbf{v}^2 + e A_0 - \frac{e}{mc} \mathbf{H} \mathbf{S} + \frac{\hbar^2}{8m} \frac{(\nabla P)^2}{P^2} + \frac{1}{2m} |\nabla \mathbf{S}|^2 \right\}. \quad (3.20)$$

In the right side, the first term is the kinetic energy, the second and third terms usual potential energy, and the last two terms the positive definite potential energy due to self-stress.

In (3.20) the contributions from spin are separated in a very simple form, and do not involve any term like kinetic energy of rotation.

It is also to be noted that the expression (3.20) provides a suitable form to find out energy eigen-states through the *variational principle* :

$$\int \mathcal{H} d\mathbf{x} = \min., \quad \left(\int P d\mathbf{x} = 1 \right). \quad (3.21)$$

iv) We come back to eq. (3.18), i.e., (IV). The relation (3.19) indicates that the stress π'_{ik} may be replaced by a pressure potential II' . On the other hand, the force that comes from σ_{ik} , i.e., the fourth term in the right side of (IV) :

$$Q_i^s \equiv - (1/mP) \partial_k (P \partial_i S_k - \partial_k S_i), \quad (3.22)$$

can be separated into two parts such that

$$Q_i^s = - (e/mc) H_k^{\text{in}} \cdot \partial_i S_k - (1/2m) \partial_i (|\nabla \mathbf{S}|^2), \quad (3.23a)$$

$$= (e/mc) \partial_i H_k^{\text{in}} \cdot S_k + (1/2m) \partial_i (|\nabla \mathbf{S}|^2), \quad (3.23b)$$

where \mathbf{H}^{in} is the internal magnetic field* defined in (2.25), and the following relation has been used :

$$(1/m) |\nabla \mathbf{S}|^2 = - (e/mc) \mathbf{H}^{\text{in}} \cdot \mathbf{S}, \quad (3.24)$$

which is simply a consequence of (I). Eq. (3.23b) is just the relation employed in the transcription from (IV) to (IV').

* As for \mathbf{H}^{in} , it is to be noted that a) it acts on particle only through magnetic moment, and b) $\text{div } \mathbf{H}^{\text{in}} \neq 0$.

By the way, Π and \mathbf{H}^{in} depend only upon the gradients of $\log P$ and \mathbf{S} .

v) Next we shall derive the equation of motion (V). First, using (3.1) we can obtain

$$\frac{\partial(\sigma_i)}{\partial t} = \gamma[(\sigma) \times \mathbf{H}]_i + \frac{i\hbar}{2m} \partial_{\underline{k}} (\psi^* \sigma_i D_{\underline{k}} \psi - \text{c.c.}). \quad (3.25)$$

Inserting the reduction formula (4.21) into the second term of the right side, (3.25) is put in the form:

$$\frac{\partial(\sigma_i)}{\partial t} + \partial_{\underline{k}} \{v_{\underline{k}} \cdot (\sigma_i)\} = \gamma[(\sigma) \times \mathbf{H}]_i + \frac{\hbar}{2m} \partial_{\underline{k}} \left\{ \frac{1}{P} [(\sigma) \times \partial_{\underline{k}}(\sigma)]_i \right\}. \quad (3.26)$$

Noting (III) again, (3.26) is readily rewritten as (V).

vi) Here we shall note that, under the constraint (I), only two of the three component equations of (V) are independent of one another, because the scalar product relation of (V') with \mathbf{S} is automatically satisfied on account of (I).

The compatibility of the subsidiary condition (II) with the equations of motion (III), (IV), and (V), may be verified directly: We can exhibit that the substantial derivative equation of (II) is actually satisfied as a consequence of (I)–(V). This calculation gives at the same time the expression for the substantial rate of change of the vorticity of our hydrodynamical field, showing that in our hydrodynamics the *vorticity does not persist*. We omit, however, to write down this calculation here, since the consistency of our formalism will also be made clear in II by finding the Lagrangian which derives (I)–(V).

vii) A feature of the equations of motion for \mathbf{j} and (σ) , (3.18) and (3.25), is that the quantum terms appear only in divergences. This is a general situation which should be needed for *Ehrenfest's theorem* to hold. Indeed, when we integrate these equations over whole space, the quantum non-linear terms drop away and we obtain simply

$$\left\{ m \frac{d}{dt} \int P v_i d\mathbf{x} = \int P \left(K_i + \frac{e}{mc} \partial_i H_{\underline{k}} \cdot S_{\underline{k}} \right) d\mathbf{x}, \quad (3.27) \right.$$

$$\left. \left\{ \frac{d}{dt} \int P S d\mathbf{x} = \frac{e}{mc} \int P [\mathbf{S} \times \mathbf{H}] d\mathbf{x}. \quad (3.28) \right. \right.$$

These results exhibit that particle moves on the average like a classical spinning particle. So far as these results are concerned, we can get them simply by taking the expectation value of Heisenberg's equation of motion for \underline{v} or \underline{s} as q -numbers.

viii) Finally, we again take up the last term of the energy density expression (3.20),

$$P|\mathbf{FS}|^2/2m, \quad (3.29)$$

which is smaller according as the spatial variation of spin directions becomes more gradual. Also, this energy term is responsible for the appearance of the "spin torque" term,

$$\frac{1}{mP}[\mathbf{S} \times \partial_k(P\partial_k\mathbf{S})] = \frac{1}{m}[\mathbf{S} \times \mathcal{A}\mathbf{S}] + \frac{1}{mP}\partial_k P \partial_k \mathbf{S}, \quad (3.30)$$

in the equation of motion (V) for \mathbf{S} . The occurrence of (3.29) or (3.30) means the action inside our fluid which tends to align the distribution of spin directions.

A noticeable fact is that our spin energy density and torque, (3.29) and (3.30) (or the first term in its right side), are of similar forms respectively to the energy density of exchange interaction and the torque due to it which have been introduced by Herring and Kittel⁽¹⁰⁾, in their phenomenological approach to *spin waves* in ferromagnetic media.

It can thus be seen for a simple example that, as a consequence of that spin torque, we may have such solutions in our hydrodynamical theory as to behave like a spin wave (see II).

It is to be noted, however, that the spin wave theory takes up the spin vector field which is originally a q -number, while our formulation represents, not the *spin* operator, but the *spinor* wave function by two kinds of c -number vector fields, i.e., spin and velocity distributions.

§ 4. Identities between bilinear quantities

i) In this section we will derive, in a systematic and simplest way, identical relations holding among bilinear quantities (2.10), which relations have provided the mathematical basis for the arguments in the preceding sections. All our treatments of the problem will be performed without recourse to any specified representation of σ_i 's.

As we deal in this section with essentially mathematical problems, it is convenient to employ the abbreviation,

$$(O) \equiv \psi^* O \psi, \quad (4.1)$$

where O means some operator; e.g.,

$$\left. \begin{aligned} (\partial_k) &\equiv \partial_k \psi^* \cdot \psi, & (\vec{\partial}_k) &\equiv \psi^* \partial_k \psi, \\ (\vec{\partial}_i \vec{\partial}_k) &\equiv \partial_i \psi^* \partial_k \psi, & (\vec{\partial}_k \sigma_\mu) &\equiv \partial_k \psi^* \sigma_\mu \psi, \dots \end{aligned} \right\} \quad (4.2)$$

We also use another abbreviation,

$$\partial_k(\sigma_\mu) \equiv i \{ (\vec{\partial}_k \sigma_\mu) - (\sigma_\mu \vec{\partial}_k) \}, \quad (4.3)$$

where $\mu = 1, 2, 3, 0$, with $\sigma_0 \equiv I$.

Our problem is then to find out the kinematical relations existing between our seven basic quantities :

$$\left. \begin{aligned} (I) &= \psi^* \psi = P, & (\sigma_k) &= \psi^* \sigma_k \psi, \\ \partial_k(I) &= i \{ (\vec{\partial}_k) - (\vec{\partial}_k) \} = (2/\hbar) P p_k; \end{aligned} \right\} \quad (4.4)$$

and also to reduce the higher tensor quantities :

$$\left\{ \begin{array}{l} i \{ (\vec{\partial}_i \sigma_k) - \text{c.c.} \} \equiv \partial_i (\sigma_k), \\ (\vec{\partial}_i \vec{\partial}_k) + \text{c.c.} \equiv \{ (\vec{\partial}_i \vec{\partial}_k) \}, \\ i \{ (\vec{\partial}_i \vec{\partial}_k \vec{\partial}_l) - \text{c.c.} \}, \\ \dots\dots\dots \end{array} \right. \quad \begin{array}{l} (4 \cdot 5a) \\ (4 \cdot 5b) \\ (4 \cdot 5c) \end{array}$$

to the basic quantities (4.4).

We have here picked up the forms, (4.5a, b, c, ...), from (2.19), on the basis of following reasons :

a) $(\vec{\partial}_i \sigma_\mu)$ and its conjugate $(\sigma_\mu \vec{\partial}_i)$ are connected by

$$(\vec{\partial}_i \sigma_\mu) + (\sigma_\mu \vec{\partial}_i) = \partial_i (\sigma_\mu), \quad (4.9)$$

therefore it is sufficient to consider $\vec{\partial}_i (\sigma_\mu)$ only.

b) As for $(\vec{\partial}_i \vec{\partial}_k)$, we have an obvious relation

$$(\vec{\partial}_i \vec{\partial}_k) - \text{c.c.} = (i/2) \{ \partial_i \partial_k (I) - [i, k] \}, \quad (4.7)^*$$

so (4.5b) alone comes into question.

c) $(\vec{\partial}_i \vec{\partial}_k)$ or $(\vec{\partial}_i \vec{\partial}_k)$ is related to $(\vec{\partial}_i \vec{\partial}_k)$ through

$$\{ (\vec{\partial}_i \vec{\partial}_k) + (\vec{\partial}_i \vec{\partial}_k) \} + \text{c.c.} = \partial_i \partial_k (I), \quad (4.8)$$

and

$$\{ (\vec{\partial}_i \vec{\partial}_k) + (\vec{\partial}_i \vec{\partial}_k) \} - \text{c.c.} = i \partial_i \partial_k (I); \quad (4.9)$$

hence it is not necessary to deal with the former quantities separately besides $(\vec{\partial}_i \vec{\partial}_k)$.

ii) The next step is to study the identities existing between the elements of Pauli matrices σ_i 's. First of all, we have the well-known basic identities :

$$\sigma_{\alpha\beta}^i \sigma_{\alpha'\beta'}^i = 2 \delta_{\alpha\beta'} \delta_{\alpha'\beta} - \delta_{\alpha\beta} \delta_{\alpha'\beta'}. \quad (A)^{**}$$

Multiplying both sides of (A) by $\sigma_{\beta'\beta''}^k$ and summing over β' , we can obtain another important formula :

$$\sigma_{\alpha\beta}^{[i} \sigma_{\alpha'\beta'}^{j]} = i \sigma_{\alpha\beta'}^{[k} \sigma_{\alpha'\beta}^{n]}, \quad (k=i \times j). \quad (B)^{***}$$

* $[i, k]$ stands for the expression obtained from its preceding term by the suffix interchange between i and k .

** $\sigma_{\alpha\beta}^i \equiv (\sigma_i)_{\alpha\beta}$. Greek suffix runs 1 and 2. The underlines below the superscripts i on the left side of (A) indicate that the summation $\sum_{i=1,2,3}$ should be taken, as before.

*** $\sigma_{[\alpha\beta\sigma\beta']}^{[i} \sigma_{\alpha'\beta'}^{j]} \equiv \sigma_{\alpha\beta\sigma\beta'}^{[i} \sigma_{\alpha'\beta'}^{j]} - [i, j]$; and

$\sigma_{[\alpha\beta\sigma\beta']}^{[k} \sigma_{\alpha'\beta'}^{n]} \equiv \sigma_{\alpha\beta\sigma\beta'}^{[k} \sigma_{\alpha'\beta'}^{n]} - \delta_{\alpha\beta\sigma\beta'}^{[k} \sigma_{\alpha'\beta'}^{n]},$ since $\sigma^0 \equiv I$.

Further, making use of (B), we can get a formula which refers to a determinant of the third rank :

$$\varepsilon_{lmn} \sigma_{\alpha\beta}^l \sigma_{\alpha'\beta'}^m \sigma_{\alpha''\beta''}^n = -2i (\delta_{\alpha\beta'} \delta_{\alpha'\beta''} \delta_{\alpha''\beta} - \delta_{\alpha\beta''} \delta_{\alpha'\beta} \delta_{\alpha''\beta'}) . \quad (C)$$

iii) By the aid of the formulae of ii), we can now derive the required identities.

First, the former identity (I) is derived, of course, by an application of (A). We also note here

$$\sum_k \partial_k \partial_k = 0, \quad (I')$$

which is a result of (I).

If we transform the expression $\partial_i(\sigma_l) \partial_k(\sigma_l)$ again by means of (A), we find the identity,

$$\partial_i(\sigma_l) \partial_k(\sigma_l) = 2(I) \{(\vec{\partial}_i \vec{\partial}_k)\} - \delta_i(I) \delta_k(I), \quad (4 \cdot 10)$$

which supplies the reduction formula for (4.5b) :

$$\{(\vec{\partial}_i \vec{\partial}_k)\} = -\frac{1}{2P} \{\partial_i P \partial_k P + \partial_i(I) \delta_k(I)\} + \frac{P}{2} \partial_i \sum_l \partial_k \sum_l . \quad (4 \cdot 10')$$

Next, we transform the expression, $(\sigma_i) \partial_i(\sigma_j) - [i, j]$, with the aid of (B) ; then we obtain another identity :

$$(\sigma_i) \partial_i(\sigma_j) - [i, j] = (\sigma_k) \delta_i(I) - (I) \delta_i(\sigma_k), \quad (4 \cdot 11)$$

$$(k = i \times j)$$

which works as the reduction formula for $\delta_i(\sigma_k)$.

Finally, we apply the formula (C) to transform the determinant,

$$\bar{J}_{ij} = \varepsilon_{lmn} (\sigma_l) \partial_i(\sigma_m) \partial_j(\sigma_n). \quad (4 \cdot 12)$$

Then we are led to

$$\bar{J}_{ij} = -2i [(I)^2 \{(\vec{\partial}_i \vec{\partial}_j) - [i, j]\} - (I) \{(\vec{\partial}_i) (\vec{\partial}_j) - [i, j]\}]. \quad (4 \cdot 13)$$

Its right side is next rearranged, by use of (4.7) and also of

$$(\vec{\partial}_i) (\vec{\partial}_j) - [i, j] = (i/2) \{\partial_i(I) \delta_j(I) - [i, j]\}. \quad (4 \cdot 14)$$

Eventually we get

$$\bar{J}_{ij} = (I)^3 \{\partial_i(\delta_j(I)/(I)) - [i, j]\}, \quad (4 \cdot 15)$$

or

$$A_{ij} \equiv \varepsilon_{lmn} \sum_l \partial_i \sum_m \partial_j \sum_n = (2/\hbar) \text{curl}_{i \times j} \mathbf{P}. \quad (4.15')$$

iv) We shall now apply the substitution :

$$\left. \begin{aligned} \vec{\partial}_i &\rightarrow D_i \equiv \vec{\partial}_i - i(e/\hbar c) A_i, \\ \vec{\partial}_i^* &\rightarrow D_i^* \equiv \vec{\partial}_i + i(e/\hbar c) A_i, \end{aligned} \right\} \quad (4.16)$$

to the foregoing relations. With this substitution, $\partial_i(\sigma_\mu)$ goes over to

$$\partial_i(\sigma_\mu) \rightarrow \delta_i(\sigma_\mu) - 2(e/\hbar c) A_i(\sigma_\mu), \quad (4.17)$$

in particular,

$$\delta_i(I) \rightarrow (2m/\hbar) j_i. \quad (4.18)$$

It is noticed that the left side expression of (4.8), and the right side ones of (4.10) and (4.11) are all invariant with respect to the substitution (4.16), (4.17). Accordingly, we obtain at once, corresponding to (4.8), (4.10'), and (4.11), the gauge-invariant formulae :

$$(\psi^* D_i D_i \psi + D_i^* \psi^* D_i \psi) + \text{c.c.} = \partial_i \partial_i P, \quad (4.19)$$

$$D_i^* \psi^* D_i \psi + \text{c.c.} = \frac{1}{2P} \partial_i P \partial_i P + \frac{2m^2}{\hbar^2} P v_i v_i + \frac{P}{2} \partial_i \sum_l \cdot \partial_l \sum_l, \quad (4.20)$$

$$\frac{1}{i} (\psi^* \sigma_k D_l \psi - \text{c.c.}) = \frac{2m}{\hbar} (\sigma_k) v_l - \frac{1}{P} \{ (\sigma_i) \partial_i (\sigma_j) - [i, j] \}. \quad (4.21)$$

($k=i \times j$)

On the other hand the right side of (4.15) is not invariant with respect to the substitution (4.17) ; the explicitly gauge-invariant re-expression of (4.15') is

$$(\hbar/2) A_{ij} = m \text{curl}_k \mathbf{v} + (e/c) H_k, \quad (k=i \times j) \quad (4.22)$$

which is nothing but the second subsidiary condition (II).

v) Finally, we will mention a few relations upon the determinant quantity A_{ij} , i.e.,

$$A_{ij} = \begin{vmatrix} \sum_1 & \sum_2 & \sum_3 \\ \partial_i \sum_1 & \partial_i \sum_2 & \partial_i \sum_3 \\ \partial_j \sum_1 & \partial_j \sum_2 & \partial_j \sum_3 \end{vmatrix}.$$

We multiply each column of the determinant by \sum_1 , \sum_2 , and \sum_3 , respectively, and sum up. Then, noting (I) and (I'), we obtain

$$A_{ij} = \frac{1}{\sum_3} \begin{vmatrix} \partial_i \sum_1 & \partial_i \sum_2 \\ \partial_j \sum_1 & \partial_j \sum_2 \end{vmatrix}, \quad (4.23)$$

which is the form adopted in (II') [§ 2].

The relation :

$$\varepsilon_{lmn} \partial_i \sum_l \partial_j \sum_m \partial_k \sum_n = 0, \quad (4 \cdot 24)$$

is confirmed also by a similar procedure. Noting this, we can easily show

$$\sum_{\text{cyclic}} \partial_k A_{ij} = 0. \quad (k=i \times j) \quad (4 \cdot 25)$$

Appendix

A. Ensemble picture

Our formalism may be interpreted in terms of an ensemble of particle motions instead of the hydrodynamical picture, which circumstance is similar as in the case of spinless particle^{1),9)}. In the present case, however, we consider an "ensemble" of innumerable motions of a classical spinning particle which in effect interact through total quantum potential II and internal magnetic field II^{in} . Such an ensemble cannot of course correspond to a classical-statistical interpretation of quantum mechanics, at least as it is.

Moreover, the ensemble is limited to a particular one such as to consist of particle motions of number ∞^3 to satisfy the subsidiary condition (II)*.

If we have a solution (P, S, \mathbf{v}) for our basic equations, individual particle motions constituting the ensemble specified by that solution may be obtained as follows: First integrating the simultaneous differential equations

$$d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}, t), \quad (\text{A} \cdot 1)$$

we find particle orbital trajectories of number ∞^3 , $\mathbf{x}(\mathbf{x}_0, t)$, (where we have chosen the integration constant to be particle initial position \mathbf{x}_0). Next, by substituting this into the spin field $S(\mathbf{x}, t)$, we get the time change of the spin direction of individual particle, as $S_t = S(\mathbf{x}(\mathbf{x}_0, t), t)$. The initial direction of spin is correlated with the initial position by $S_0 = S(\mathbf{x}_0, 0)$.

B. Expectation values and probability distributions

Our picture reproduces correctly the quantum-mechanical expectation values of most of the significant dynamical quantities, as the mean values over the ensemble,—or as the total values taken by the hydrodynamical field.

a) Especially, our picture reflected the quantum-mechanical probability distribution of particle position precisely by the density $P(\mathbf{x})$. Hence it also does so for any quantities which are functions of the position only.

b) The picture also reflected the expectation value of spin component correctly (see § 2). In this picture, however, spin component is distributed in space taking continuous

* Previously¹⁾ we have called an ensemble "elementary", that obeys such a sort of restriction.

values between $-\hbar/2$ and $\hbar/2$, in disagreement with the quantum-mechanical probability distribution of particle spin. Hence, e.g., we would be given the mean value of the magnitude of spin to be $\int S(x)^2 P(x) dx = \hbar^2/4$, differing from $\langle s_z^2 \rangle_{qu} = (3/4)\hbar^2$.

c) As for a momentum component, the expectation value was correctly given by our picture [§ 3-iii)], such that

$$\langle \mathbf{p} \rangle_{qu} = \int \psi^* (\hbar/i) \nabla \psi dx = \int P(x) \mathbf{p}(x) dx. \quad (B.1)$$

Hence, noting a), $\langle \mathbf{v} \rangle_{qu}$ is also reproduced:

$$\langle \mathbf{v} \rangle_{qu} = \int \mathbf{v}(x) P(x) dx. \quad (B.2)$$

Our picture, however, does not reflect the probability distribution of p_i faithfully, so it cannot reproduce, e.g., the expectation value of p_i^2 . This is a consequence due to that our formulation does not stand on equal footing with respect to particle position and momentum, but it starts from the coordinate representation¹⁾ of quantum mechanics of spinning particle.

d) The picture, nevertheless, reproduced the quantum-mechanical expectation value of particle energy. It was connected with the fact that the energy distribution \mathcal{H} in our picture takes account of the contribution from the potential energy due to self-stress, which is, in fact, a part of kinetic energy in the original quantum mechanics [§ 3-iii)]: First, for kinetic energy, we have

$$\begin{aligned} \langle H_{kin} \rangle_{qu} &= \int \psi^* (1/2m) (\mathbf{p} - e\mathbf{A}/c)^2 \psi dx \\ &= -(\hbar^2/2m) \int \psi^* \mathbf{D}^2 \psi dx = (\hbar^2/2m) \int \mathbf{D}^* \psi^* \mathbf{D} \psi dx. \end{aligned} \quad (B.3)$$

Hence for total energy, we obtain

$$\langle H \rangle_{qu} = \langle H_{kin} \rangle_{qu} + \int \{eA_0 - (e/mc) \mathbf{H} \mathbf{S}\} P dx. \quad (B.4)$$

Comparing (B.3), (B.4) with (3.20), we find at once

$$\langle H \rangle_{qu} = \int \mathcal{H} dx. \quad (B.5)$$

* We denote the expectation value of a quantum-mechanical quantity \underline{F} by $\langle \underline{F} \rangle_{qu}$.

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On the Recoil Correction to Adiabatic Nuclear Potential

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Recoil correction to adiabatic nuclear potential in the symmetrical pseudoscalar meson theory with pseudovector coupling is calculated up to the first order in the nucleon velocities by means of the F-S-T formalism¹⁾ which is formulated by taking into account the normalization condition for the Tamm-Dancoff amplitude. Divergences which appear in the course of the calculations are subtracted by a sort of non-covariant renormalization procedure. Resultant correction consists of two parts, spin-orbit coupling part and velocity-independent part. The former is found to be of the correct sign and suitable magnitude as required by the theory of nuclear shell model.

§ 1. Introduction

It is well known that the theory of nuclear shell structure requires the spin-orbit interactions in the shell model potential.²⁾ Though the two-body tensor force yields such an interaction in the quasi-atomic model, the resultant doublet splitting of the nuclear fine structure is not only too small in magnitude but also of regular order³⁾, in contrast with the experimental evidences (which indicate the inverted doublet). In this connection, it must also be noted that the well-known Thomas spin-orbit term, which arises from the relativistic correction to the static interaction Hamiltonian and is of the order $(1/M^2)$, yields the doublet inversion, but it is insufficient in its order of magnitude³⁾. It seems therefore to be of some interest to examine the direct two-body spin-orbit coupling afforded by the meson theory, although it may be possible that the spin-orbit interaction in the shell model is derived from the velocity-independent two-body forces by a future refinement of the treatment of the many-body problem. Recently, Araki* has pointed out the possibility of interpreting the observed doublet interval ${}^2D_{5/2} - {}^2D_{3/2}$ of the O^{17} nucleus as due to the spin-orbit coupling derived meson-theoretically. However, the meson potential adopted by him was incomplete for the required order of approximation.

In the meson theory, the velocity-dependent forces can be derived only by taking into account the effect of the nucleon recoil accompanying emission and absorption of the virtual mesons. With the usual canonical transformation method, however, the recoil corrections to the nuclear forces cannot be obtained without ambiguity, since the transformation

*) G. Araki, Prog. Theor. Phys. **13** (1955), 13. One of the present authors (I. S.) is indebted to Professor G. Araki for sending his manuscript before publication.

operator is not uniquely determined by the condition alone that the transformed Hamiltonian should be diagonal with respect to the meson number. On the other hand, the "potential operator" derived along the line of Tamm-Dancoff approximation is not adequate in the strict meaning of the word owing to its non-hermiticity (especially in the recoil correction part) and the fact that the T-D amplitude is not normalized to unity.¹⁾ For our purpose, therefore, it is necessary to fix a certain standard method.

Recently, Fukuda, Sawada and Taketani (F-S-T) have given a general method of constructing the potential in the quantum field theory.¹⁾ According to their formalism, the potential operator can be analyzed into its normal part and the probability operator. In the meson theory, the probability operator is related to the probability of the nucleons dissociating into the bare ones and the meson cloud.* F-S-T have also shown that the nuclear potentials previously calculated by several authors²⁾ can be obtained from the F-S-T potential by various approximate replacements of the probability operator. In view of this point, the F-S-T method seems most promising at the present stage.

For these reasons, we shall here investigate the effect of the nucleon recoil on the two-nucleon interaction along the F-S-T formalism**, and especially, emphasis will be laid upon the derivation of the spin-orbit coupling. The symmetrical pseudoscalar meson theory with pseudovector coupling [$PS(PV)$] will be adopted, and actual calculations will be carried out to the first order in the nucleon velocities.

§ 2. Derivation and renormalization of the potential formula

Eliminating the components of the state vector other than the zero-meson component Ψ_0 from the ordinary Schrödinger equation for the system of two nucleons and meson field, we obtain

$$(E - K) \Psi_0 = \langle HJ(E) \rangle \Psi_0, \quad (2.1)$$

with

$$J(E) = [1 - (E - K - H_0)^{-1} (1 - v) H]^{-1}, \quad (2.2)$$

where E , K , H , and H_0 are the energy eigenvalue of the system, the kinetic energy operator of two nucleons, the interaction Hamiltonian between the nucleons and the meson field, and the free Hamiltonian of the meson field, respectively. v is the projection operator to the zero-meson state, and accordingly, Ψ_0 is connected to the original state vector Ψ of the system by $\Psi_0 = v\Psi$. In eq. (2.1), we have introduced an abbreviation $\langle A \rangle$ for vAv , A being an arbitrary operator.

Owing to the relation¹⁾

¹⁾ Two of the present authors have also found the importance of the probability operator independently of F-S-T, and have investigated the electromagnetic properties of deuteron taking this into consideration. Cf. I. Sato and K. Itabashi, *Prog. Theor. Phys.* **13** (1955), 341.

²⁾ The static part of the F-S-T potential has already been calculated by Sawada. Cf. K. Sawada, *Soryushiron-Kenkyu* (Mimeographed Circular in Japanese), **7** (1954), 115.

$$\Psi = J(E) \Psi_0,$$

we have

$$(\Psi_0, \langle J(E) * J(E) \rangle \Psi_0) = 1. \quad (2.3)$$

For the calculations, it is necessary to expand the right-hand side of (2.1) in powers of the coupling constant f . For this purpose, we expand $\langle HJ(E) \rangle$ up to the sixth order in H , assuming the latter to be linear in the meson field variables. That, for our purpose, such an assumption is permissible in the case of $PS(PV)$ theory will be shown in the next section. Correspondingly, we expand $\langle J(E) * J(E) \rangle$ in (2.3) up to the fourth order in H , taking into account the fact that the first term in the expansion of $\langle J(E) * J(E) \rangle$ is of the zeroth order, while that of $\langle HJ(E) \rangle$ is of the second order.

Then, each term in these expansions contains the factor $(E - K - H_0)^{-1}$. In the static approximation, this factor is replaced by $(-H_0)^{-1}$, or otherwise, K is put to zero. However, this is not a good approximation since $(E - K)/H_0$ is not less than f^2 in order of magnitude, as is seen from (2.1). [Since K in $(E - K - H_0)^{-1}$ is the kinetic energy of the nucleons in an intermediate state, $(E - K)/H_0$ is of the same order as or much larger than f^2 , according to whether the changes in momenta of the nucleons due to the emission or absorption of virtual mesons are small or large.] Therefore, we further expand the second order, the fourth order, and the sixth order terms in the above expansion of $\langle HJ(E) \rangle$ up to the second order, the first order, and the zeroth order in $(E - K)/H_0$, respectively. Similarly, we expand the second order and the fourth order terms in the previous expansion of $\langle J(E) * J(E) \rangle$ up to the first order and the zeroth order in $(E - K)/H_0$, respectively.

In order that the above procedure may give a good approximation to the potential for the internucleon distances of the order of the meson Compton wavelength, the changes in energies of the nucleons must be at most of the order of $(\mu c^2 f^2/4\pi)$ for the emission or the absorption of virtual mesons with the momenta of the order of μc . Since $f^2/4\pi \sim \mu/M$, this requires that the nucleon momenta must be at most of the order of μc at those internucleon distances, and therefore that $E \lesssim \mu^2 c^2/M$. In what follows, we assume that this condition is satisfied, and thus regard $(E - K)/H_0$ of the same order as $f^2/4\pi$ or μ/M .

Now, the equations which are deduced from (2.1) and (2.3) by the above expansions contain explicitly the energy eigenvalue E . However, since Ψ_0 obeys (2.1), we can eliminate E from those equations by the repeated uses of (2.1). We write the equations thus obtained as

$$(E - K) \Psi_0 = U \Psi_0, \quad (2.4)$$

and

$$(\Psi_0, P \Psi_0) = 1. \quad (2.5)$$

U and P can be written as

$$U = U^{(2,0)} + U^{(4,0)} + U^{(2,1)} + U^{(6,0)} + U^{(4,1)} + U^{(2,2)}, \quad (2.6)$$

and

$$P = 1 + P^{(2,0)} + P^{(4,0)} + P^{(2,1)}, \quad (2.7)$$

where the superscript (m, n) means that the term with it is of the m -th order in H and of the n -th order in K/H_0 . [As is previously mentioned, we have assumed that H is linear in the meson field variables.] Both P and U no longer contain the energy eigenvalue E . However, \mathcal{V}_0 is not normalized to unity [eq. (2.5)], and moreover U is not hermitean. Therefore, we cannot regard U as the potential operator.

We now introduce the normalized wave function χ by

$$\chi = P^{1/2} \mathcal{V}_0, \quad (2.8)$$

then, we obtain from (2.4) as the equation for χ

$$(E - K)\chi = W\chi, \quad (2.9)$$

with

$$W = P^{-1/2} V P^{-1/2} - K, \quad (2.10)$$

and

$$V = P(K + U). \quad (2.11)$$

We expand also V in powers of H and K/H_0 , and approximate it by*

$$V = K + V^{(2,0)} + V^{(4,0)} + V^{(2,1)} + V^{(6,0)} + V^{(4,1)} + V^{(2,2)}, \quad (2.12)$$

with the same meaning of the notation (m, n) as above. Then, V is found to be hermitian, and this is also the case for W .** We therefore adopt W as the potential operator.

Now, P and V contain the divergences of the nucleon self-interaction type. To remove these divergences, we shall perform a sort of non-relativistic renormalization. First, we replace E and H in (2.1) by

$$\epsilon = E - \delta E \quad \text{and} \quad H' = H - \delta E$$

respectively. δE denotes sum of the self-energies of the nucleons, and is determined by the equation obtained from (2.9) by expanding W in powers of the coupling constant and K/H_0 .

Then, by proceeding in the same way as above, we are led to the equation (2.9) with ϵ in place of E . However, P and V are now lacking in the strongest divergences that were previously contained in them. We shall show that the remaining divergences in P and V can also be removed by the renormalization of the coupling constant. For this purpose, we write $P^{(m,n)}$ and $V^{(m,n)}$ in (2.7) and (2.12) as

$$P^{(m,n)} = P_c^{(m,n)} + P_\infty^{(m,n)}$$

and

$$V^{(m,n)} = V_c^{(m,n)} + V_\infty^{(m,n)},$$

where $P_c^{(m,n)}$ and $V_c^{(m,n)}$ are the convergent parts, and $P_\infty^{(m,n)}$ and $V_\infty^{(m,n)}$ the divergent parts.

[†]) Apart from the reason which is explained previously with respect to the expansions of $\langle HJ(E) \rangle$ and $\langle J(E)^\dagger J(E) \rangle$, also by an analogy with the case of the neutral scalar meson theory, it seems to be reasonable that V is calculated up to the order higher by 2 than P . See K. Sawada, loc. cit.

^{††}) Cf. ref. 1).

Then, it can be proved by the straightforward calculation that

$$V = K + (1 + 2\lambda) (V_c^{(2,0)} + V_c^{(2,1)}) + V_c^{(4,0)} + V^{(6,0)} + V_c^{(4,1)} + V_\infty'^{(4,1)} + V_c^{(2,2)} \\ + (1/2) \{K, (P_\infty^{(2,1)} + P_\infty^{(4,0)} - 2\lambda P_c^{(2,0)} + P_\infty^{(2,1)})\}, \quad (2.13)$$

where

$$V_\infty'^{(4,1)} = -\langle H_1 \frac{1}{H_0} H_2^{(3,1)} \rangle - \langle H_2^{(3,1)} \frac{1}{H_0} H_1 \rangle \\ - \langle H_2 \frac{1}{H_0} H_1^{(3,1)} \rangle - \langle H_1^{(3,1)} \frac{1}{H_0} H_2 \rangle,$$

λ is the divergent constant defined by

$$\lambda H_i = \sum_k \omega_k^{-2} H_{ik} H_i H_{ik}^*$$

and $H_i^{(3,1)}$ is given by

$$H_i^{(3,1)} = \sum_k \omega_k^{-2} \{[K_i, H_{ik}] H_i H_{ik}^* - H_{ik} H_i [K_i, H_{ik}^*]\},$$

where H_i and K_i are respectively the interaction Hamiltonian and the kinetic energy operator of the i -th nucleon, and H_{ik} is defined by

$$H_i = \sum_k (H_{ik} a_k + H_{ik}^* a_k^*),$$

a_k and a_k^* are the annihilation and the creation operators for the meson k , whose energy we have denoted by ω_k . The index k specifies the momentum and the charge of the meson.

On the other hand, (2.7) can be written as

$$P = 1 + (1 + 2\lambda) P_c^{(2,0)} + P_c^{(4,0)} + P_c^{(2,1)} + P_\infty^{(2,0)} \\ + P_\infty^{(4,0)} - 2\lambda P_c^{(2,0)} + P_\infty^{(2,1)}. \quad (2.14)$$

Comparing (2.13) with (2.14), we introduce P' and V' defined by

$$P = \mathfrak{w} P' \mathfrak{w} \text{ and } V = \mathfrak{w} V' \mathfrak{w}, \quad (2.15)$$

with

$$\mathfrak{w} = 1 + (1/2) (P_\infty^{(2,0)} + P_\infty^{(4,0)} - 2\lambda P_c^{(2,0)} + P_\infty^{(2,1)}) - (1/8) (P_\infty^{(2,0)})^2. \quad (2.16)$$

Then, it follows from (2.13) ~ (2.16) that, to the present order of approximation,

$$P' = 1 + (1 + 2\lambda - P_\infty^{(2,0)}) P_c^{(2,0)} + P_c^{(4,0)} + P_c^{(2,1)}, \quad (2.17)$$

and

$$V' = K + (1 + 2\lambda - P_\infty^{(2,0)}) (V_c^{(2,0)} + V_c^{(2,1)}) + (1 + 4\lambda - 2P_\infty^{(2,0)}) V_c^{(4,0)} + \bar{V}^{(6,0)} + V_c^{(4,1)} \\ + \bar{V}_\infty^{(4,1)} + V_c^{(2,2)}, \quad (2.18)$$

where

$$\bar{V}^{(6,0)} = V^{(6,0)} + (P_\infty^{(2,0)} - 4\lambda) V_c^{(4,0)} - [P_\infty^{(4,0)} - (P_\infty^{(2,0)})^2 \\ - \lambda (P_c^{(2,0)} - 2P_\infty^{(2,0)})] V_c^{(2,0)}$$

and

$$\bar{V}_{\infty}^{(4,1)} = V'_{\infty}{}^{(4,1)} - (1/2) \{P_{\infty}^{(2,1)}, V_c^{(2,0)}\}.$$

It is now obvious that, to the present order of approximation, (2.17) and (2.18) are transformed by the renormalization of the coupling constant into

$$P' = 1 + P_c^{(2,0)} + P_c^{(4,0)} + P_c^{(2,1)} \quad (2.19)$$

and

$$V' = K + V_c^{(2,0)} + V_c^{(4,0)} + V_c^{(2,1)} + \bar{V}^{(6,0)} + V_c^{(4,1)} + \bar{V}_{\infty}^{(4,1)} + V_c^{(2,2)}. \quad (2.20)$$

Now, we introduce the new wave function φ by

$$\varphi = P'^{1/2} W F_0. \quad (2.21)$$

Then, it follows from (2.8) ~ (2.10) that

$$(\epsilon - K)\varphi = W'\varphi, \quad (2.22)$$

where

$$W' = P'^{-1/2} V' P'^{-1/2} - K. \quad (2.23)$$

It is easily seen from (2.5), (2.15) and (2.21) that

$$(\varphi, \varphi) = 1. \quad (2.24)$$

(2.23) is our working formula for the potential operator.

Our purpose is to calculate the recoil part of W' , the exact definition of which is W' minus what is obtained from W' by putting all nucleon velocities equal to zero. However, we confine ourselves to the calculation of that part of W' which is of the first order in the nucleon velocities. Thus, we omit $V_c^{(2,2)}$ in (2.20), since this is of the second order in the nucleon velocities. In the next section, it will be shown that P' does not depend on the nucleon velocities, and therefore $V^{(6,1)}$ does not contribute to the recoil part. In what follows, we further omit $V_{\infty}^{(4,1)}$, since it can easily be verified that its velocity-dependent part vanishes for the symmetrical $PS(PV)$ theory which will be adopted in the following. The desired recoil part of W' is thus given by

$$W'_R = P'^{-1/2} V'_R P'^{-1/2} - K, \quad (2.25)$$

where

$$V'_R = K + \text{the part of } (V_c^{(2,0)} + V_c^{(4,0)} + V_c^{(2,1)} + V_c^{(4,1)}) \text{ which} \\ \text{is of the first order in the nucleon velocities.} \quad (2.26)$$

We shall here give the necessary expressions:

$$P_c^{(2,0)} = \langle H_1 \frac{1}{H_0^2} H_2 \rangle + \langle H_2 \frac{1}{H_0^2} H_1 \rangle, \quad (2.27)$$

$$P_c^{(4,0)} = \text{the convergent part of} \left[\langle H \frac{1}{H_0^2} H \frac{1-v}{H_0} H \frac{1}{H_0} H \rangle \right. \\ \left. + \langle H \frac{1}{H_0} H \frac{1-v}{H_0^2} H \frac{1}{H_0} H \rangle + \langle H \frac{1}{H_0} H \frac{1-v}{H_0} H \frac{1}{H_0^2} H \rangle \right]$$

$$-\left\{ \left\langle H \frac{1}{H_0} H \right\rangle, \left\langle H \frac{1}{H_0^2} H \right\rangle \right\}, \quad (2.28)$$

$$V_c^{(2,1)} = \text{the convergent part of} \left[\left\{ K, \left\langle H \frac{1}{H_0^3} H \right\rangle \right\} - 2 \left\langle H \frac{K}{H_0^3} H \right\rangle \right], \quad (2.29)$$

$$V_c^{(2,0)} = - \left\langle H_1 \frac{1}{H_0} H_2 \right\rangle - \left\langle H_2 \frac{1}{H_0} H_1 \right\rangle, \quad (2.30)$$

$$V_c^{(4,0)} = \text{the convergent part of} \left[- \left\langle H \frac{1}{H_0} H \frac{1-v}{H_0} H \frac{1}{H_0} H \right\rangle \right], \quad (2.31)$$

$$V_c^{(2,1)} = \left\langle H_1 \frac{K}{H_0^2} H_2 \right\rangle + \left\langle H_2 \frac{K}{H_0^2} H_1 \right\rangle, \quad (2.32)$$

$$\begin{aligned} V_c^{(4,1)} = \text{the convergent part of} & \left[\left\langle H \frac{K}{H_0^2} H \frac{1-v}{H_0} H \frac{1}{H_0} H \right\rangle \right. \\ & + \left\langle H \frac{1}{H_0} H \frac{K(1-v)}{H_0^2} H \frac{1}{H_0} H \right\rangle + \left\langle H \frac{1}{H_0} H \frac{1-v}{H_0} H \frac{K}{H_0^2} H \right\rangle \\ & \left. - \left\langle H \frac{1}{H_0} H \right\rangle \left\langle H \frac{1}{H_0^3} H \right\rangle K - K \left\langle H \frac{1}{H_0^3} H \right\rangle \left\langle H \frac{1}{H_0} H \right\rangle \right]. \end{aligned} \quad (2.33)$$

§ 3. Calculations and final results

In the following, we shall work with the symmetrical $PS(PV)$ meson theory. Then, the interaction Hamiltonian is given by

$$H = - (f/\mu) \sum_{i=1,2} \tau_{\alpha i} [\sigma_i \nabla \phi_\alpha(\mathbf{r}_i) + \rho_{1i} \pi_\alpha(\mathbf{r}_i)]. \quad (3.1)$$

However, it is convenient to eliminate the odd operator ρ_{1i} by a unitary transformation. Then, to the first order in M^{-1} , we have

$$H = H_\sigma + H_p + H_\pi, \quad (3.2)$$

where

$$H_\sigma = - (f/\mu) \sum_i \tau_{\alpha i} \sigma_i \nabla \phi_\alpha(\mathbf{r}_i), \quad (3.3)$$

$$H_p = - (1/2M) (f/\mu) \sum_i [(\mathbf{F}_i \cdot \sigma_i) \pi_\alpha(\mathbf{r}_i) + \pi_\alpha(\mathbf{r}_i) (\mathbf{p}_i \cdot \sigma_i)] \tau_{\alpha i}, \quad (3.4)$$

and

$$H_\pi = (1/2M) (f/\mu)^2 \sum_i \pi_\alpha(\mathbf{r}_i)^2. \quad (3.5)$$

As a consequence of this transformation, we can take the nonrelativistic expression for K :

$$K = (1/2M) (\mathbf{p}_1^2 + \mathbf{p}_2^2). \quad (3.6)$$

It should be noted that H_σ itself is not invariant under the Galilei transformation. In fact, with H_σ alone, we are led to the potential involving the center-of-mass momentum. As one of the present authors (S.S) has already pointed out⁶⁾, H is made Galilei invariant

by including H_p , and the dependence of the potential on the center-of-mass momentum is cancelled out by the contribution of H_p .

Now, the term H_π which is quadratic in the meson field variables contributes only to the terms in P' and V' of the fourth or higher order in f . In fact, it is easily found that the largest contribution of H_π to V' is of the order of $f^4 \mu^2/M$ for the internucleon distances of the order of μ^{-1} , while the fourth order contribution of H_σ is of the order of $f^4 \mu$. Therefore, the contribution of H_π is of the same order as $V_c^{(4,1)}$ or $V_c^{(6,0)}$ in its magnitude. However, the former can easily be found to be velocity-independent, and moreover, a simple calculation shows that its convergent part actually vanishes. Its remaining diverging part does not cause any essential modification in our renormalization procedure, and our working formulas (2.23) ~ (2.33) for the potential operator remain unaltered to the desired order of approximation. [The largest contribution of H_π to P' is found to be of the order of $f^4 \mu/M$, and accordingly, can be neglected in our approximation.] Thus, we can omit H_π entirely and take

$$H = H_\sigma + H_p \quad (3.7)$$

instead of (3.2). This confirms the validity of our assumption that H is considered to be linear in the meson field variables, of which we have made full use in the preceding section.

Since, H_p is of the first order in the nucleon velocities, it is sufficient for the calculation of W'_R to replace H by H_σ in $V_c^{(2,1)}$, $V_c^{(4,1)}$ and $P_c^{(2,1)}$. Then we have

$$P_c^{(2,1)} = 0,$$

owing to the relation

$$\langle H_{\sigma 1} \frac{1}{H_0^3} H_{\sigma 2} \rangle = \langle H_{\sigma 2} \frac{1}{H_0^3} H_{\sigma 1} \rangle.$$

As was explained in § 2, we have assumed that the nucleon velocities are of the order of μ/M or $f^2/4\pi$ at the internucleon distances of the order of μ^{-1} . Under this assumption, the contribution of H_p to $P_c^{(4,0)}$ is of the order of $(f^2/4\pi)^3$, and therefore can be neglected in our approximation. Thus, the velocity-dependent part of P' is only that of $P_c^{(2,0)}$. According to (2.27), the latter is

$$\left[\langle H_{\sigma 1} \frac{1}{H_0^2} H_{\rho 2} \rangle + \langle H_{\rho 1} \frac{1}{H_0^2} H_{\sigma 2} \rangle \right] + [1 \leftrightarrow 2], \quad (3.8)$$

to the first order in the nucleon velocities. However, a simple calculation shows that (3.8) vanishes. Thus, P' becomes entirely velocity-independent to the present order of approximation. [i.e., it agrees with that in the case of static approximation.] Therefore, $V_c^{(2,0)}$ and $V_c^{(4,0)}$ contribute to W'_R only through the parts of (2.30) and (2.31) which are linear in H_p .

After these preliminaries, the calculations of P' and V'_R are straightforward, and the results are as follows:

$$P' = A(x) + B(x) \sum_{12} \quad (3.9)$$

and

$$V'_R = (1/2) \{K, P'\} + (\mu^2/M) [C(x) + D(x) (\sigma_1 \cdot \sigma_2) + E(x) \sum_{12} + (1/x^2) L(x) (S \cdot L)], \quad (3 \cdot 10)$$

where

$$\sum_{12} = \left(\sigma_1 \cdot \frac{\mathbf{r}}{r} \right) \left(\sigma_2 \cdot \frac{\mathbf{r}}{r} \right), \quad S = (1/2) (\sigma_1 + \sigma_2),$$

$$L = (1/2) [\mathbf{r} \times (\mathbf{p}_1 - \mathbf{p}_2)] = -i[\mathbf{r} \times \nabla_{\mathbf{r}}],$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \text{and } x = \mu r.$$

The functions $B(x), \dots, L(x)$ involve the operator $(\tau_1 \cdot \tau_2)$, and $A(x)$ contains not only $(\tau_1 \cdot \tau_2)$, but also $(\sigma_1 \cdot \sigma_2)$. The expressions of these functions will be given later.

Finally, we have to calculate W'_R , which involves the factor $P'^{-1/2}$. However, this factor does not cause any serious difficulty, since P' is velocity-independent. Indeed, in virtue of the fact that the operators $(\tau_1 \cdot \tau_2)$ and $(\sigma_1 \cdot \sigma_2)$ contained in $A(x)$ and $B(x)$ are commutable with \sum_{12} , we can use the formula

$$P'^{-1/2} = (1/2) \{ [(A+B)^{-1/2} + (A-B)^{-1/2}] + [(A+B)^{-1/2} - (A-B)^{-1/2}] \sum_{12} \}. \quad (3 \cdot 11)$$

With this formula, the calculation of W'_R is straightforward, and we finally obtain

$$W'_R = W_{SL} + W_0, \quad (3 \cdot 12)$$

where W_{SL} is the spin-orbit coupling part, and W_0 is the velocity-independent part. These are respectively given by

$$W_{SL} = (\mu^2/M) (1/x^2) [2 + (L-2A) (A^2 - B^2)^{-1/2}] (S \cdot L) \quad (3 \cdot 13)$$

and

$$\begin{aligned} W_0 = & -(\mu^2/M) \{ (1/4) (A^2 - B^2)^{-2} [(A'^2 + B'^2) (A^2 + B^2) - 4A'B'AB] \\ & - (1/x^2) [2 + (L-2A) (A^2 - B^2)^{-1/2}] \\ & - (A^2 - B^2)^{-1} [A(C - (L/x^2)) - B(D+E)] \} \\ & - (\mu^2/M) \{ (1/6) (A^2 - B^2)^{-2} [A'B' (A^2 + B^2) - (A'^2 + B'^2) AB] \\ & - (1/3x^2) [2 + (L-2A) (A^2 - B^2)^{-1/2}] \\ & - (1/3) (A^2 - B^2)^{-1} [A(E + 3D - (L/x^2)) + B(2D - C)] \} (\sigma_1 \cdot \sigma_2) \\ & - (\mu^2/M) \{ (1/6) (A^2 - B^2)^{-2} [A'B' (A^2 + B^2) - (A'^2 + B'^2) AB] \\ & + (1/6x^2) [2 + (L-2A) (A^2 - B^2)^{-1/2} - L(A-B)^{-1}] \\ & - (1/3) (A^2 - B^2)^{-1} [AE - B(C+D) - (L/x^2)] \} S_{12}, \end{aligned} \quad (3 \cdot 14)$$

where

$$A' = \frac{dA}{dx}, \quad B' = \frac{dB}{dx}$$

and

$$S_{12} = 3 \sum_{12} - (\sigma_1 \cdot \sigma_2).$$

The explicit forms of the functions A , B , etc., are as follows:

$$A(x) = 1 + F(x) + (\sigma_1 \cdot \sigma_2) \{ (f^2/4\pi) (\tau_1 \cdot \tau_2) k_1(x) + G(x) \}, \quad (3 \cdot 15)$$

$$B(x) = - (f^2/4\pi) (\tau_1 \cdot \tau_2) \{ k_0(x) + 2k_1(x) \} + H(x), \quad (3 \cdot 16)$$

$$C(x) = \frac{1}{4} (f^2/4\pi)^2 \left[2(3 + 2 \tau_1 \cdot \tau_2) \left(\frac{1}{x} + \frac{5}{x^2} + \frac{16}{x^3} + \frac{32}{x^4} + \frac{36}{x^5} + \frac{18}{x^6} \right) \exp[-2x] \right. \\ \left. + (3 - 2 \tau_1 \cdot \tau_2) \left\{ (x^2 + 4 + \frac{28}{x^2}) k_1^2 + 2(1 + \frac{14}{x^2}) k_0 k_1 + \frac{7}{x^2} k_0^2 \right\} \right], \quad (3 \cdot 17)$$

$$D(x) = \frac{1}{2} (f^2/4\pi)^2 \left[(3 - 2 \tau_1 \cdot \tau_2) \left\{ 2(1 + 4/x^2) k_1^2 + (1 + 8/x^2) k_0 k_1 + \frac{2}{x^2} k_0^2 \right\} \right. \\ \left. - (3 + 2 \tau_1 \cdot \tau_2) \left(\frac{2}{x^2} + \frac{9}{x^3} + \frac{20}{x^4} + \frac{24}{x^5} + \frac{12}{x^6} \right) \exp[-2x] \right], \quad (3 \cdot 18)$$

$$E(x) = \frac{1}{2} (f^2/4\pi)^2 \left[(3 + 2 \tau_1 \cdot \tau_2) \left(\frac{2}{x^2} + \frac{11}{x^3} + \frac{28}{x^4} + \frac{36}{x^5} + \frac{18}{x^6} \right) \exp[-2x] \right. \\ \left. - (3 - 2 \tau_1 \cdot \tau_2) \left\{ 2(1 + 6/x^2) k_1^2 + (1 + 12/x^2) k_0 k_1 + (3/x^2) k_0^2 \right\} \right], \quad (3 \cdot 19)$$

$$L(x) = (f^2/4\pi)^2 \left[(3 - 2 \tau_1 \cdot \tau_2) (k_0 + 2k_1)^2 - 2(3 + 2 \tau_1 \cdot \tau_2) \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3} + \frac{1}{x^4} \right) \right] \\ \times \exp[-2x], \quad (3 \cdot 20)$$

where

$$k_0(x) = (2/\pi) K_0(x) \quad \text{and} \quad k_1(x) = (2/\pi) (1/x) K_1(x) \quad (3 \cdot 21)$$

with K -functions defined by

$$K_\nu(\mu r) = \frac{\Gamma(\nu + 1/2) (2\mu)^\nu}{r^\nu \Gamma(1/2)} \int_0^\infty \frac{\cos kr \, dk}{(k^2 + \mu^2)^{\nu+1/2}}.$$

In eqs. (3·15) and (3·16), we have used the notations $F(x)$, $G(x)$ and $H(x)$ for the following functions:

$$F(x) = (f^2/4\pi)^2 \left[4 \tau_1 \cdot \tau_2 \left(\frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3} + \frac{2}{x^4} \right) \exp[-2x] \right. \\ \left. + (3/2) (1 - \tau_1 \cdot \tau_2) (k_0^2 + 2k_0 k_1 + 3k_1^2) \right], \quad (3 \cdot 22)$$

$$G(x) = (f^2/4\pi)^2 \left[-6 \left(\frac{2}{x^2} + \frac{3}{x^3} + \frac{2}{x^4} \right) \exp[-2x] + ((9/2) - 2 \tau_1 \cdot \tau_2) (k_0 k_1 + k_1^2) \right], \quad (3 \cdot 23)$$

$$H(x) = (f^2/4\pi)^2 \left[6 \left(\frac{2}{x^2} + \frac{5}{x^3} + \frac{4}{x^4} \right) \exp[-2x] - ((9/2) - 2\tau_1 \cdot \tau_2) (k_0 k_1 + 2k_1^2) \right]. \quad (3.24)$$

In Figs. 1, 2 and 3, we plot the coefficient of $(\mathbf{S} \cdot \mathbf{L})$ in W_{SL} and the velocity-independent correction W_0 as the functions of x with the value 0.086 for $(f^2/4\pi)$. For comparison, we show in Fig. 4 also the spin-orbit coupling part obtained by the approximation in which V' is expanded only up to the fourth order in the coupling constant (in our sense*) and P' only up to the second order. To this approximation, not only H_π , but also H_ρ do not give any contribution, and the recoil part of the potential takes the form

$$W_R^{(1)} = P^{(1)-1/2} (1/2) \{K, P^{(1)}\} P^{(1)-1/2}, \quad (3.25)$$

where

$$P^{(1)} = A^{(1)}(x) + B^{(1)}(x) \sum_{112}, \quad (3.26)$$

$$A^{(1)}(x) = 1 + (f^2/4\pi) (\tau_1 \cdot \tau_2) k_1(x) (\sigma_1 \cdot \sigma_2), \quad (3.27)$$

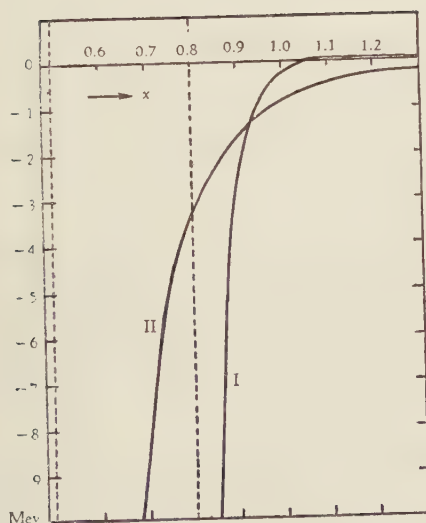


Fig. 1. Coefficient of $(\mathbf{S} \cdot \mathbf{L})$ in W_{SL} .

I: triplet even

II: triplet odd

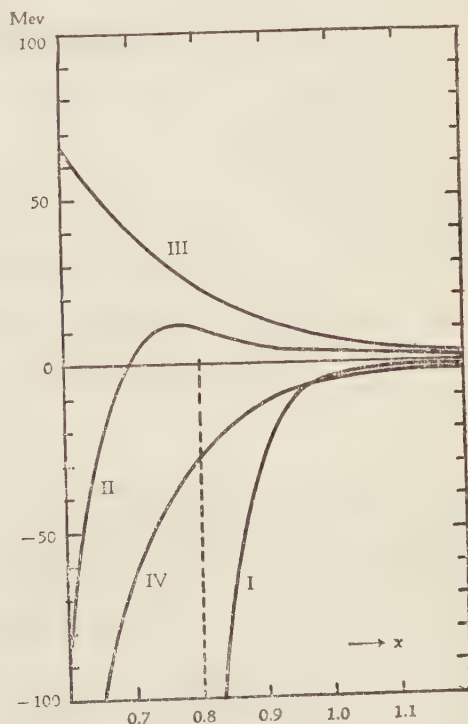


Fig. 2. W_0 , central part.

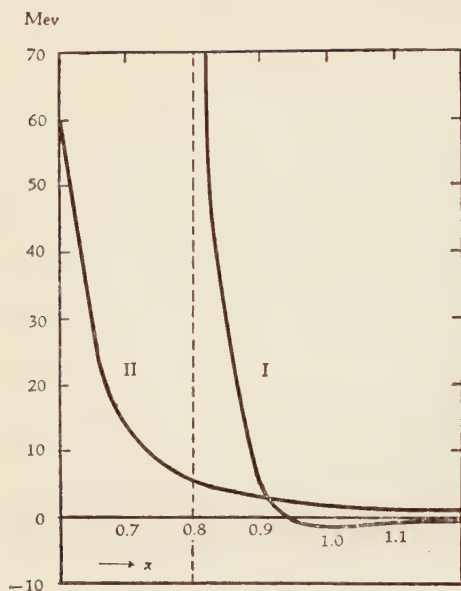
I: triplet even

II: triplet odd

III: singlet even

IV: singlet odd

*) I.e., with an understanding that K/H_0 is also regarded as the second order quantity.

Fig. 3. W_0 , tensor part.

I: triplet even

II: triplet odd

and

$$B^{(1)}(x) = -(f^2/4\pi) (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \{k_0(x) + 2k_1(x)\}. \quad (3 \cdot 28)$$

(3.25) yields the following spin-orbit coupling part*:

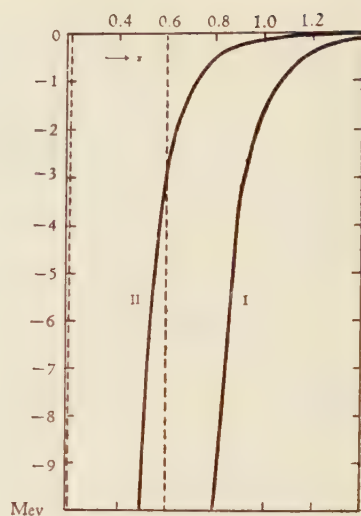
$$W_{SL}^{(1)} = (\mu^2/M) (2/x^2) [1 - A^{(1)}(A^{(1)2} - B^{(1)2})^{-1/2}] (\mathbf{S} \cdot \mathbf{L}). \quad (3 \cdot 29)$$

§ 4. Discussions

We have already mentioned in § 2 that our approximation can be good only when $\epsilon \lesssim \mu^2/M$. We now show that there exists another limit of validity in our result.** From the definition, the exact P' must obviously be positive-definite, and accordingly $P'^{-1/2}$ must have only real eigenvalues. In other words, it should always hold that

$$A+B > 0 \text{ and } A-B > 0 \text{ (in eigenvalues).}$$

However, for A and B calculated by us, this holds only for x larger than a certain critical value x_c . x_c takes different values according to the order of approximation and also to whether the state in question is a spin (or charge) singlet or triplet one. We have shown in Figs. 1~4 also the corresponding values of x_c (dotted line). The appearance of such

Fig. 4. Coefficient of $(\mathbf{S} \cdot \mathbf{L})$ in $W_{SL}^{(1)}$.

I: triplet even

II: triplet odd

*) This was reported by two of us (I.S. and K.I.) at the Kyoto meeting of the Physical Society of Japan held in March, 1955.

**) A similar discussion on this point has already been made by two of us, cf., I. Sato and K. Itabashi, loc. cit.

a critical distance is due to the fact that we have broken off the expansion of P' in the coupling constant at the term of a finite order. Thus, these values of x_c give the measures of the applicability regions of our approximations. Of course, it might be possible to make the eigenvalues of $(A+B)$ and $(A-B)$ positive everywhere by applying an adequate cut-off procedure in momentum space (instead of our renormalization) to the integrations over the virtual meson momenta.* However, not only such a procedure necessitates terribly cumbersome calculations, but also the results calculated by means of any method which has been proposed for the problem of nuclear force until nowadays would be, in essence, not trustworthy for the internucleon distances smaller than the meson Compton wave length.

As for our final result, there are two features of particular interest. First, it should be noted that the spin-orbit coupling obtained by the present formalism is of the first order in (μ/M) . This is an encouraging feature in connection with the theory of nuclear shell model (see § 1). In addition to this, we notice that the spin-orbit coupling derived in the $PS(PS)$ theory (with the usual formalism)⁷⁾ is of the order $(\mu/M)^2$. The second interesting feature is the sign of W_{SL} . As is well known, a spin-orbit coupling with negative coefficient (in the single-particle model) is phenomenologically required by the theory of nuclear shell model. Of course, we must not confuse the spin-orbit coupling $(\sigma \cdot L)$ in the single-particle model with the present two-body $(S \cdot L)$ coupling. However, it is possible to deduce the former from the two-body $(S \cdot L)$ coupling by summing the latter over the constituent nucleons of the closed shells. Referring to the several calculations⁸⁾ of this sort, it would be well expected that both of our W_{SL} and $W_{SL}^{(1)}$ yield the spin-orbit couplings with negative coefficients in the single-particle model. Thus, our results (both of W_{SL} and $W_{SL}^{(1)}$) which are suitably strong (order μ/M) and of the correct sign seem promising or, at least, worthy to be tested in connection with the nuclear shell structure.

Finally, we shall briefly mention the results obtained by expanding $P'^{-1/2}$ in powers of the coupling constant [instead of using the formula (3.11)]. As reported previously by one of the present authors (S-S)⁶⁾, the resultant spin-orbit part obtained by such an expansion method is, up to the fourth order in f ,

$$-2\mu(\mu/M)(f^2/4\pi)^2\{(1+x)(1+x+x^2)/x^6\}\exp[-2x](3+2\tau_1 \cdot \tau_2)(S \cdot L). \quad (4.1)$$

The coefficient of $(S \cdot L)$ in (4.1) is negative everywhere for the charge triplet state. However, in contrast with our result W_{SL} (§ 3), it is positive everywhere for the charge singlet state.** Thus, we see that the difference between the results of this method [in which $P'^{-1/2} = (A+B\sum_{12})^{-1/2}$ is expanded in powers of f] and our method which has been explained in § 3 is very large. In this connection, we want to notice that, corresponding

*) The calculation of the static part of the nuclear potential with such a cut-off procedure has been reported by Inoue, Toyoda, and Taketani at Kyoto Meeting of the Physical Society of Japan, March 29, 1955.

**) However, according to the work of Elliott and Lane⁸⁾, the contribution of the $(\tau_1 \cdot \tau_2)$ part of the two-body $(S \cdot L)$ coupling to the spin-orbit interaction in the nuclear shell model is very small. Therefore, the positive sign of (4.1) in the charge singlet state does not necessarily mean the disagreement with the requirement of the theory of nuclear shell structure.

to our $W_{SL}^{(1)}$, the former method [expansion of $P^{(1)}$ in powers of f] gives vanishing recoil part of the potential to the second order in f and first order in the nucleon velocities.

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β -Ray Spectrum of RaE

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It is shown that the famous beta ray spectrum of RaE can be explained in the Fermi theory in terms of the scalar plus tensor interaction for the selection rule, $\Delta J = \pm 1$, yes. In this attempt, the simultaneous contribution to the electron wave function of the finite de Broglie wave length effect and the finite nuclear radius effect is reasonably taken into account for the first time. As far as the present calculation is concerned, the relative sign of the weight of the scalar and tensor interaction can not be determined.

§ 1. Introduction

The study of the β -ray spectrum of RaE has historically been of important bearing on the theory of the interaction for the β -decay. It is well known that the decay does not accompany the emission of the successive γ -quanta and the shape of its spectrum is one of the best investigated, experimentally and theoretically. The incentive deviation of its spectrum shape from that predicated by the Fermi theory enforced Konopinski and Uhlenbeck to suggest the following two alternatives. i) The one¹⁾, known as the $K-U$ interaction, involves the derivatives of the wave-function, exclusively for the neutrino, and not the derivatives for the electron. According to this theory, the electron spectrum should have in general the so-called $K-U$ shape for the allowed transition. Later, the evidence for the so-called Fermi shape has been found one by one in the allowed beta-emitters and the $K-U$ interaction lost general acceptance. ii) In 1941, Konopinski and Uhlenbeck developed the forbidden theory in the Fermi theory as the second alternative²⁾ and the result was applied to the interpretation of RaE. In this theory, the spectrum is subject to correction due to effects of the higher angular momentum shared by the emitted particles. The correction factors which are defined as corrections to the allowed shape show in general appreciable dependence on the energy of the electron. They showed that these correction factors, if suitably interfere each other, could reproduce the peculiar deviation of the spectrum of RaE from the allowed shape. At that time, the $\log ft = 8.01$ assigned to RaE was classified to the second forbidden transition. In view of this classification, they used the correction factors in the second forbidden transition of the Fermi theory, in terms of the tensor or vector interaction. In their formulation, it was shown explicitly that the Fermi theory has a wide variety in its implication for the forbidden transitions. A great effort has been made to derive an over-all general formula for the spectrum shapes and the ft

values in terms of the forbidden Fermi theory. On the other hand, several beta spectra have been found which are certainly of the forbidden transitions. The Fermi theory³⁾, in principle, could explain many of them fairly well:

1) Unique forbidden shape. This corresponds to the special case when the theoretical correction factor (in the Gamow-Teller interaction) is only one, obeying the selection rule $\Delta J = \pm (n+1)$. (ΔJ is the spin change and n is the degree of forbiddenness.)

2) Non unique forbidden shape. In this case, we have many correction factors to contribute the spectrum of the transitions, $\Delta J = \pm n$. The interferences which might occur among them will give the quite different shape to the original one.

3) Allowed shape. If the nuclear matrix element is considerably small in the allowed transition, $\Delta J = \pm 1, 0$, no, or if the interferences of the correction factors do not happen in the first forbidden transition, $\Delta J = \pm 1, 0$, yes, it is not unreasonable to get the spectra which are almost of the allowed transition in spite of their forbidden life time.

The results thus obtained confirm the basic view point of $K-U$'s forbidden theory in the Fermi interaction. However, since many assumptions are involved in these analyses, it seemed necessary to make the more systematic investigations before the final answer is drawn. One of the authors (Taketani)⁴⁾ attacked the problem in the classification of ft values and reexamination of the forbidden spectra, especially of the non unique shape, such as RaE. In view of the later developments on the shell structure of the nuclei and the classification of the ft values, it became clear that the 4.8 day decay of RaE belongs to the first forbidden transition⁵⁾. Thus the parity of RaE is odd, since the 83rd proton is in the odd parity orbit ($h_{9/2}$ or $f_{7/2}$ etc.) and the 127th neutron is in the even parity orbit ($g_{9/2}$ or $i_{11/2}$ etc.); the final nucleus, RaF, belongs to the ground state of an even-even nuclei and its parity should be even. Therefore, it is clear that the parity change of the beta-decay in question is yes. Keeping in mind that the $\log ft$ of RaE is 8.01 which is reinterpreted to be in the first forbidden region, the possible spin of RaE is certainly to be one of the cases of 2, 1 and 0. The shell model does not provide an unambiguous assignment for the spin. The case of spin 2 was clearly ruled out, since the only one term which can be responsible for the correction factor of the spectrum in the T or A interaction has quite a different form from the experiment. It seemed, on the other hand, difficult to give complete answer for the case of spin 1. It was expected that, although each of the correction factors in the first forbidden transition shows almost energy independent shape, interference of them might reproduce the required shape for RaE spectrum which exhibits a strong decrease with energy of the electron. In terms of the tensor or vector interaction, however, it was found out that the required shape could not be obtained.⁵⁾ This failure was considered to be difficulties of the Fermi theory, since several evidences in favor of the tensor interaction were accumulated so far. If this problem is to be solved in the frame work of the Fermi theory, one may proceed in the more general forms of the interaction, i.e., the possible linear combinations of the five forms of the Fermi couplings, scalar, S , vector, V , tensor, T , pseudo (axial) vector, A and pseudoscalar, P . By taking

account of the finite nuclear size effect* in the tables of Rose and Holmes⁶⁾ and the finite de Broglie wave length effect**⁷⁾ in succession, Petschek and Marshak⁸⁾ concluded that none of the possible linear combinations can afford the required shape for the spectrum for the case $\Delta J = \pm 1$, yes. Naturally, the case $0- \rightarrow 0+$ remains exclusively and they showed that the combined interaction (P, T) can give the experimental shape quite nicely. At this time, this seemed to be an interesting conclusion. The pseudoscalar interaction is now necessary in addition to the Gamow-Teller interaction, T (or A) and the Fermi interaction, S (or V).*** Many laborious works followed as regards the role of this new interaction P .

Ahrens, Feenberg and Primakoff⁹⁾ called attention to the fact that the pseudoscalar nuclear matrix element $\int \beta \gamma_5$ is vanishingly small in the non-relativistic approximation. Therefore, it became quite doubtful to evaluate the radial part of the wave function for the lepton at the surface of the nucleus as was made conventionally. The higher order terms involving derivatives of the lepton wave functions do certainly contribute. The work of Petschek and Marshak should be subject to further more detailed study.

At this stage, the principal idea of our approach was that before any definite conclusion is drawn as regards the RaE spectrum, one needs the most general point of view in evaluating the correction factors. Along this line of attack, the following investigations have been made in succession in our group¹⁰⁾:

1) The complex spectrum explanation, the effect of the mesonic correction and the contribution from the radiative correction were investigated, but they were all proved to be fruitless.

2) More rigorous treatments in evaluating the effects of the charge of the nucleus on the electron wave have been undertaken. Since the final expression for the spectrum is given after cancellation among several correction factors by a factor of 10–100, it is possible that a slight inaccuracy in each factors may overlap and lead to an entirely opposite results. In this connection, we have dealt with the two problems¹⁰⁾.

i) Is it really possible to fit with the experiment in terms of the tensor T plus pseudoscalar P interaction?

One of the authors¹¹⁾ (Takebe) derived the correction factors by taking into account the finite nuclear size effect and the finite de Broglie wave length effect simultaneously. In fitting the experiment with these correction factors, we had to face another difficulty; the pseudoscalar matrix element $\int \beta \gamma_5$ is vanishingly small in the non-relativistic approximation and a next order terms involving the $\sigma \cdot \nabla$ term may be effective. There remain, however, possibilities that relativistic effects such as nucleon pair formation might enhance the values of $\int \beta \gamma_5$. Therefore, it may be more general to take the ratio of $\int \beta \gamma_5$ to the

* The finite nuclear size effect, i.e., the correction to the solution for the point charge by taking into account the extension of the charge over the nucleus.

** The "finite de Broglie wave length effect" means the contribution from the higher terms in the hypergeometric solution for the Coulomb field of the final nucleus.

*** Later, A is excluded on evidences from the He^6 recoil experiment¹³⁾ and V is excluded from the Ne^{19} recoil experiment¹⁴⁾.

matrix element $\int \beta \sigma \mathbf{r}$ in the correction factor as an unknown parameters. In this manner, it was shown that the spectrum of RaE can really be explained by means of (P, T) combination, if the suitable range of parameters is assumed.

ii) Is it really impossible to fit with the experiment in terms of any of the linear combinations other than the (P, T) group?

Yamada¹²⁾ pointed out that if the finite de Broglie wave length effect is taken into account correctly, the cases (S, T) and (V, A) which were formerly excluded by Petschek and Marshak are, however, capable of explaining the spectrum of RaE. Yamada insisted that his conclusion needs no alteration from the consideration of the finite nuclear size effect which was not explicitly expressed in his formulation. However, this point should have been cleared out in more explicit formulation.

Recently, Smith¹⁵⁾ determined the spin of RaE to be 1 by his ingenious experiment. Now, we have only one case, the transition $\Delta J = \pm 1$, yes, to be explained in terms of scalar plus tensor interaction.

In this paper, we first try to interpret the spectrum of RaE by using the rigorous formula for the correction factors mentioned above. In this attempt, we suffer ambiguities involved in the ratios of the nuclear matrix elements and have to deal with two of these as arbitrary parameters. We have shown that the required shape can be reproduced in the limited range of these parameters. Secondly, we discuss the relative weight of scalar and tensor interaction from the information of the values of these parameters which we have now determined.

§ 2. Method of calculation

As is well known, the β -ray spectrum of RaE deviates from the statistical shape. It seems to be possible to explain this fact only if the radiations by different coupling types interfere in a way to bring about almost complete destruction. As such a destruction conceals the usually largest contributions, small terms in the correction factor which are usually negligible become effective. As a consequence, we should use a more accurate correction factor than those used commonly. Namely, in the power series expansion of the lepton radial wave function with respect to $p\rho$ or $q\rho$, it is necessary to take into account the terms which are of higher order than those given by many authors^{2,3,5,*}. Here, p is the electron momentum, q is the neutrino momentum and ρ is a length of the order of the nuclear radius.

Certainly, the β -decay of RaE is the first forbidden transition. In the correction factor, however, there are some terms which are classified as those of the third forbidden transition but have the same selection rule as the first forbidden transition, and these might have some effects on the RaE transition.

* Rose and Perry⁷⁾ have pointed out that, in some cases, even if there is no destructive interference, these higher order terms may be effective. They have called this effect "the finite de Broglie wave length effect".

Moreover, Rose and Holmes⁽¹⁾ and Malcolm⁽¹⁰⁾ have shown the possibility that the spread of the nuclear charge affects the result appreciably. As RaE is a highly charged nucleus ($Z=83$ for RaE, thus $Z=84$ for the daughter element ^{210}Po), and as in our case the large terms of the correction factor cancel out each other, the above effect might be important.

Heretofore the wave function of the electron in a Coulomb field caused by a point source has been used to derive the correction factor. But, as was mentioned above, in the analysis of the β -transition of RaE we should use those in an electrostatic field caused by a finite source.

The correction factor which contains all the above effects is given by eq. (24) of the reference 20)*. We have analysed the β -ray spectrum of RaE by this formula.

Although this formula is an exact one, we have made some approximations in the following calculation.

i) The radial wave functions of the electron are usually evaluated at a value ρ of r , which has the order of the magnitude of the nuclear radius. (I·24) is an exact correction factor without such an approximation. This correction factor is expressed by a series through Taylor expansion of the lepton radial wave function around R , the radius of the nuclear charge spread. This expansion was possible, because, owing to the nuclear charge spread, no electron radial wave function is singular at the origin. If we retain only the first terms of the expansions (I·20) equating r to R , the correction factor (I·24) reduces to the one which resembles to the commonly used correction factor, but differs from this in having been corrected for nuclear charge spread and in containing the correctives of the n -th forbidden transition by $(n+2m)$ -th forbidden transitions. We have adopted this approximation to equate r in (I·20) to R . To make such an approximation corresponds to disregarding the r dependence of the lepton radial wave functions inside the nucleus. We note here that, as can be seen from § 4 of I, R in our correction factor is the radius of the nuclear charge spread, while ρ in the usual correction factor, at which the lepton radial wave functions are evaluated, is regarded only as a constant of the order of the magnitude of the nuclear radius, and has no physical meaning.

ii) We have further neglected all the contributions from the above mentioned $(n+2m)$ -th forbidden transitions, namely, the corrections to the first forbidden transition from the third or the higher order transitions, which are estimated to be roughly of the order of $R^4 \approx (1/50)^4$ or higher.

* Hereinafter, we shall denote the reference 20) as I. For example, eq. (24) of this reference will be written as (I·24). In this reference, in addition to the corrigenda mentioned in Prog. Theor. Phys. 13 (1955), 127, the following corrections are desirable. In eqs. (2) and (7a, b) sums are also to be performed over ρ . In eq. (7a, b), $\mathcal{J}_{u(k\nu)}^{-\mu\nu-\mu}(\sigma_p, \mathbf{r})$ should read $\mathcal{J}_{u(k\nu)}^{-\mu\nu-\mu}(\sigma_p, \mathbf{r}_p)$. The left-hand side of eq. (11) should be replaced by

$$\sum_p \int \mathcal{P}^* Q_p \omega_p \times \mathcal{J}_{u(k\nu)}^{-\mu\nu-\mu}(\sigma_p, \mathbf{r}_p) \mathcal{O} d\Omega_p d\tau'$$

where $d\tau' = d\mathbf{r}_1 \dots d\mathbf{r}_{p-1} d\mathbf{r}_{p+1} \dots d\mathbf{r}_n$. Throughout this reference $(J \| Q_p \omega_p \times \mathcal{J}_{u(k\nu)}(\sigma_p, \mathbf{r}) \| J')$ is to be replaced by $(J \| Q_p \omega_p \times \mathcal{J}_{u(k\nu)}(\sigma, \mathbf{r}) \| J')$. In (A4a) and (A4b), $-(W-1-\alpha Z/R^2) D^{(i)}$ and $-(W+1-\alpha Z/R^2) \cdot 1, D^{(i)}$ should be replaced by $\alpha Z/R^2 \cdot D^{(i)}$ and $-\alpha Z/R^2 \cdot 1/D^{(i)}$.

iii) In accordance with Konopinski and Uhlenbeck's formulae, we have retained only the first terms of the neutrino parts which may as well be of the order of $R^2 \approx (1/50)^2$. (I. C2)

When a large cancellation occurs in the correction factor as is the case for RaE, e.g. the magnitude of the final expression reduces to as small as $\sim 1/500$ of that of the main terms, we can not ensure that the errors introduced by these approximations, especially by iii), are necessarily small.

On the other hand, we have exactly calculated the first three terms in the power series expansion of L_κ , M_κ and N_κ in the correction factor, and have taken into account the effect of the nuclear charge spread except to equate r to R . As will be shown later, the effect of the nuclear charge spread has appreciable effect on the analysis of the β -ray spectrum of RaE.

Owing to the above approximations i) and ii), the correction factor reduces to the sum of C_S , C_T and $C_{S,T}$; namely, the sum of the correction factors for the first forbidden transition of scalar and tensor interaction given by Konopinski and Uhlenbeck², and interference term of these two interactions for the first forbidden transition given by Smith¹⁵⁾. In our treatment, however, due to the effect of the nuclear charge spread, L_κ , M_κ and N_κ in this sum must be replaced as follows:

$$\left. \begin{aligned} L_\kappa &\rightarrow L_\kappa(-1, -1; RR), \\ M_\kappa &\rightarrow M_\kappa(-1, -1; RR), \\ N_\kappa &\rightarrow N_\kappa(-1, -1; RR), \end{aligned} \right\} \quad (1)$$

Here we have used the index $\kappa = \pm(j + \frac{1}{2})$ for $j = l \mp \frac{1}{2}$ which is more convenient than the above authors index*. The above $L_\kappa(-1, -1; RR)$ is L_κ as defined by radial wave functions of the electron in the electrostatic field caused by the finite nuclear charge spread, and are evaluated at $r=R$. This applies also to $M_\kappa(-1, -1; RR)$ and $N_\kappa(-1, -1; RR)$. They are given by (I.17), (I.20) and (I.21):

$$\begin{aligned} L_\kappa(-1, -1; RR) &= \frac{1}{2p^2 F_1(R) R^2} \cdot \mathcal{Q}_{\kappa 00}(-1, -1) \\ &= \frac{1}{2p^2 F_1(R) R^{2\kappa}} \left[\frac{X_\kappa^2 + Y_{-\kappa}^2}{2} 2l_\kappa^{(L)} + \frac{X_\kappa^2 - Y_{-\kappa}^2}{2} 2l_\kappa^{(S)} \right] \\ &= \frac{X_\kappa^2 + Y_{-\kappa}^2}{2} L_\kappa + \frac{X_\kappa^2 - Y_{-\kappa}^2}{2} L_\kappa^-. \end{aligned} \quad (2a)$$

* Our notations L_1 , L_2 , M_1 , N_1 , L_1^- , M_1^- and N_1^- correspond to Konopinski and Uhlenbeck's L_0 , L_1 , M_0 , N_0 and Smith's L_0^- , M_0^- and N_0^- , respectively.

$$\begin{aligned}
M_{\kappa}(-1, -1; RR) &= \frac{1}{2p^2 F_1(R) R^{2\kappa+2}} \mathfrak{M}_{\kappa 00}(-1, -1) \\
&= \frac{1}{2p^2 F_1(R) R^{2\kappa+2}} \left[\frac{X_{-\kappa}^2 + Y_{\kappa}^2}{2} \cdot 2m_{\kappa}^{(L)} + \frac{X_{-\kappa}^2 - Y_{\kappa}^2}{2} 2m_{\kappa}^{(S)} \right] \\
&= \frac{X_{-\kappa}^2 + Y_{\kappa}^2}{2} M_{\kappa} + \frac{X_{-\kappa}^2 - Y_{\kappa}^2}{2} M_{\kappa}^{-}.
\end{aligned} \tag{2b}$$

$$\begin{aligned}
N_{\kappa}(-1, -1; RR) &= \frac{1}{2p^2 F_1(R) R^{2\kappa+1}} \mathfrak{N}_{\kappa 00}(-1, -1) \\
&= \frac{1}{2p^2 F_1(R) R^{2\kappa+1}} \left[\frac{X_{-\kappa} Y_{-\kappa} + X_{\kappa} Y_{\kappa}}{2} 2n_{\kappa}^{(L)} + \frac{X_{-\kappa} Y_{-\kappa} - X_{\kappa} Y_{\kappa}}{2} 2n_{\kappa}^{(S)} \right] \\
&= \frac{X_{-\kappa} Y_{-\kappa} + X_{\kappa} Y_{\kappa}}{2} N_{\kappa} + \frac{X_{-\kappa} Y_{-\kappa} - X_{\kappa} Y_{\kappa}}{2} N_{\kappa}^{-}.
\end{aligned} \tag{2c}$$

Here L_{κ} , M_{κ} , N_{κ} , L_{κ}^{-} , M_{κ}^{-} and N_{κ}^{-} are those defined by Konopinski and Uhlenbeck²¹⁾, Smith¹²⁾ and Pursey¹³⁾ for a point nuclear charge. The first three terms of each of the above in the expansion with respect to pR can be calculated by the appendix B of I. $X_{\pm\kappa}$ ($Y_{\pm\kappa}$) are the ratios at $r=R$ of the electron radial wave functions for the finite nuclear charge spread $f_{\kappa}(r)$ ($g_{\kappa}(r)$) to those for the point nuclear charge $f_{\kappa}^c(r)$ ($g_{\kappa}^c(r)$):

$$X_{\kappa} = \frac{f_{\kappa}(R)}{f_{\kappa}^c(R)}, \quad Y_{\kappa} = \frac{g_{\kappa}(R)}{g_{\kappa}^c(R)}.$$

They are given by (I.22). In numerical calculation we have equated A_{κ} to unity in (I.22) and assumed uniform nuclear charge distribution.

Heretofore, the radius $R=r_0 A^{1/3}$ of the nuclear charge spread was evaluated with $r_0 = 1.45 \times 10^{-13}$ cm. But recent experiments on the X-ray spectroscopy of μ -mesonic atoms²¹⁾ indicate the value of r_0 as about 1.2×10^{-13} cm, especially for $Z=82$ as 1.17×10^{-13} cm. As this value of $Z(=82)$ is very near to our $Z(=84)$, we have adopted this value of r_0 in our calculation. The values of $X_{\pm\kappa}$, $Y_{\pm\kappa}$ and $L_{\kappa}(-1, -1; RR)$, etc., for this r_0 (1.17×10^{-13} cm) are shown in Tables 1, 2 and Figs. 1, 2.

To examine whether the results alter or not when r_0 varies, we have also studied the RaE spectrum for $r_0 = 1.45 \times 10^{-13}$ cm. That is, we have studied it for two values of r_0 , which seems to be the smallest (1.17×10^{-13} cm) and the largest (1.45×10^{-13} cm). Of course, $X_{\pm\kappa}$ and $Y_{\pm\kappa}$ ought to depend on r_0 . But, for the sake of convenience, for the values of $X_{\pm\kappa}$ and $Y_{\pm\kappa}$ in the case of $r_0 = 1.45 \times 10^{-13}$ cm, the values of $X_{\pm\kappa}$ and $Y_{\pm\kappa}$ in the case of $r_0 = 1.17 \times 10^{-13}$ cm have been used.

On the other hand, as the values of L_{κ} , M_{κ} , N_{κ} , etc., differ seriously for various r_0 , we have calculated the values for $r_0 = 1.45 \times 10^{-13}$ cm in the proper way. The results are shown in parentheses in Table 2.

To examine the influence of the nuclear charge spread, we have calculated the correction factor for $X_{\pm\kappa} = Y_{\pm\kappa} = 1$ when r_0 is 1.17×10^{-13} cm, that is, for the case of the point

nuclear charge. In this case, as the coefficients of L_{κ}^{-} , M_{κ}^{-} and N_{κ}^{-} are all zero, $L_{\kappa}(-1, -1; RR)$, etc., reduce to Konopinski and Uhlenbeck's L_{κ} , etc.. The R for such a treatment has not the meaning of the radius of the nuclear charge spread any longer, and is only a magnitude of the order of the nuclear radius. As only the ratios of the nuclear matrix elements affect the spectrum shape, it is convenient to define:

$$x = ig_s \int \beta \mathbf{r} / g_T \int \beta \boldsymbol{\sigma} \times \mathbf{r}, \quad y = \int \beta \alpha / \int \beta \boldsymbol{\sigma} \times \mathbf{r}. \quad (3)$$

We have tried to make the Kurie plot of the RaE spectrum straight by adjusting these two parameters.

The data of conventional Kurie plots for the electron energies $W_0=1.2, 1.4, \dots, 3.0$ were obtained by interpolation from Table 1 of Plassmann and Langer's paper²², and are shown in Table 3.

The condition that the three points on the Kurie plot for electron energies W_0 (=maximum energy), W_1 and W_2 should lie on a straight line is represented by a quadratic of x and y . The graphs of these quadratics for various values of W_1 and W_2 are shown in Fig. 3 by dotted-lines. To make the Kurie plot straight, we should adopt x and y which correspond to the points where the hyperbolas best coincide.

We have made Kurie plots for various pairs of x and y on these hyperbolas. The plots of y against the corresponding values of x which yield the best fit are shown in Fig. 3 by solid lines. As can be seen from this figure, the ranges of x and y are confined as follows:

$$0.2 \leq x \leq 5.5 \text{ and } 18.8 \leq y \leq 104.0 \\ \text{for } r_0 = 1.17 \times 10^{-13} \text{ cm } (X_{\pm\kappa} \approx 1, Y_{\pm\kappa} \approx 1), \quad (4a)$$

$$0.2 \leq x \leq 10.0 \text{ and } 15.4 \leq y \leq 147.0 \\ \text{for } r_0 = 1.45 \times 10^{-13} \text{ cm } (X_{\pm\kappa} \approx 1, Y_{\pm\kappa} \approx 1), \quad (4b)$$

$$0.2 \leq x \leq 50.0 \text{ and } 24.8 \leq y \leq 1088 \\ \text{for } r_0 = 1.17 \times 10^{-13} \text{ cm } (X_{\pm\kappa} = Y_{\pm\kappa} = 1). \quad (4c)$$

For various values x , the values of y which make the Kurie plots most straight are shown in Table 4. The corresponding Kurie plots are given in Fig. 4.

§ 3. Conclusion

We have shown, in the preceding section, that the beta-ray spectrum of RaE can be reproduced, in a limited range of parameters used, in terms of the linear combination of the scalar and the tensor interactions in the Fermi theory. In our attempt, the relevant transition is assumed to be of the first forbidden transition, obeying the selection rule $\Delta J = \pm 1$, yes, which is in accordance with the recent finding by Smith¹⁵⁾ that the spin of RaE is 1. Our emphasis is, as already mentioned, on introducing a consistent contribution

of the finite nuclear size effect and the finite de Broglie wave length effect, both of which are surely felt by the emitted electrons. This can be realized in our final expression for the correction factor $C(Z, W, R)$ on the forbidden spectrum in question:

$$\begin{aligned} C(Z, W, R) &= g_s^2 C_s + g_T^2 C_T + g_s g_T C_{sT} \\ &= \{x^2 (\frac{1}{3} q^2 \bar{L}_1 + 2 \bar{L}_2 + \bar{M}_1 + \frac{2}{3} q \bar{N}_1) + \gamma^2 \bar{L}_1 + (\frac{1}{3} q^2 \bar{L}_1 + \frac{1}{2} \bar{L}_2 + \bar{M}_1 - \frac{2}{3} q \bar{N}_1) \\ &\quad - 2\gamma (\frac{1}{3} q \bar{L}_1 - \bar{N}_1) - 2x (\bar{L}_2 - \bar{M}_1) + 2x\gamma (\frac{1}{3} q \bar{L}_1 + \bar{N}_1)\} g_T^2 \cdot \{\beta \sigma \times \mathbf{r}. \end{aligned} \quad (5)$$

We have used abbreviations:

$$\bar{L}_1 = L_1(-1, -1; R, R),$$

$$\bar{L}_2 = L_2(-1, -1; R, R),$$

$$\bar{M}_1 = M_1(-1, -1; R, R),$$

$$\bar{N}_1 = N_1(-1, -1; R, R).$$

\bar{L}_1 , \bar{L}_2 , \bar{M}_1 and \bar{N}_1 are the modified of L_0 , L_1 , M_0 and N_1 of Konopinski and Uhlenbeck's²⁰ paper respectively by the simultaneous consideration of charge spread over the nucleus and of the higher terms in the hypergeometric solution for the electron in the nuclear field. The modifications are found quite appreciable and variant among each member of them. In view of this fact, it seems likely that the very well fits with the experimental curve by using either the former values of L_0, \dots, N_0 , or by partially introducing one of the effects mentioned above, were rather accidental.

As was shown already in the relation (3), x and γ involve the ratio of nuclear matrix elements which are not precisely known. x and γ are constant over the spectrum and used as arbitrary parameters. Our results may be considered to be a detailed refinement of parameters which were already estimated by some former workers^{12, 22) - 25)}. However, since they have neglected one of the important effects in the formula, their reports would be far from final one as far as the actual values of parameters, x and γ , are concerned. The present calculation of ours has also neglected some of the higher order terms, which might as well be ignored. Our results may be modified by further research, but could be regarded as a first approximation to the true values.

As is well known, one can infer the sign of g_s/g_T from the values of x in the relation (3),

$$g_s/g_T = \varepsilon x, \quad (6)$$

if the ratio of the nuclear matrix elements

$$1/\varepsilon \equiv i \int \beta \mathbf{r} / \int \beta \sigma \times \mathbf{r}$$

can be evaluated. Recently, Lee-Whitung²⁶⁾ concluded that, for the beta decay of RaE, $g_s/g_T < 0$ is to be ruled out. His process of argument, however, seems to us so limited for the following two reasons:

A) The lower limit of the allowable region for x , i.e.,

$$x \geq 0.35$$

which Lee-Whitung has determined from the spectrum analysis is changed to

$$x \geq 0.20$$

by the present calculation if we take $R = 1.17 \times 10^{-13} A^{1/3}$ cm. In order to know the true value of x , we shall have to make more precise estimation of the correction factors and also of the radius of charge spread.

B) The value of $1/\varepsilon$ which he has evaluated by using the relation²⁵⁾

$$1/\varepsilon = j_p(j_p + 1) - j_n(j_n + 1) - l_p(l_p + 1) + l_n(l_n - 1) \quad (7)$$

seems not to be final, since this value might be modified by more general considerations, e.g., the mesonic correction²⁷⁾, the configuration mixing²⁸⁾ and break of the assumption that the neutron distribution is the same as the proton distribution²⁹⁾, etc..

In view of these considerations, we shall denote amendments to x and ε as Δx and $\Delta \varepsilon$ respectively which might possibly be required by the above two reasons. Thus we have instead of (6),

$$g_s/g_T = x(1 + \Delta x) \cdot \varepsilon(1 + \Delta \varepsilon). \quad (8)$$

Furthermore, difficulties to get an equivocal prediction for the nuclear configuration of RaE have so far prevented us to determine the value of ε .

On the other hand, we know

$$|g_s/g_T| \approx 1$$

from various analyses of beta decay phenomena. It is a keen interest whether the sign of g_s/g_T is positive or negative. Unfortunately, we can not at present draw any conclusion for this question from our result, since much ambiguities are attached to the values of Δx and $\Delta \varepsilon$. (See Table 5.)

As for the values of γ , our results, (4), agree within a factor of ~ 2 with the theoretical estimation of Ahrens and Feenberg²⁶⁾,

$$\gamma \sim 13.$$

Our final conclusions are as follows :

1) The deviation of the beta ray spectrum of RaE from the allowed spectrum can be reproduced in terms of (S, T) interaction combination in the Fermi theory,

$$H = g_S S + g_T T,$$

assuming the selection rule $\Delta J = \pm 1$, yes $(1 \rightarrow 0+)$.

2) The current question whether $g_s/g_T \cong 1^{26)}$ or $-1^{30)}$ can not be answered in the present calculation, since there remains some ambiguities concerning the ratios of the nuclear matrix elements. (See (8) and Table 5.) In order to determine the sign of g_s/g_T , we have to make more systematic investigation for the forbidden spectra, angular correlations and the allied phenomena of beta decay.

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Table 1

W	X_1	X_{-1}	X_2	X_{-2}	Y_1	Y_{-1}	Y_2	Y_{-2}
1.2	0.965380	0.715336	0.997618	0.916232	0.727934	0.960202	0.909112	0.997734
1.4	0.964617	0.715340	0.997596	0.916632	0.727832	0.959514	0.909462	0.997716
1.6	0.963879	0.715345	0.997574	0.916990	0.727727	0.958837	0.909806	0.997697
1.8	0.963150	0.715356	0.997552	0.917384	0.727631	0.958176	0.910146	0.997679
2.0	0.962434	0.715340	0.997530	0.917800	0.727541	0.957522	0.910483	0.997662
2.2	0.961735	0.715333	0.997507	0.918181	0.727454	0.956882	0.910814	0.997644
2.4	0.961047	0.715329	0.997486	0.918560	0.727367	0.956257	0.911183	0.997626
2.6	0.960374	0.715304	0.997463	0.918935	0.727288	0.955643	0.911470	0.997607
2.8	0.959716	0.715321	0.997440	0.919305	0.727211	0.955044	0.911791	0.997590
3.0	0.959077	0.715316	0.997416	0.919672	0.727137	0.954455	0.912100	0.997571

Values of $X_{\pm\kappa} = f_{\pm\kappa}(R)/f_{\pm\kappa}^c(R)$ and $Y_{\pm\kappa} = g_{\pm}(R)/g_{\pm\kappa}^c(R)$ when the nuclear charge is distributed uniformly and r_0 is 1.17×10^{-13} cm.

Table 2. 1.

W	$L_1^{(L)}$	$L_1^{(S)}$	$L_1(-1, -1; R, R)$
1.2	0.868995	-0.569585	0.802696
	(0.862840)	(-0.564868)	(0.797014)
1.4	0.865514	-0.485397	0.798717
	(0.858520)	(-0.480681)	(0.792266)
1.6	0.801938	-0.422471	0.794569
	(0.854117)	(-0.417807)	(0.787363)
1.8	0.858300	-0.373149	0.790324
	(0.849621)	(-0.368435)	(0.782337)
2.0	0.854620	-0.333820	0.786014
	(0.845077)	(-0.329096)	(0.777241)
2.2	0.8509125	-0.301720	0.781672
	(0.840499)	(-0.297008)	(0.772110)
2.4	0.847184	-0.274935	0.777316
	(0.835897)	(-0.270224)	(0.766965)
2.6	0.843441	-0.252271	0.772956
	(0.831279)	(-0.247561)	(0.761814)
2.8	0.839688	-0.232845	0.768600
	(0.826650)	(-0.228136)	(0.756670)
3.0	0.835929	-0.216006	0.764259
	(0.822014)	(-0.211298)	(0.751542)

Values of $L_1^{(L)}$, $L_1^{(S)}$ and $L_1(-1, -1; R, R)$ for $r_0 = 1.17 \times 10^{-13}$ cm. Those for $r_0 = 1.45 \times 10^{-13}$ cm are shown in parentheses.

Table 2. 2.

W	$L_2^{(L)}$	$L_2^{(S)}$	$L_2(-1, -1; R, R)$
1.2	0.0375896 (0.0307461)	-0.0298008 (-0.0297011)	0.0374116 (0.0305999)
1.4	0.0668780 (0.0666307)	-0.0454286 (-0.0452567)	0.0665593 (0.0663132)
1.6	0.102196 (0.110177)	-0.0607206 (-0.0604474)	0.101706 (0.101279)
1.8	0.143633 (0.142959)	-0.0758338 (-0.0754487)	0.142939 (0.142268)
2.0	0.191273 (0.190278)	-0.0908596 (-0.0903452)	0.190343 (0.189353)
2.2	0.245200 (0.243798)	-0.105855 (-0.105193)	0.243998 (0.242603)
2.4	0.305490 (0.303585)	-0.120858 (-0.120031)	0.303982 (0.302086)
2.6	0.372217 (0.369700)	-0.135890 (-0.134880)	0.370365 (0.357861)
2.8	0.445449 (0.442200)	-0.150967 (-0.149755)	0.443214 (0.439982)
3.0	0.525343 (0.521233)	-0.166129 (-0.164695)	0.522687 (0.518598)

Values of $L_2^{(L)}$, $L_2^{(S)}$ and $L_2(-1, -1; R, R)$ for $r_0 = 1.17 \times 10^{-13}$ cm. Those for $r_0 = 1.45 \times 10^{-13}$ cm are shown in parentheses.

Table 2. 3.

W	$M_1^{(L)}$	$M_1^{(S)}$	$M_1(-1, -1; R, R)$
1.2	324.094 (211.100)	205.869 (132.914)	166.921 (108.735)
1.4	326.825 (213.299)	175.395 (113.059)	168.605 (110.051)
1.6	329.259 (215.255)	152.531 (98.159)	170.067 (111.196)
1.8	331.474 (217.034)	134.744 (86.567)	171.369 (112.219)
2.0	333.538 (218.691)	120.641 (77.3935)	172.549 (113.150)
2.2	335.492 (219.529)	108.857 (69.6905)	173.653 (114.004)
2.4	337.363 (221.757)	99.144 (63.3558)	174.695 (114.848)
2.6	339.168 (223.202)	90.921 (57.992)	175.691 (115.636)
2.8	340.921 (224.604)	83.8694 (53.3910)	176.648 (116.394)
3.0	342.630 (225.970)	77.7548 (49.4002)	177.574 (117.129)

Values of $M_1^{(L)}$, $M_1^{(S)}$ and $M_1(-1, -1; R, R)$ for $r_0 = 1.17 \times 10^{-13}$ cm. Those for $r_0 = 1.45 \times 10^{-13}$ cm are shown in parentheses.

Table 2. 4.

W	$N_1^{(L)}$	$N_1^{(S)}$	$N_1(-1, -1; R, R)$
1.2	-16.7805 (-13.4945)	-10.8330 (-8.67018)	-11.5732 (-9.30719)
1.4	-16.8172 (-13.5303)	-9.23231 (-7.37860)	-11.6025 (-9.33517)
1.6	-16.8443 (-13.5566)	-8.03186 (-6.41000)	-11.6220 (-9.35401)
1.8	-16.8650 (-13.5764)	-7.09825 (-5.65673)	-11.6350 (-9.36667)
2.0	-16.8809 (-13.5914)	-6.35142 (-5.05419)	-11.6430 (-9.37459)
2.2	-16.8934 (-13.6027)	-5.74043 (-4.56126)	-11.6477 (-9.37932)
2.4	-16.9030 (-13.6113)	-5.23132 (-4.15055)	-11.6498 (-9.38155)
2.6	-16.9105 (-13.6175)	-4.80058 (-3.80309)	-11.6500 (-9.38181)
2.8	-16.9162 (-13.6217)	-4.43142 (-3.50532)	-11.6486 (-9.38048)
3.0	-16.9203 (-13.6243)	-4.11152 (-3.24729)	-11.6459 (-9.37781)

Values of $N_1^{(L)}$, $N_1^{(S)}$ and $N_1(-1, -1; R, R)$ for $r_0=1.17 \times 10^{-13}$ cm. Those for $r_0=1.45 \times 10^{-13}$ cm are shown in parentheses.

Table 3

W	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
$[N/p^2 F]^{1/2}$	0.1613	0.1344	0.1121	0.09184	0.07446	0.05857	0.04428	0.03178	0.02056	0.01076

Values of $[N/p^2 F]^{1/2}$ obtained by interpolation from Plassmann and Langer's table⁽¹¹⁾.

Table 4

x	0.20	0.40	0.60	0.80	1.00	1.20	1.50	2.00	3.00	4.00	5.00	5.50
y	18.8	22.4	25.7	28.9	32.35	35.6	40.4	48.7	64.8	81.0	97.0	104.0

(a) $r_0=1.17 \times 10^{-13}$ cm ($X_{\pm\kappa} \neq 1$, $Y_{\pm\kappa} \neq 1$)

x	0.22	0.25	0.27	0.30	0.40	0.60	0.80	1.00	2.00	4.00	4.50	5.00	8.00	10.00
y	15.7	16.2	16.5	17.0	18.4	21.2	23.9	26.8	40.5	67.0	73.4	80.0	120.8	147.0

(b) $r_0=1.45 \times 10^{-13}$ cm ($X_{\pm\kappa} \neq 1$, $Y_{\pm\kappa} \neq 1$)

x	0.20	0.40	0.60	0.80	1.00	2.00	3.00	4.00	10.00	20.00	30.00	40.00	50.00
y	24.8	29.5	33.9	38.2	42.6	64.2	85.6	106.6	233.0	448.0	660.0	873.0	1088.0

(c) $r_0=1.17 \times 10^{-17}$ cm ($X_{\pm\kappa} = Y_{\pm\kappa} = 1$)

The values of y which make the Kurie plots most straight for corresponding values of x .

Table 5

Possible configuration of RaE.	$1/\varepsilon$ (from eq. (7))	g_S/g_T ($\Delta x=0$)	g_S/g_T $\Delta x=-0.5$
$(h_{9/2}, g_{9/2})1-$	-0.1	$\leq -2(1+\Delta\varepsilon)$	$-(1+\Delta\varepsilon)$
$(h_{9/2}, i_{11/2})1-$	1	$\geq (1+\Delta\varepsilon)$	$0.5(1+\Delta\varepsilon)$

Possible configuration of RaE, if it is of a pure state, is cited from the discussion of Lee-Whitung²⁵⁾ and Pryce²⁶⁾. $26.5 \geq \Delta x \geq 0$ corresponds to the results of the present calculation, and $\Delta x = -0.5$ is only a guess which might be introduced by the consideration of B) in the section 3. The sign of g_S/g_T can not be determined from our results. For instance, if $\Delta x = 0$, $\Delta\varepsilon = -0.5$ for the configuration $(h_{9/2}, g_{9/2})1-$, or if $\Delta x = 0$, $\Delta\varepsilon = -2$ for the configuration $(h_{9/2}, i_{11/2})1-$, we have $g_S/g_T = -1$. Also if $\Delta x = 0$, $\Delta\varepsilon = 0$ for the configuration $(h_{9/2}, i_{11/2})1-$, we have $g_S/g_T = +1$.

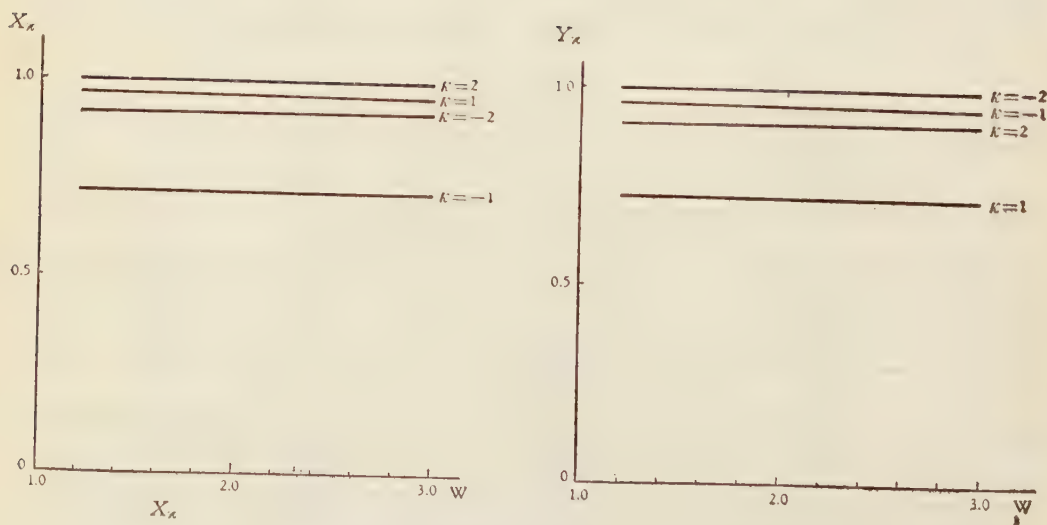


Fig. 1 $X_{\pm\kappa}$ and $Y_{\pm\kappa}$ for $Z=84$. r_0 was equated to 1.17×10^{-13} cm, and the nuclear charge distribution was assumed uniform.

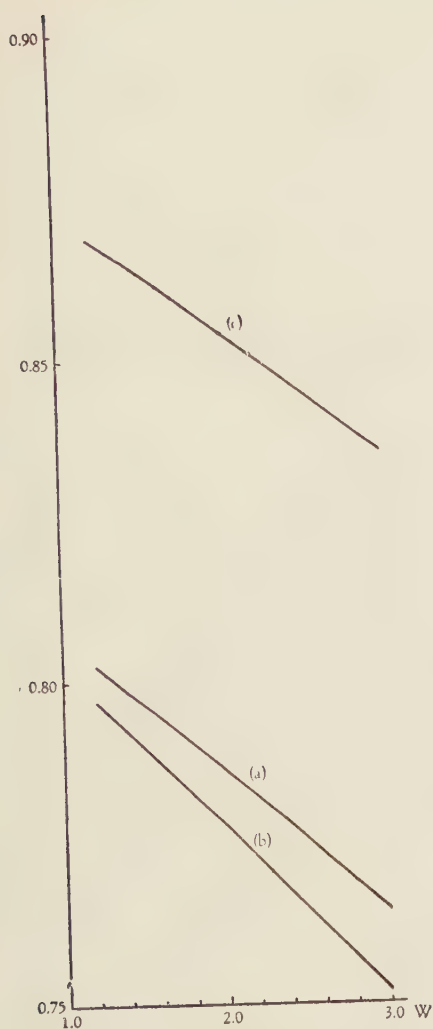


Fig. 2. a. $L_1(-1, -1; RR)$ for
 a) $r_0 = 1.17 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} \neq 1, Y_{\pm\pi} \neq 1)$;
 b) $r_0 = 1.45 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} \neq 1, Y_{\pm\pi} \neq 1)$;
 c) $r_0 = 1.17 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} = Y_{\pm\pi} = 1)$.

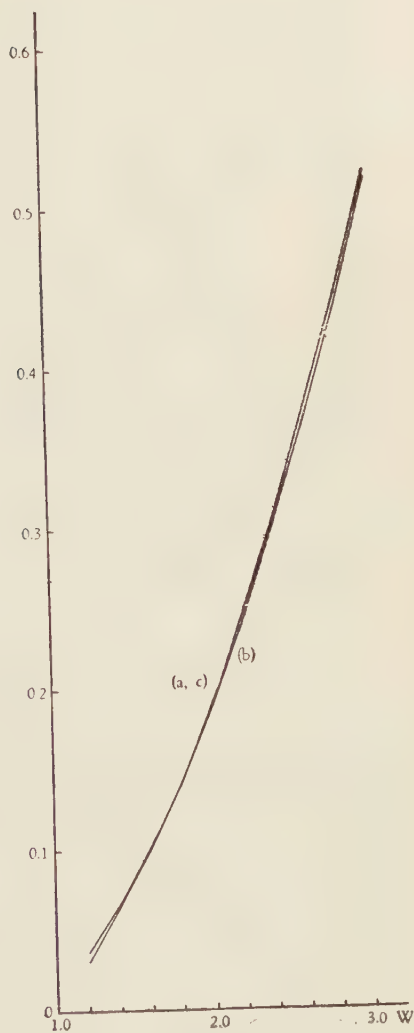


Fig. 2. b. $L_2(-1, -1; RR)$ for
 a) $r_0 = 1.17 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} \neq 1, Y_{\pm\pi} \neq 1)$;
 b) $r_0 = 1.45 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} \neq 1, Y_{\pm\pi} \neq 1)$;
 c) $r_0 = 1.17 \times 10^{-13} \text{cm}$, $(X_{\pm\pi} = Y_{\pm\pi} = 1)$.

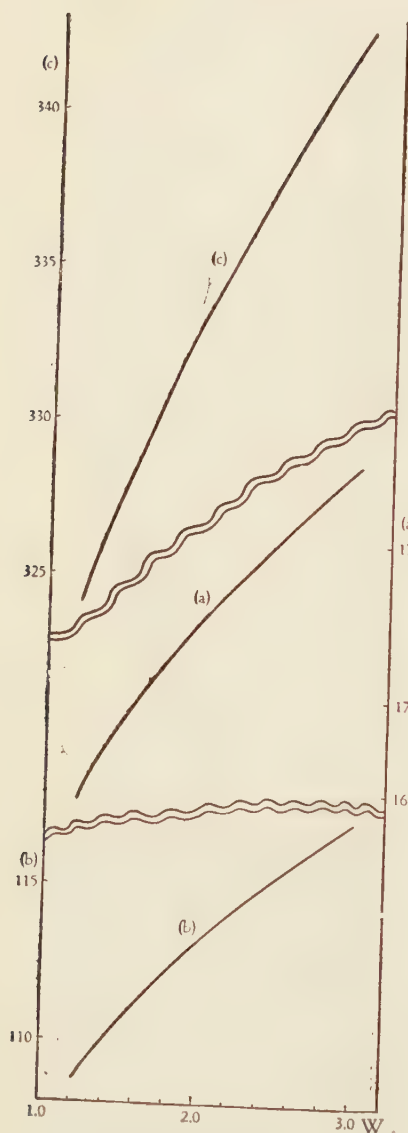


Fig. 2. c. $M_1(-1, -1; RR)$ for
 a) $r_0 = 1.17 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} \neq 1$, $Y_{\pm\pi} \neq 1$);
 b) $r_0 = 1.45 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} \neq 1$, $Y_{\pm\pi} \neq 1$);
 c) $r_0 = 1.17 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} = Y_{\pm\pi} = 1$).

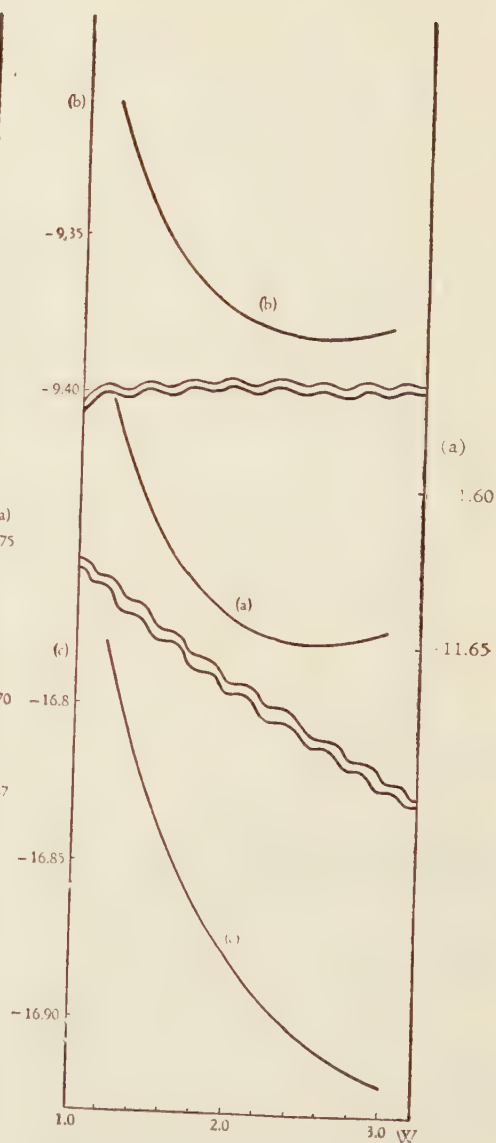


Fig. 2. d. $N_1(-1, -1; RR)$ for
 a) $r_0 = 1.17 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} \neq 1$, $Y_{\pm\pi} \neq 1$);
 b) $r_0 = 1.45 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} \neq 1$, $Y_{\pm\pi} \neq 1$);
 c) $r_0 = 1.17 \times 10^{-13} \text{cm}$, ($X_{\pm\pi} = Y_{\pm\pi} = 1$).

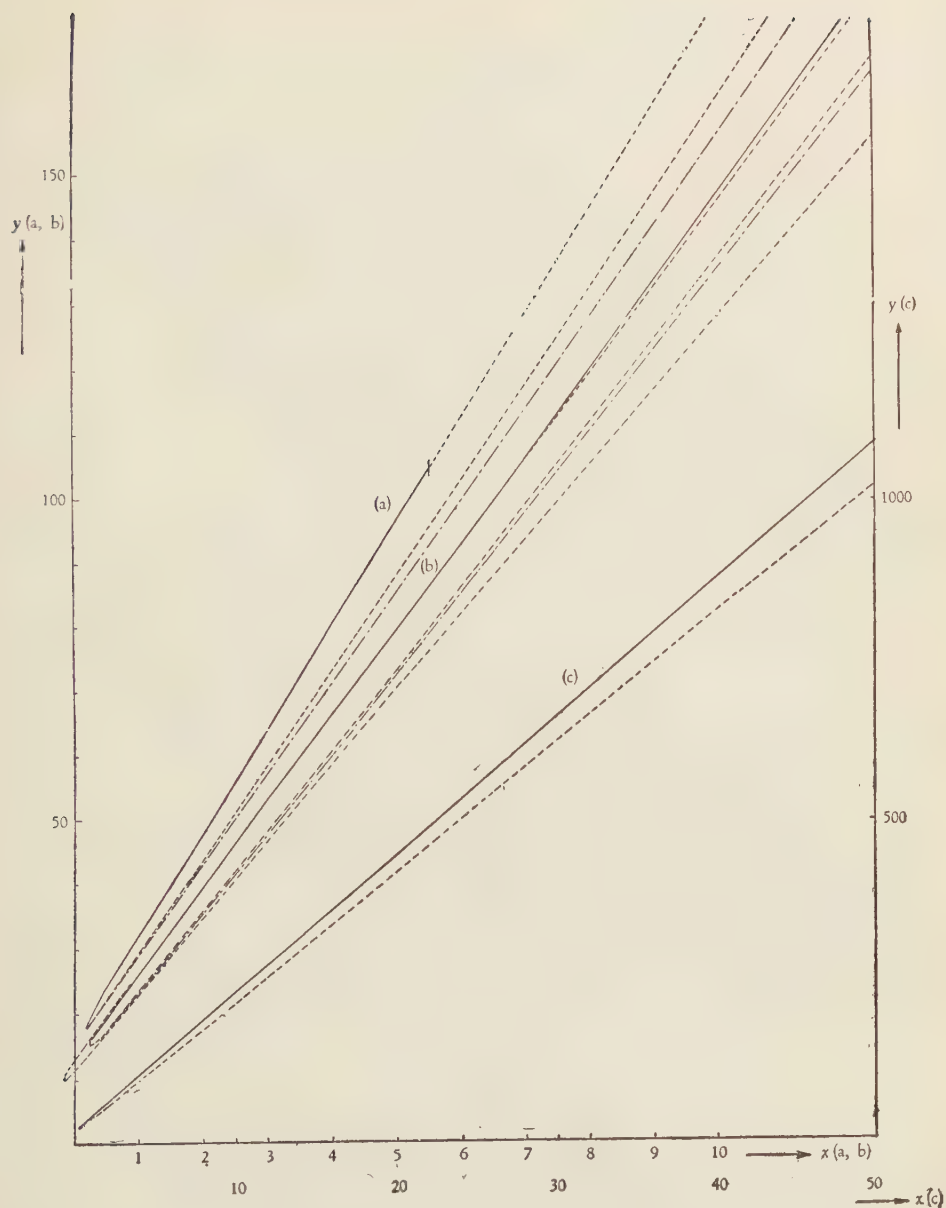


Fig. 3 Dotted-lines represent the quadratics of x and y which are the conditions to place three points on the Kurie plot on a straight line. Solid lines are the plots of y against x which yield the best fit.

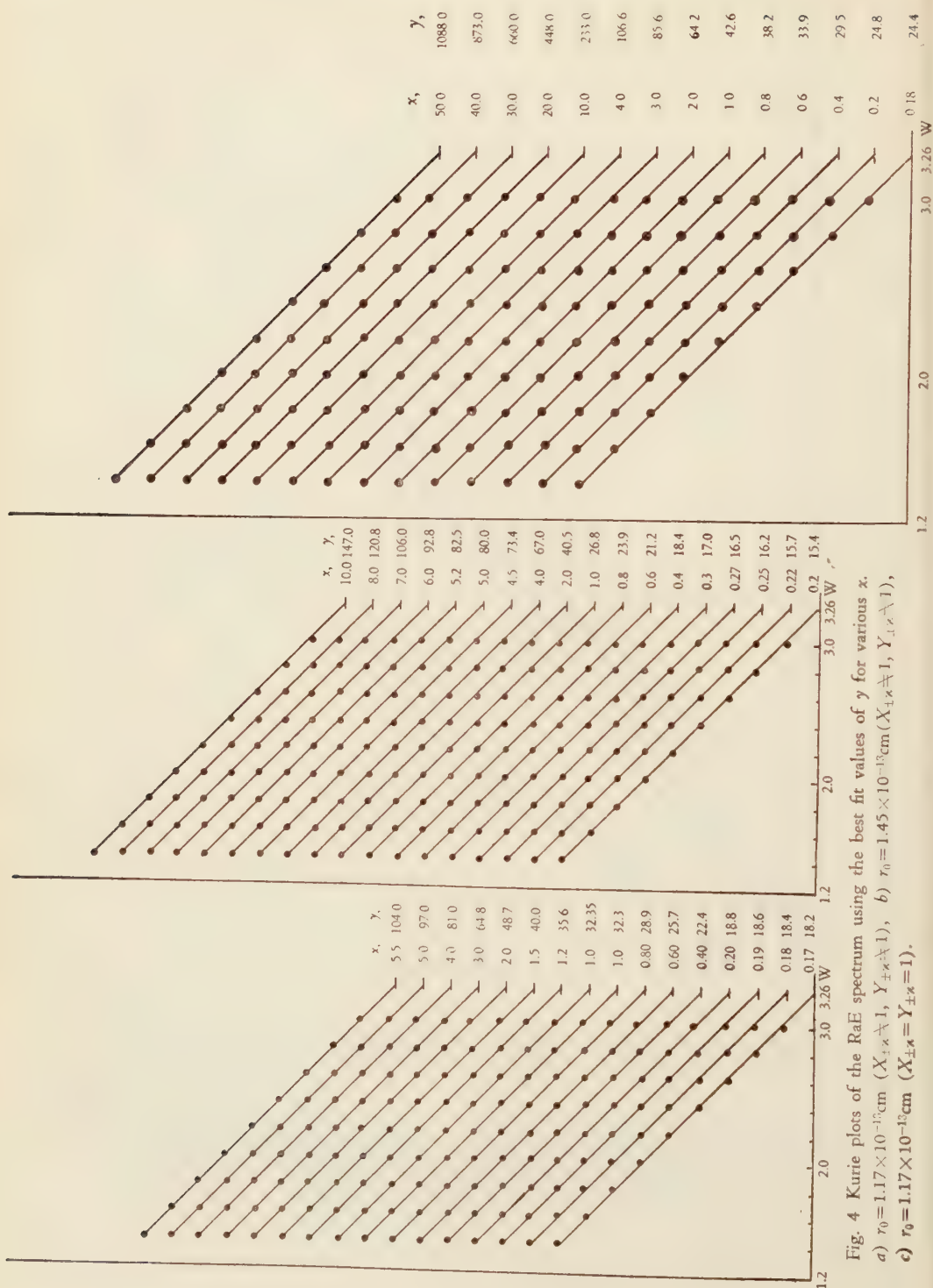


Fig. 4 Kurie plots of the RaE spectrum using the best fit values of γ for various x .

a) $r_0 = 1.17 \times 10^{-15} \text{ cm}$ ($X_{\pm\alpha} \approx 1$, $Y_{\pm\alpha} \approx 1$), b) $r_0 = 1.45 \times 10^{-15} \text{ cm}$ ($X_{\pm\alpha} \approx 1$, $Y_{\pm\alpha} \approx 1$),

c) $r_0 = 1.17 \times 10^{-15} \text{ cm}$ ($X_{\pm\alpha} = Y_{\pm\alpha} = 1$).

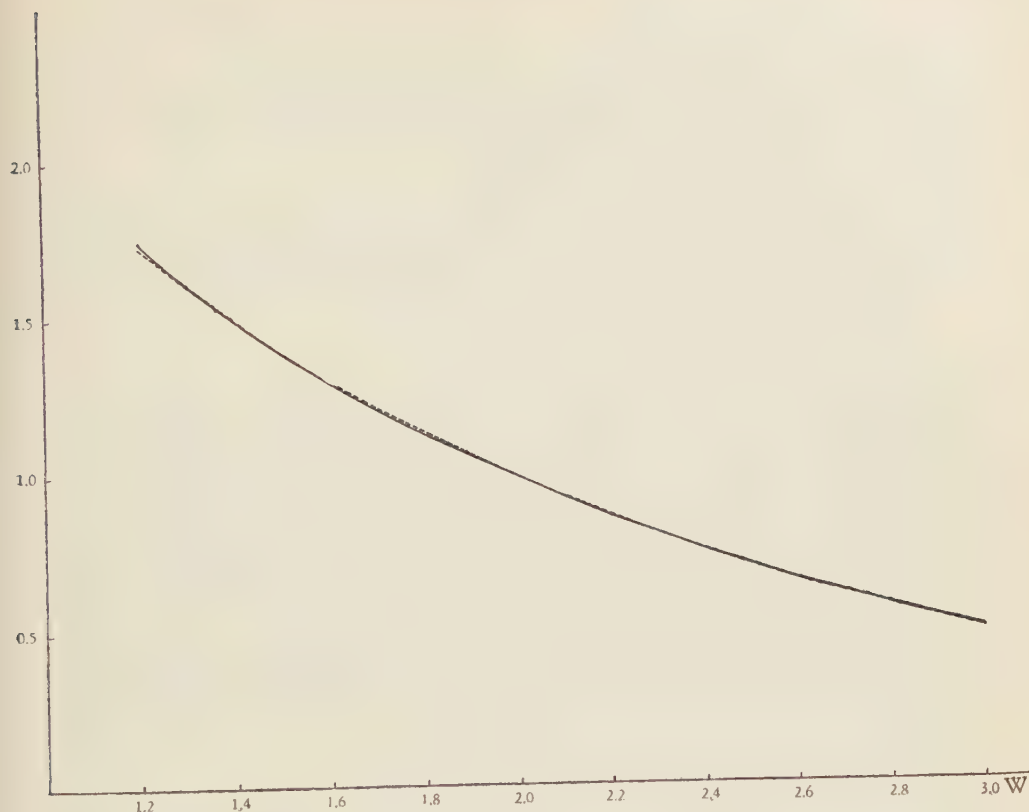


Fig. 5 The theoretical correction factor for $(\alpha=1, \gamma=32.35)$ is represented by the dotted line. The correction factor inferred from the experiment of Plassmann and Langer is given by the solid line. As far as Kurie plots, each of the two sets of parameters, $(\alpha=1, \gamma=32.35)$ and $(\alpha=1, \gamma=32.3)$ could yield a straight line equally well (See Fig. 4.), but the former set gives a better fit with the experiment than the latter, if we interpret them in the graph of correction factors. Anyhow, the differences between them may be covered by the experimental error or by the slight modification of R . Therefore, we have neglected the differences in the fourth effective figures of γ in this two sets. For any other sets of parameters (α, γ) , the differences in γ of this order are also neglected.

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On the Bose-Einstein Liquid Model for Liquid Helium, IV

— *A Revised Model* —

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The discrepancies between the modified Bose-Einstein theory developed in previous papers and some experimental facts are shown to be ascribed partly to the erroneous experimental results concerning the lambda-temperature shifts in mixtures, on which the theory was based, and partly to the inapplicability of an ideal gas model to pure liquid He^3 . According to the recent experiments on lambda-temperature shifts, the energy gap must be taken as proportional to the 0.4th power of the number density of He^4 "particles", instead of as proportional to the number density. It is also shown that the above relation between the energy gap and the number density is applicable not only for the mixture problem but also for the high pressure effect. In particular, the thermal expansion of liquid He^4 calculated on the present model is found to be negative and of the right order of magnitude. Application of a similar model for liquid He^3 predicts the fall with pressure of the Fermi-Dirac degeneracy temperature while the usual ideal-gas model predicts the pressure shift towards the inverse direction.

§ 1. Introduction

In a series of the present author's papers^{1,2,3,4,5)*} the modified Bose-Einstein liquid theory of He^4 was extended to the case of He^3 - He^4 mixtures and to the case of pure liquid He^4 at high pressures. The relation between the energy gap, Δ , and the mean number density of He^4 "particles", n_4/ν , was taken so as to fit the observed dependence of T_λ on the He^3 concentration in mixtures. Several authors^{6,7,8)} have recently shown that the earlier observations on lambda-temperature shift due to He^3 admixture were largely in error. If we take the relation between Δ and n_4/ν so as to fit those new experiments, it turns out to be $\Delta = \Delta_0 \{ (n_4/\nu) / (n_4^0/\nu_0) \}^{0.4}$ instead of a previously proposed one, $\Delta = \Delta_0 (n_4/\nu) / (n_4^0/\nu_0)$. The revised relation removes all the discrepancies between the earlier calculations and experiments in the cases of the specific heat and the velocity of second sound in mixtures. The discrepancy between the observed and previously calculated vapour pressures of solutions is shown to be due to the erroneous Δ vs. n_4 relation, on one hand, and to the non-ideal gas behaviour of pure liquid He^3 , on the other. The vapour pressures of dilute solutions calculated on the revised model are found to be in good agreement with observations when the former are derived by making use of the observed values for an arbitrarily chosen standard solution instead of those for pure He^3 .

*) The first three will be quoted as *I*, *II* and *III* respectively. The notations in these papers are kept in the present note.

The influence of pressure on the various properties of pure liquid He^1 can also be explained by applying the same relation between Δ and n_i/ν as that for mixtures. The value of ν is then a function of pressure, which is to be determined so as to give the lambda-temperature shift correctly. The thermal expansion coefficient calculated on the same model is negative and of the right order of magnitude at temperatures between 1.3°K and the lambda-point.

Applying a modified Fermi-Dirac gas model to liquid He^3 , it is suggested that the temperature of degeneration will be lowered by the application of high pressure.

§ 2. $\text{He}^3\text{-He}^4$ mixtures

The mass factor, ν , is assumed as independent of the concentration while the energy gap, Δ , between the ground and the lowest excited states is assumed to vary with the concentration through the relation

$$\Delta = \Delta_0 \{N_4 V_4 / (N_3 V_3 + N_4 V_4)\}^\kappa = \Delta_0 C'^\kappa, \quad (1)$$

where V_3 and V_4 are the molecular volumes in the mixture, and κ is determined so that the calculated value of $dT_\lambda/d\tilde{\epsilon}_3$ may be equal to the observed one. Table I shows the calculated values of $dT_\lambda/d\tilde{\epsilon}_3$ for different values of κ together with those observed by different authors. The ratio of the molecular volumes, V_3/V_4 , is taken as equal to 1.44,

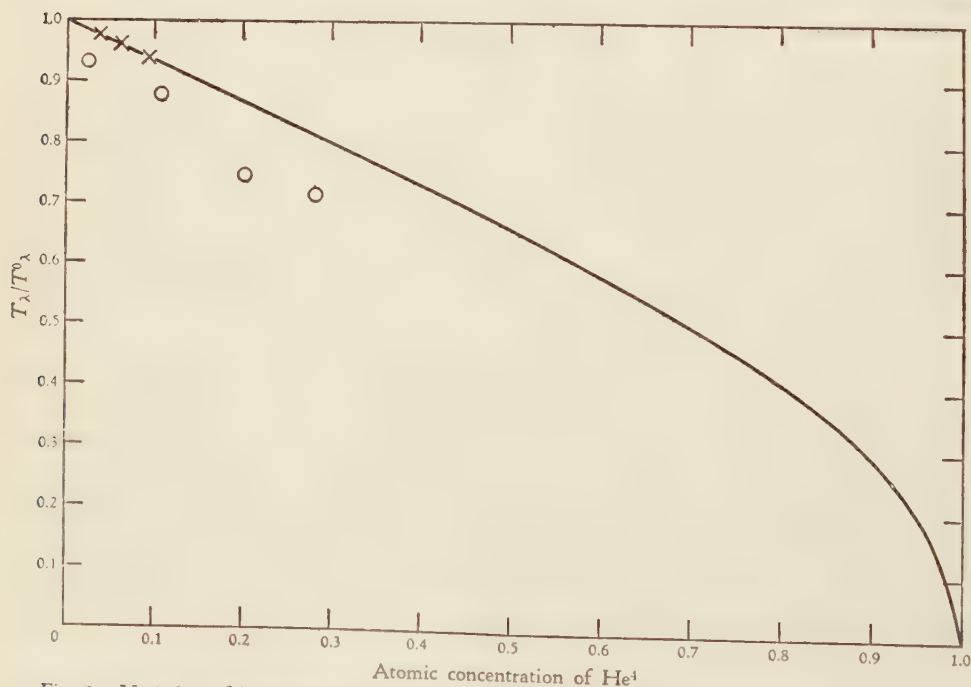


Fig. 1. Variation of lambda-point with He^3 concentration. \times : Experiments by Dash and Taylor.⁸⁾
 \circ : Experiments by Abraham *et al.*⁹⁾

Table I. Initial slopes $dT_\lambda/d\xi_3$, calculated for various values of κ and observed by various authors.

κ Observer $dT_\lambda/d\xi_3$ (deg/mole)	Theoretical			Experimental			
	1	2/3	0.4	Abraham <i>et al.</i> ⁽⁹⁾ -2.8	King & Fairbank ⁽⁶⁾ -1.50	Dokoupil <i>et al.</i> ⁽⁷⁾ -1.48	Dash & Taylor ⁽⁵⁾ -1.47

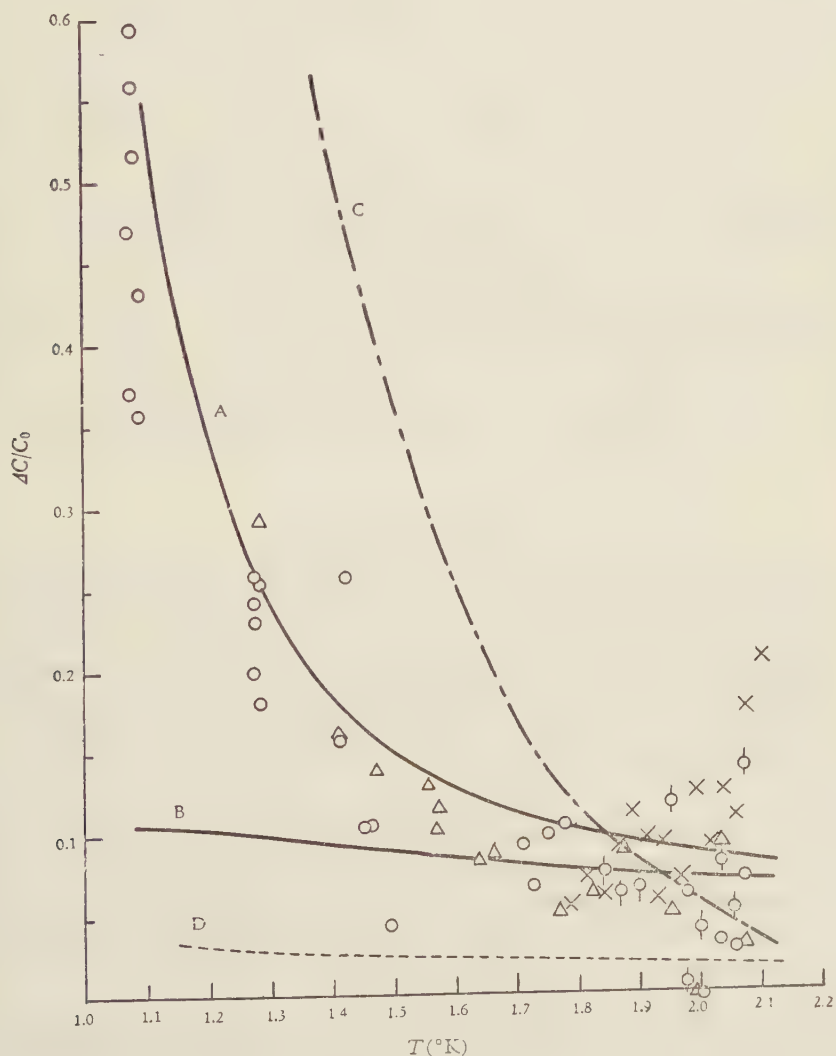


Fig. 2. The relative increase in the specific heat of a 2.5 percent solution. Curves A and B: modified B.E. theory assuming eq. (2) with and without including the contribution of pure He^3 . Curve C: de Boer-Gorter-Taconis theory. Curve D: Heer-Daunt theory. Experimental points: Dokoupil *et al.*⁽⁷⁾

which corresponds to the ratio of those of pure liquids at $T=2.186^\circ\text{K}$, the normal lambda-point of pure He^4 . The value of $dT_\lambda/d\hat{\epsilon}_3$ obtained by Abraham *et al.*⁽⁹⁾, on which the author's previous works were based, differs considerably from those of recent workers. As clearly seen from the table, κ should be 0.4 instead of unity. Thus eq. (1) leads to

$$\Delta = \Delta_0 C_4^{10.4}. \quad (2)$$

Fig. 1 shows the calculated $T_\lambda\text{-}\hat{\epsilon}_3$ plot compared with experiment. The value of V_3/V_4 is taken as 1.40 in calculating the whole $T_\lambda\text{-}\hat{\epsilon}_3$ diagram, because it seems reasonable for this purpose to use a value applicable to the widest range of concentration.

Various properties of mixtures will be discussed on the basis of the revised model, but, as to details of the calculation procedure, readers are referred to previous papers.

(i) *The specific heat* The calculations of the specific heat of a 2.5 percent solution based on the modified Bose-Einstein theory has already been reported¹¹⁾ and compared with the experiment of Dokoupil *et al.*⁽⁷⁾ However, the dependence of Δ on C_4' was there erroneously assumed as $\Delta = \Delta_0 C_4'^{2/3}$ and, moreover, the contribution from the phonon excitations was not included. The results assuming eq. (2) are shown in Fig. 2. The upper solid curve indicates the case, where the contribution of the He^3 component is represented by that of an ideal, nondegenerate Fermi-Dirac gas having the same particle density and mass as those for He^3 atoms. The lower solid curve shows the result when the contribution of pure He^3 is wholly neglected. Both include the phonon parts, which are considered as equal in pure and diluted He^4 . The phonon part is approximately obtained by the Debye T^3 -law

$$C_{ph} = 0.021 \times T^3 \text{ joule/g deg},$$

which is known to be exact below 0.6°K . Possible deviations from this in the high temperature region considered would change the result little, because the contribution of phonons to the total specific heat is not large there. The agreement of the upper curve with the experimental points of Dokoupil *et al.*⁽⁷⁾ is very satisfactory. For comparison, the results according to the de Boer-Gorter-Taconis theory¹¹⁾ and the Heer-Daunt ideal B.E. gas theory¹¹⁾ are also plotted. It is evident that the modified B.E. model assuming eq. (2) and an ideal, classical gas behaviour of He^3 is the best among the various models for the mixture.

(ii) *The velocity of second sound* The velocity of second sound in a solution, calculated in I assuming $\Delta = \Delta_0 C_4'$, was in qualitatively good agreement with experiment, but, as pointed out in that paper, there were unexplained discrepancies between theory and experiment. When the revised relation between Δ and C_4' represented by eq. (2) is adopted, the square of the velocity of second sound is approximately given by

$$v_{II}^2 = \frac{1}{1 + (x_3/x) - x_3} \left[\frac{1-x}{1-x_0} v_{II}^{02} - \frac{0.8}{3} \frac{V_3}{V_4} x_3 (1-x) (\Delta + T) \frac{1+5T/\Delta+25T^2/4\Delta^2}{1+5T/\Delta+15T^2/4\Delta^2} - \frac{RT}{3} (1-x_3) (1-x) \frac{x_3}{x} \right] \quad (3)$$

which corresponds to eq. (72), I. Applying a similar correction as in I to the theoretical values of α , we obtain the result plotted in Fig. 3. Agreement with the experiment of Lynton and Fairbank¹²⁾ has been much improved. Some authors¹³⁾ stated that the observed

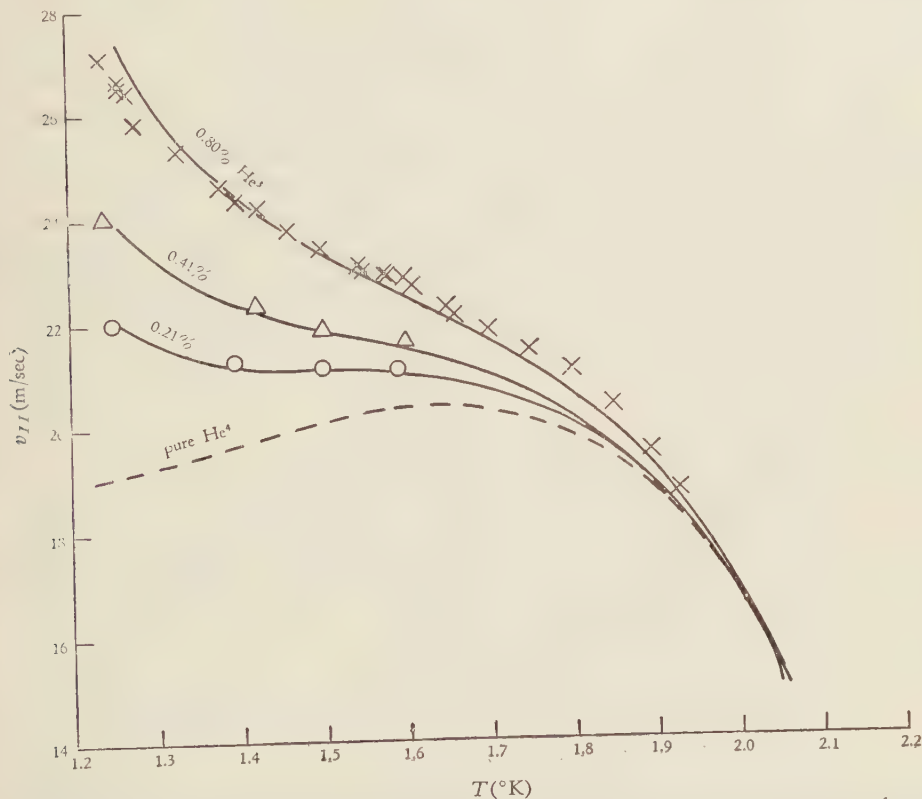


Fig. 3. The velocity of second sound in solutions as a function of temperature. Theoretical curves are calculated by eq. (3). Experimental points show the measurements by Lynton and Fairbank.¹²⁾

change of the velocity of second sound in a solution cannot be explained quantitatively without assuming an effective mass for He^3 particles largely different from the atomic one. The calculations presented here indicate, however, that it is unnecessary to assume a mass different from the true atomic one. Thus, the change of the velocity of second sound in solutions as well as that of the specific heat can be satisfactorily explained by adopting eq. (2) and assuming an ideal gas behaviour of the He^3 component.

(iii) *The vapour pressure* It will be shown that, in discussing the vapour pressure of a solution, eq. (2) leads to better results than assuming Δ as proportional to C_4' . We shall compare the partial pressures of He^3 calculated in different theories with those derived from the observed total vapour pressure. The method of the derivation of empirical values of p_3 was described in III. Since recent experiments on the nuclear magnetic susceptibility¹⁴⁾ and the specific heat¹⁵⁾ showed that pure liquid He^3 cannot be represented by an ideal Fermi-Dirac

gas with the number density and the mass of particles as equal to the atomic number density and the atomic mass (in this connection, see also a recent note by the author¹⁶⁾), the formulae for p/p_3^0 derived in **I** according to the modified B.E. theory are not justified. In the present note, let us first calculate p_i for any solution, not relative to p_3^0 , but relative to some standard solution, for which the vapour pressures are experimentally known as a function of temperature. Then p_3 can be calculated for a solution of an arbitrary concentration by using the empirical values of the standard solution, which can be obtained from the observed total pressure in the way described in **III**. As illustrations, we have calculated p_3 for solutions containing 9.49, 13.0, 20.3 and 25.5 atomic percent of He^3 , taking as the standard a 1.98 percent solution, on which Sommers¹⁷⁾ made accurate measurements. It is convenient to choose as the standard rather a low concentration solution, because it has a high lambda-temperature and, in consequence, a smooth p_3 - T curve in a wider range of temperature, on one hand, and because the validity of the present model has already been confirmed for a dilute solution by discussing the specific heat and the velocity of second sound, on the other hand.

The formulae employed are:

$$p_3 = [p_{3,s}] \frac{C'_3}{C'_{3,s}} \exp \left[C'_4 \left\{ 1 - \frac{1}{\nu} \frac{V_3}{V_4} \left(1 + \frac{0.4D}{T} \right) x \right\} \right] \\ \times \exp \left[-C'_{4,s} \left\{ 1 - \frac{1}{\nu} \frac{V_3}{V_4} \left(1 + \frac{0.4D_s}{T} \right) x_s \right\} \right], \quad T \leq T_\lambda (< T_{\lambda,s})$$

$$p_3 = [p_{3,s}] \frac{C'_3}{C'_{3,s}} \exp \left[C'_4 \left\{ 1 - \frac{1}{\nu} \frac{V_3}{V_4} \left(1 + \frac{0.4D}{T} \right) \right\} \right] \\ \times \exp \left[-C'_{4,s} \left\{ 1 - \frac{1}{\nu} \frac{V_3}{V_4} \left(1 + \frac{0.4D_s}{T} \right) x_s \right\} \right], \quad T_\lambda \leq T \leq T_{\lambda,s}$$

where those quantities with suffix s refer to a standard solution, and $[p_{3,s}]$ means the observed value. The value of V_3/V_4 seems to range from about 1.36 in the low temperature region to about 1.44 at the lambda-temperature of the solution. In the practical calculations, however, a fixed value of 1.40 was adopted for simplicity. The curves A in Fig. 4 show p_i calculated by the above procedure as a function of temperature. The empirical p_i - T plots are those derived from the observed total vapour pressure of Sommers¹⁷⁾ and Weinstock *et al*¹⁸⁾. The large discrepancies between theory and experiment as found in the earlier calculations do not exist in the present case. (Compare Fig. 4 with Figs. 1 and 2 in **III**.) Agreement is satisfactory and, in particular, the general aspects in the theoretical and experimental plots are very similar even in the vicinity of lambda-temperatures. If we apply the same procedure as described in **III** but assuming eq. (2), the discrepancies are found to be even larger than those found in **III**. This clearly shows that pure liquid He^3 cannot be represented by an ideal gas with a common energy spectrum while He^3 in a solution can be well represented by such a model at least up to 25 percent.

The similar calculations according to the de Boer-Gorter-Taconis and the Heer-

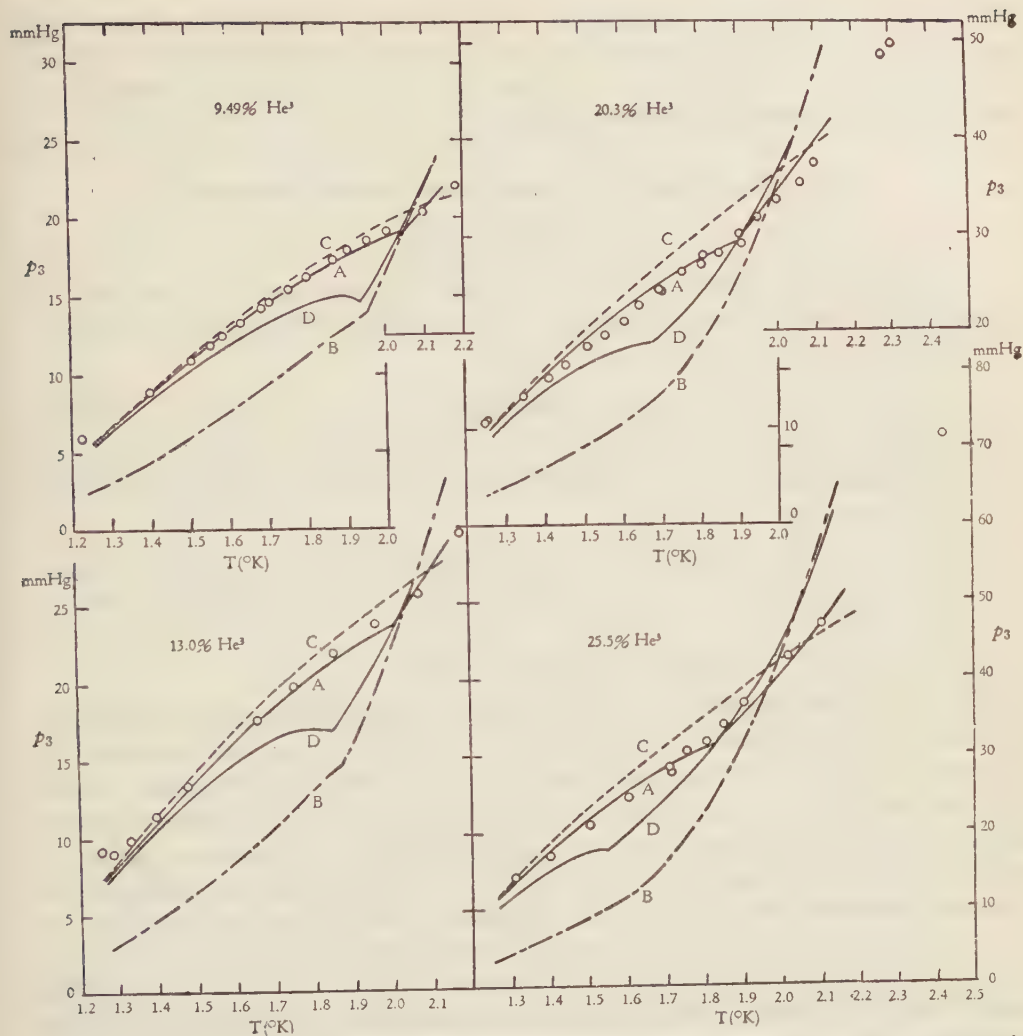


Fig. 4. The partial vapour pressure of He^3 as a function of temperature calculated from the experimental data for a 1.98 percent solution. A: modified B. E. theory assuming $\Delta = \Delta_0 C_4^{0.4}$ B: de Boer-Gorter-Taconis theory. C: Heer-Daunt theory. D: modified B. E. theory assuming $\Delta = \Delta_0 C_4'$. Experimental points for 9.49 and 13.0 percent He^3 are derived from the total vapour pressure data of Sommers¹⁷⁾, and those for 20.3 and 25.5 percent He^3 are derived from the data of Weinstock *et al.*¹⁸⁾

Daunt (ideal Bose-Einstein) models have also been done, and the results are plotted in Fig. 4 by dot-dashed and dotted curves respectively (Curves B and C). The former deviate from experiment considerably, and the latter are in moderate agreement but cannot predict the breaks in empirical plots. In a previous paper, *III*, it was pointed out that these breaks were found to occur at considerably higher temperatures than the lambda-temperatures deduced from the superflow experiment by Abraham *et al.*⁹⁾ It is now evident that the break-points in the experimental plots of p_3 vs. T nearly coincide with the lambda-tempera-

tures deduced from the recent experimental results concerning the shift of lambda-points with the admixture of He^3 . In order to indicate that the agreement in p - T plots is violated if the earlier relation between Δ and C_4' is used, the results of similar calculations on this assumption are also included (Curves D in Fig. 4).

§ 3. The pressure effect and the thermal expansion (pure liquid He^4)

In the preceding section, it was shown that the energy gap, Δ , in the liquid He^4 diluted with He^3 can be represented by eq. (2) as a function of the apparent density of He^4 particles. It may also be written in the form

$$\Delta = \Delta_0 \{ (n_4/\nu) / (n_4^0/\nu_0) \}^{0.4}. \quad (4)$$

In a previous paper, **II**, we used the relation $\Delta = \Delta_0 (n_4/\nu) / (n_4^0/\nu_0)$ instead of eq. (4), and an additional factor $(\nu/\nu_0)^{1/3}$ was needed in order to extend it so as to apply to liquid He^4 at high pressures. Thus we got

$$\Delta = \Delta_0 (\nu/\nu_0)^{1/3} (n_4/\nu) / (n_4^0/\nu_0). \quad (5)$$

Similarly, let us first put

$$\Delta = \Delta_0 (\nu/\nu_0)^\gamma \{ (n_4/\nu) / (n_4^0/\nu_0) \}^{0.4},$$

and then choose the value of γ so that the calculated values of the lambda-temperature and the velocity of second sound at some high pressure may be in as good agreement with experiment as possible. This procedure leads to the result that γ may be taken as zero. It means that the energy gap in pure He^4 under high pressure can be represented by the same function of the number density of He^4 particles, n_4/ν , as in mixtures. It is a remarkable fact that the energy gap depends on ν only through the number density of particles, but not explicitly on it. The effect of pressure on Δ can thus be expressed by

$$\Delta = \Delta_0 (\rho/\rho_0)^{0.4} (\nu/\nu_0)^{-0.4}, \quad (6)$$

where the quantities with suffix 0 correspond to any standard state. If we take the normal lambda-point as a standard, we have

$$\Delta_0 = 8.609^\circ\text{K}, \quad \nu_0 = 8.8, \quad \rho_0 = 0.1462 \text{ g/cc.}$$

Applying eq. (6) and requiring that the calculated lambda-temperatures are in agreement with the observed ones, we get the values for ν and Δ as listed in Table II. Comparison of these values with those listed in Table II, **II** shows that the two sets of ν and Δ differ only slightly from each other, and suggests that almost the same results as obtained in previous papers^{2,1)} can be expected when the values listed in Table II of this note are used. The velocity of second sound calculated by employing these values, when plotted as a function of pressure and temperature, can indeed be represented by nearly the same figure as found in **II** (Fig. 1).

There is, however, one particular case, where eqs. (4) and (5) lead to considerably

different results: it is the coefficient of thermal expansion, of which we have not spoken so far.

As simple thermodynamics shows, the coefficient of thermal expansion in the present model assuming eq. (4) or (6) is given by

$$\begin{aligned}\alpha &= -\rho(\partial\mathcal{S}/\partial p)_T \\ &= -\rho\mathcal{S}_R\left\{\frac{3}{2} + \frac{0.4\Delta}{T}\left(1 - \frac{1}{\Delta/T + 5/2}\right)\right\}\frac{1}{\nu}\frac{d\nu}{dp} \\ &\quad + \mathcal{S}_R\left\{1 + \frac{0.4\Delta}{T}\left(1 - \frac{1}{\Delta/T + 5/2}\right)\right\}\frac{1}{v_I^2} + \alpha_{ph},\end{aligned}\quad (7)$$

where \mathcal{S}_R is the entropy due to the Bose-Einstein excitations, v_I is the velocity of ordinary sound, and α_{ph} means the coefficient of expansion due to the excitations of longitudinal Debye waves. This formula can easily be obtained from eq. (2), II and eq. (6). The

Table II. The mass factor, ν , and the energy gap, Δ , as functions of pressure. They are determined from observed pressure-shift of lambda-temperature by using eq. (6).

p (atm)	0.05	5	10	15	20	25
ν	8.800	9.448	10.146	10.852	11.649	12.554
Δ (°K)	8.609	8.546	8.453	8.341	8.201	8.038

phonon part can be expressed by

$$\alpha_{ph} = \frac{16\pi^5 k^4}{15h^3\nu_I^3}\left(\frac{1}{\nu_I}\frac{\partial\nu_I}{\partial p} + \frac{1}{3\rho\nu_I^2}\right)T^3, \quad (8)$$

as shown by Atkins and Edwards¹⁹⁾. They also showed that, if the dependence on density of the three parameters is properly assumed in Landau's 1947 model, the calculated coefficient of expansion can be fitted to their measurements below 1.6°K. Our modified Bose-Einstein model is formally identical with Landau's 1941 roton model, but we have only one adjustable parameter $\nu^{-1}d\nu/dp$ because we have already imposed a relation between Δ and ν represented by eq. (6). The values of this parameter, moreover, cannot be chosen arbitrarily, as it should be consistent with the dependence of ν on p deduced from the shift of lambda-temperature due to pressure.

The theoretical curve in Fig. 5 shows the coefficient of thermal expansion *vs.* temperature relation obtained from eq. (7) by assuming

$$d\nu/dp = 0.144 \text{ atm}^{-1}. \quad (9)$$

For \mathcal{S}_R in eq. (7), we used the experimental values of Hercus and Wilks²⁰⁾ corresponding to $\rho = 0.1465 \text{ g/cc}$ minus the theoretical phonon entropy. In order to estimate \mathcal{S}_{ph} and α_{ph} at different temperatures, the T^3 -law is assumed but the observed values²¹⁾ were used at respective temperatures for ρ , v_I and $\partial v_I/\partial p$. A slight change of Δ with density was neglected.

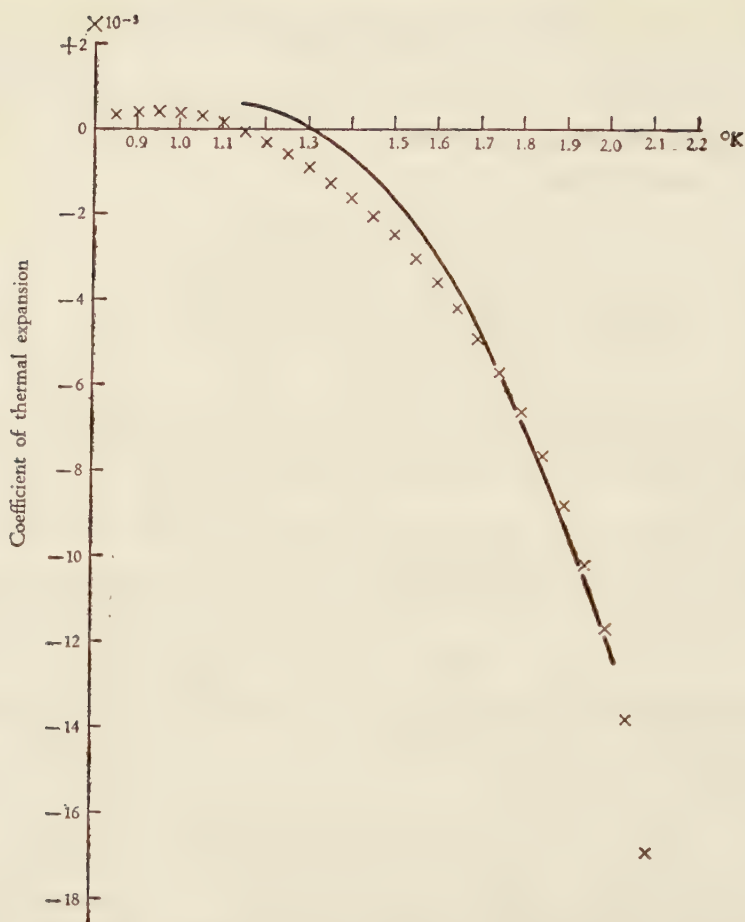


Fig. 5. The coefficient of thermal expansion of liquid He⁴ under its saturated vapour pressure as a function of temperature. The theoretical curve assumes $d\nu/dp=0.144 \text{ atm}^{-1}$. Crosses are the experimental points of Atkins⁽²²⁾.

In order to show that the value of $d\nu/dp$ required for explanation of observed thermal expansion is consistent with the dependence of ν on pressure obtained before, a straight line representing

$$\nu = 8.8 + 0.144(p - 0.05), \quad (p \text{ in atm}) \quad (10)$$

is drawn in Fig. 6a together with the points indicating the values of ν listed in Table II. The figure clearly shows that the different determinations are wholly consistent. For comparison, the result of a similar procedure assuming eq. (5) instead of eq. (4) or (6) is plotted in Fig. 6b. The value of $d\nu/dp$ required for a good fit of thermal expansion data is considerably larger than that consistent with the pressure dependence of ν determined from the lambda-point data. It may be said that the relation (4) yields better results than eq. (5) also in discussing the influence of pressure, for the thermal expansion can

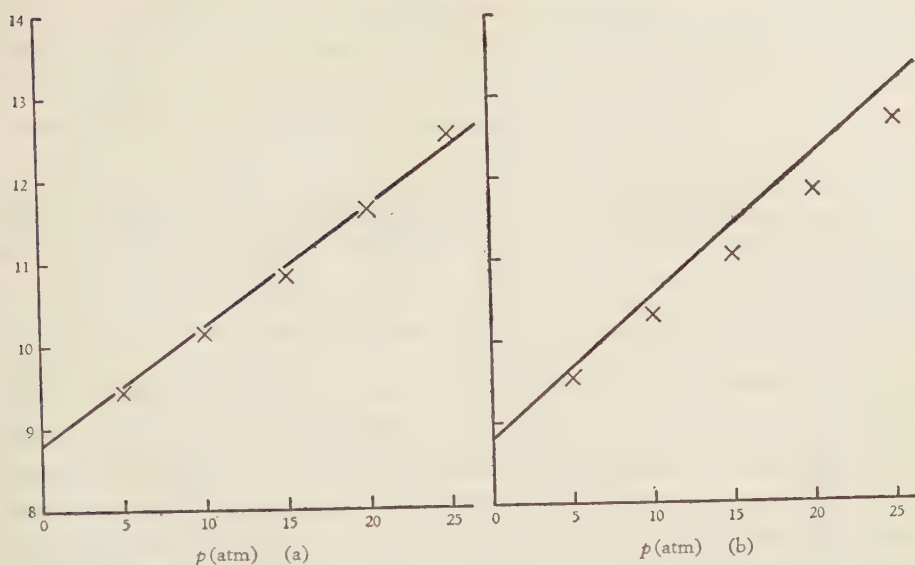


Fig. 6. ν as a function of pressure. The straight lines represent (a) $\nu=8.8+0.144 (p-0.05)$ and (b) $\nu=8.8+0.175 (p-0.05)$, which give the best fit of thermal expansion when eq. (4) and eq. (5) are assumed respectively. Crosses in (a) indicate the values listed in Table II, and those in (b) the values listed in Table II, II.

be regarded as an effect of pressure on entropy. Possible relations between the energy gap and the number density of particles will be further discussed in the last section.

The effect of pressure on the specific heat has already been dealt with in previous papers^{2,4)}. But, after they were published, a report on detailed measurements by Hercus and Wilks²⁰⁾ has appeared. It seems of value to compare their experiments with the present theory in the revised form.

In Table III, the theoretical and experimental ratios of the specific heats at a constant density, $\rho=0.1675$ g/cc, to that at nearly the saturated vapour pressure are listed. The theoretical values correspond to the specific heats due to the Bose-Einstein excitations, and the experimental ones are the ratios of the total specific heats minus the estimated phonon parts. In all the calculations, the change of pressure on the liquid with temperature and the change of sound velocity with pressure were taken into account. In the calculations of phonon parts the T^3 -approximation was used. The estimated pressure for $\rho=0.1675$ g/cc listed in Table III was derived from the isopycnal data of Keesom. The values of ν used for calculating theoretical specific heats were obtained by inserting the estimated pressure value into eq. (10). Moderate agreement between theory and experiment can be seen from the table. Similar results are also obtained for each of the other densities, for which the experimental data are available.

Table III. Relative increase of the specific heat of liquid He^4 due to pressure. The phonon contributions are subtracted from the observed specific heat. The theoretical values concern only the contributions of the Bose-Einstein excitations.

$$C_v(\rho=0.1675 \text{ g/cc})/C \text{ (sat. vap. pressure)}$$

$T(^{\circ}\text{K})$	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$\rho \text{ (atm)}$	19.1	19.0	18.9	18.7	18.4	18.1	17.7
Observed (Hercus & Wilks ²⁰)	1.39	1.42	1.40	1.37	1.38	1.39	1.44
Theoretical	1.57	1.53	1.48	1.44	1.39	1.36	1.34

§ 4. The pressure effect on the degeneracy temperature of liquid He^3

It has been shown in preceding sections that a modified gas model can successfully be applied to liquid He^4 . This fact suggests that a similar model can also be applied to liquid He^3 , apart from the difference of statistics. In fact, the author¹⁶⁾ has shown that the low degeneracy temperature and the temperature-dependence of the specific heat can consistently be explained by assuming $\nu_3 \approx 4$ for He^3 . The existence of an energy gap is not essential in this case, because the lowest state plays no important role in the Fermi-Dirac statistics. It seems worth noting that the degeneracy temperature would be shifted further to lower temperature by applying high pressure, while an ideal gas model predicts the shift to the inverse direction.

The degeneracy temperature is, in fact, obtained from

$$T_e = h^2 / 8\nu_3 m_3 k \cdot (3n_3 / \pi\nu_3)^{2/3},$$

and, if the relative increase of ν_3 due to pressure is greater than that of the atomic number density as was the case for He^4 , T_e must decrease with pressure. It may also be suggested that the high-temperature limiting value of the excess specific heat over the Debye one would decrease with pressure, because it is given by $3R/2\nu_3$ per mole.

§ 5. Discussion

The assumption that ν has the same value both in pure and diluted liquid He^4 has been shown in § 2 to give satisfactory results in predicting various properties of mixtures. But it should be remarked that the assumption of concentration-dependent ν and constant J would give nearly as good results as the above mentioned in all problems discussed in this paper, provided that ν is determined as a function of concentration so as to give the lambda-temperature correctly. This requires us, however, to assume a large increase of ν with concentration, which seems unreasonable because ν seems to become unity in a very concentrated solution of He^3 in He^4 , i.e., in a very dilute solution of He^4 in He^3 . Besides, this model is not easy to extend in a physically reasonable way to discuss the high pressure effects. The difference between the predictions according to the above two models seems large enough to be experimentally investigated only in the case of the normal fluid fraction,

α . The calculated values of this quantity for mixtures containing 10 and 20 percent He^3 are listed in Table IV, second and third columns. The normal fluid fraction ratio increases with decreasing temperature in one case, while it remains constant in the other.

The IV. Normal fluid fraction in mixtures calculated on different models.

α/x_0 for a 10 percent solution					α/x_0 for a 20 percent solution				
T (°K)	mod. B. E. $\nu = \text{const.}$	mod. B. E. $\lambda = \text{const.}$	Heer- Daunt ^b	deBoer- Gorter	T (°K)	mod. B. E. $\nu = \text{const.}$	mod. B. E. $\lambda = \text{const.}$	Heer- Daunt [*]	deBoer- Gorter
1.2	1.76	1.46	1.11	7.12	1.2	3.04	2.25	1.24	11.57
1.3	1.71	1.46	1.11	5.58	1.3	2.86	2.26	1.24	8.94
1.4	1.66	1.46	1.11	4.48	1.4	2.66	2.26	1.24	7.02
1.5	1.62	1.46	1.11	3.66	1.5	2.58	2.26	1.24	5.66
1.6	1.59	1.46	1.11	3.06	1.6	2.48	2.26	1.24	4.63
1.7	1.56	1.46	1.11	2.62	1.7	2.39	2.26	1.24	3.84
1.8	1.53	1.46	1.11	2.28	1.8	2.32	2.26	1.24	
1.9	1.51	1.46	1.11	2.03					
2.0	1.49	1.46	1.11						

* These values were calculated by inserting empirical values of lambda-temperatures into $\alpha/x_0 = (T_\lambda^\circ/T_\lambda)^{3/2}$.

The experimental determination of this fraction from the measurement of the velocity of second sound or the Rayleigh disc torque is probably difficult in a mixture because of the complexity of the phenomena. The difference between the predictions according to the above two models seems, however, to be large enough to be examined by an experiment similar to that of Andronikashvili⁽²³⁾ or Hollis-Hallett⁽²³⁾ if a solution of rather a high concentration is used. This type of experiment in mixtures is now being conducted at Los Alamos by Dash and Taylor,⁽⁸⁾ whom the author would like to thank for informing him of this fact.

In Table IV the predictions according to other well-known theories⁽¹¹⁾ of mixtures are also included for the sake of comparison.

Finally, it is to be remarked that the decrease of ν with He^3 concentration does not seem improbable, because, as has already been pointed out, the value of ν is supposed to become unity in highly diluted He^4 , as is the case for He^3 . If the shift of lambda-temperature is kept in agreement with experiment, this dependence of ν on concentration changes very little the properties of our model solutions unless it is very strong, and it seems difficult to find it experimentally. Until any satisfactory molecular theory is established, it may be allowed to adopt the simplest assumption among all the alternatives which yield practically the same results in most problems.

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A New Approach to Quantum-Statistical Mechanics

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A new method of calculating the grand partition function of many-body system is developed, adopting extensively the techniques of calculus in quantum field theory. It is shown that the grand partition function, which is a trace of the density matrix expressed in terms of field operators, can be evaluated in a way almost parallel with the evaluation of the vacuum expectation value of the S-matrix in quantum field theory, provided that appropriate modifications in notation and definitions are made. As an example, the method is applied to electron-phonon system. Further, basing on this new formalism, various non-perturbational methods are discussed.

§ 1. Introduction

Stimulated by the studies of cooperative phenomena in quantum statistical system such as ferro- and antiferromagnetism, superconductivity, the λ -transition in liquid helium etc., various methods for the calculation of the partition function of many-body system with interaction have been put forward by many authors. These methods of attack have each its own merit, and in some cases they have been fairly successfully applied to practical problems. For instance, Kubo established an expansion theorem of the density matrix and applied it to ferro- and antiferromagnetism.¹⁾ Schafroth, in his theory of Meissner effect in superconductors, derived a formula in which the density matrix was expressed in powers of the interaction Hamiltonian.²⁾ Essentially the same formula for the expansion of the density matrix was also obtained by a different method by Chester, who made use of it to discuss the Bose-Einstein condensation of imperfect Bose gas.³⁾ A quite different way than others to handle the density matrix was invented by Feynman (the method of integral over trajectories), and he applied it to the problem of liquid helium⁴⁾. More recently, Friedman and Butler introduced another technique of manipulating the density matrix and thereby discussed the transition in liquid helium quantitatively.⁵⁾

Generally speaking, however, it seems that major efforts have been made so far to overcome the difficulties encountered in treating the interaction in many-body system, so that there remain unsolved difficulties in taking account of the effect of statistics, especially for Fermion system such as electrons in a superconductor and liquid helium 3, in both of which the role of Fermi statistics seems to be important. To remedy this point, it may be promising to use the number representation of the second quantization theory for the calculation of the trace of the density matrix. From this view point Ichimura developed a method of expanding the grand partition function in powers of the coupling constant using the number representation.⁶⁾ It appears, however, to the present author that his method is unsatisfactory

in the following two points: First it will not be practical in evaluating higher order corrections, because troublesome calculation of an enormous number of terms are needed; Therefore, application of this method will be restricted only to cases in which the effect of higher order perturbations is unimportant. Secondly, while the number representation may be most convenient to take into account the effect of statistics, it has such a defect that it is difficult to treat by this method the problem in configuration space, namely, it is not easy by this method to utilize physical pictures connected with the configuration space. For instance, one cannot utilize the quantity such as the molecular distribution function, which has been useful for the understanding of the cooperative phenomena in classical system.

In this paper we shall present a new approach which seems to be free from the above mentioned shortcomings of the n -representation. We introduce explicitly the quantized field of particles and utilize the various techniques of operator calculus in quantum field theory as far as possible in evaluating the quantum-statistical average of the field quantities. In § 2 and § 3 we give a general formulation of our theory for an example of electron-phonon system. Various results obtained by means of this new method for electron-phonon system are illustrated in § 4. In § 5 non-perturbational treatments are discussed, starting from the formulation given in § 2 and § 3. The last section is devoted to a possible extension of our method to other systems.

§ 2. General formulation

We suppose that the Hamiltonian of a system in question can be divided into two parts

$$H = H_0 + H_1, \quad (2.1)$$

each of which is expressed in terms of field quantity as

$$\begin{aligned} H_0 &= \int H_0(\mathbf{x}) d^3\mathbf{x}, \\ H_1 &= \int H_1(\mathbf{x}) d^3\mathbf{x}. \end{aligned} \quad (2.2)$$

We shall call $H_0(\mathbf{x})$ the Hamiltonian density of free field and $H_1(\mathbf{x})$ the density of interaction Hamiltonian. The density matrix of a canonical ensemble $\rho = \exp(-\beta H)$ has to satisfy the Bloch equation

$$-\partial\rho/\partial\beta = (H_0 + H_1) \cdot \rho, \quad \beta = 1/kT. \quad (2.3)$$

If we put

$$\exp(-\beta H) = \exp(-\beta H_0) \cdot S(\beta), \quad (2.4)$$

the equation for $S(\beta)$ becomes

$$-\partial S(\beta)/\partial\beta = H_1(\beta) \cdot S(\beta), \quad (2.5)$$

where

$$H_1(t) = \exp(tH_0) \cdot H_1 \cdot \exp(-tH_0). \quad (2.6)$$

The solution of (2.5) with initial condition $S(0)=1$ may be written as

$$S(\beta) = \sum_{n=0}^{\infty} (-1)^n \int_0^{\beta} dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \dots \int_0^{t_{n-1}} dt_n H_1(t_1) H_1(t_2) \dots H_1(t_n) \\ = \sum_{n=0}^{\infty} (-1)^n / n! \int \dots \int P[H_1(t_1) \dots H_1(t_n)] dt_1 \dots dt_n, \quad (2.7)$$

where P is an ordering operator⁷ which re-arranges the operators in the bracket in such an order that the arguments t in them are decreasing in magnitude, that is,

$$P[H_1(t_1) \dots H_1(t_n)] = H_1(t_1') H_1(t_2') \dots H_1(t_n') \\ t_1' > t_2' > \dots > t_n'. \quad (2.8)$$

If we regard

$$\exp(tH_0) \cdot H_1(x) \cdot \exp(-tH_0) = H_1(x, t) \equiv H_1(x) \quad (2.9)$$

as an operator in the four dimensional space with coordinates $x=(x, t)$, then (2.7) can be put into another form

$$S(\beta) = \sum_{n=0}^{\infty} (-1)^n / n! \int \dots \int P[H_1(x_1) \dots H_1(x_n)] d^4x_1 \dots d^4x_n, \quad (2.10)$$

in which $d^4x = d^3x dt$ and the integrations are to be taken over the whole volume with respect to x_i and over the range $(0, \beta)$ with respect to t_i . P is now an operator arranging the operators in the bracket in such an order that the fourth components of coordinates in them are decreasing in magnitude.

The grand partition function of the system is defined by

$$\Xi = Tr[\exp(-\alpha N - \beta H)], \quad (2.11)$$

where N is an operator representing the total number of particles, say, of electrons, and α a selector which is related to the chemical potential per one particle μ through

$$\alpha = -\beta\mu. \quad (2.12)$$

Introducing the following notation

$$\Xi_0 = Tr[\exp(-\alpha N - \beta H_0)], \quad (2.13)$$

$$\langle \dots \rangle = Tr[\exp(-\alpha N - \beta H_0) \dots] / Tr[\exp(-\alpha N - \beta H_0)],$$

we rewrite (2.11) in the following forms:

$$\Xi / \Xi_0 = \langle S(\beta) \rangle \quad (2.14a)$$

$$= 1 + \xi_1 + \xi_2 + \xi_3 + \dots \quad (2.14b)$$

$$= \exp(C_1 + C_2 + C_3 \dots), \quad (2.14c)$$

where

$$\xi_n = (-1)^n / n! \int \dots \int \langle P[H_1(x_1) \dots H_1(x_n)] \rangle d^4x_1 \dots d^4x_n. \quad (2.15)$$

The relations between ξ_n 's and C_k 's are essentially the same as those between the moments and Thiele's semi-invariants in the theory of probability, that is,

$$\begin{aligned}\xi_n &= \sum_{\{m_l\}=n} \Pi(C_l)^{m_l}/m_l!, \\ C_k &= \sum_{\{m_l\}=k} (-1)^{(\sum m_l-1)} (\sum_l m_l-1)! \Pi(\xi_l)^{m_l}/m_l!. \end{aligned} \quad (2.16)$$

Ξ represents the grand partition function of the free system in which the interactions are absent. $\langle \dots \rangle$ means the quantum-statistical average of a given field quantity denoted by dots referred to the thermal equilibrium realized in the free system.

In order to facilitate the explanation of our further analyses, we consider hereafter as an example the electron-phonon system, whose Hamiltonian is given by⁸⁾

$$H = H_0 + H_1, \quad (2.17a)$$

$$H_0 = \sum_k \epsilon_k a_k^* a_k + \frac{1}{2} \sum_w \hbar \omega_w (b_w^* b_w + b_w b_w^*), \quad (2.17b)$$

$$\begin{aligned} H_1 &= g \sum_{k,w} (\hbar \omega_w / 2V)^{1/2} (a_{k+n}^* a_k b_w + a_{k-n}^* a_k b_w^*) \\ &\quad + (g'/2) \sum_w \hbar \omega_w (b_w^* b_w + b_w b_w^* + b_w b_{-w} - b_w^* b_{-w}^*), \end{aligned} \quad (2.17c)$$

where a_k^* and a_k represent, respectively, the creation and annihilation operators of the electron with momentum k and energy ϵ_k , b_w^* and b_w are the corresponding operators for the phonon with momentum w and energy $\hbar \omega_w$, g and g' are, respectively, a coupling constant and a renormalization constant given by

$$g = (VC^2/NMs^2)^{1/2}, \quad g' = (s_0^2 - s^2)/2s^2. \quad (2.18)$$

The meaning of the symbols appearing in (2.18) is as follows: V is the volume of the system, M the mass of an ion, N the total numbers of ions, C the usual interaction constant between electron and lattice, s_0 the sound velocity of free phonons, s , is generally different from the real velocity s because there are interactions between electrons and phonons.

Now let us define quantized wave functions of electrons and phonons by

$$\begin{aligned}\psi^*(x) &= V^{-1/2} \sum_k a_k^* e^{-ik \cdot x}, \\ \psi(x) &= V^{-1/2} \sum_k a_k e^{ik \cdot x}, \\ \varphi(x) &= \sum_w (b_w / 2V)^{1/2} (b_w e^{iw \cdot x} + b_w^* e^{-iw \cdot x}).\end{aligned} \quad (2.19)$$

Making use of the commutation relations

$$[a_k, a_{k'}^*]_+ = \delta_{k,k'}, \quad [b_w, b_{w'}^*]_- = \delta_{w,w'},$$

four dimensional fields $\psi^*(x) = \exp(iH_0) \psi^*(x) \exp(-iH_0)$ etc. are easily shown to become

$$\psi^*(x) = V^{-1/2} \sum_k a_k^* e^{-ik \cdot x + \epsilon_k t},$$

$$\psi(x) = V^{-1/2} \sum_K a_K e^{iK \cdot x - \varepsilon_K t}, \quad (2.20)$$

$$\varphi(x) = \sum_w (\hbar w s / 2V)^{1/2} (b_w e^{i w \cdot x - \hbar w s t} + b_w^* e^{-i w \cdot x + \hbar w s t}).$$

Furthermore one can verify by a direct calculation that

$$e^{tH_0} H_1 e^{-tH_0} \equiv H_1(t) = g \int \psi^*(x) \psi(x) \varphi(x) d^3x + g' \int \varphi(x) \varphi(x) d^3x, \quad (2.21)$$

or

$$H_1(x) = g \psi^*(x) \psi(x) \varphi(x) + g' \varphi(x) \varphi(x).$$

From (2.14), (2.15) and (2.21), it can be seen that those which we have to know are rules for calculating the averages such as

$$\langle P[\psi^*(x_1) \psi(x_1) \psi^*(x_2) \psi(x_2) \cdots \psi^*(x_n) \psi(x_n)] \rangle \quad (2.22)$$

and

$$\langle P[\varphi(x_1) \varphi(x_2) \cdots \varphi(x_n)] \rangle.$$

We want to emphasize here that a remarkable similarity exists between the evaluation of E/Ξ_0 and that of the vacuum expectation of the so-called S-matrix in quantum field theory. In fact, it will be shown in the next section that all the rules of calculations of the vacuum expectation of the field quantities in quantum field theory can be used in the present case with only slight modifications.

§ 3. Computation rules⁽⁷⁾⁽⁹⁾⁽¹⁰⁾

It will be found convenient in later analyses to use in place of the operator P in (2.22) another operator T defined by⁹⁾

$$T = \delta_p P, \quad (3.1)$$

where δ_p takes 1 or -1 according as the character of the permutation of the electron operators involved is even or odd in going from the written order to the one re-arranged by P . Of course it holds that

$$P[\mathfrak{S}(\psi^* \psi)] = T[\mathfrak{S}(\psi^* \psi)], \quad (3.2)$$

if $\mathfrak{S}(\psi^* \psi)$ is any functional of a product of $\psi^*(x_i) \psi(x_i)$'s as in (2.22). Now let us find the computation rules for $\langle T[\mathfrak{S}(\psi^* \psi)] \rangle$ and $\langle P[\mathfrak{F}(\varphi)] \rangle$, where $\mathfrak{F}(\varphi)$ is any functional of a product of $\varphi(x)$'s. In the first place, we decompose $\psi^*(x)$, $\psi(x)$ and $\varphi(x)$ into two parts respectively according to

$$\begin{aligned} \psi^*(x) &= \psi_+^*(x) + \psi_-^*(x), \\ \psi(x) &= \psi_+(x) + \psi_-(x), \\ \varphi(x) &= \varphi_+(x) + \varphi_-(x). \end{aligned} \quad (3.3)$$

For a given product $X_1 X_2 \cdots X_n$, where X_i is any one of the components introduced in (3.3), we define an N -product by

$$N[X_1 X_2 \cdots X_n] = \delta_p X_{i_1} X_{i_2} \cdots X_{i_n}, \quad (3.4)$$

in which the right hand side is a product of the same factors $X_1 X_2 \cdots X_n$ but ordered in such a manner that all the operators with suffix $-$ stand to the left of all the operators with suffix $+$ and, among the electron operators with the same suffix, all the operators with $*$ stand to the left of those without $*$. δ_p determines the sign of the permutation in the same way as in (3.1).*) For instance,

$$\begin{aligned} N[\psi^*(x) \psi(x')] &= N[\psi_-^*(x) \psi_+(x') + \psi_+^*(x) \psi_-^*(x') + \psi_-^*(x) \psi_+(x') + \psi_+^*(x) \psi_-^*(x')] \\ &= \psi_+^*(x) \psi_+(x') - \psi_-(x') \psi_+^*(x) + \psi_-^*(x) \psi_+(x') + \psi_-^*(x) \psi_-(x'), \\ N[\psi^*(x_1) \psi^*(x_2) \psi(x_3)] &= \psi_-^*(x_1) \psi_-^*(x_2) \psi_-(x_3) + \psi_-^*(x_1) \psi_-^*(x_2) \psi_+(x_3) \\ &\quad - \psi_-^*(x_1) \psi_-(x_3) \psi_+^*(x_2) + \psi_-^*(x_1) \psi_+^*(x_2) \psi_+(x_3) + \psi_-^*(x_2) \psi_-(x_3) \psi_+^*(x_1) \\ &\quad - \psi_-^*(x_2) \psi_+^*(x_1) \psi_+(x_3) + \psi_-(x_3) \psi_+^*(x_1) \psi_+^*(x_2) + \psi_+^*(x_1) \psi_+^*(x_2) \psi_+(x_3), \\ N[\varphi(x_1) \varphi(x_2)] &= \varphi_-(x_1) \varphi_-(x_2) + \varphi_-(x_1) \varphi_+(x_2) + \varphi_-(x_2) \varphi_+(x_1) + \varphi_+(x_1) \varphi_+(x_2), \\ N[\varphi(x_1) \varphi(x_2) \varphi(x_3)] &= \varphi_-(x_1) \varphi_-(x_2) \varphi_-(x_3) + \varphi_-(x_1) \varphi_-(x_2) \varphi_+(x_3) \\ &\quad + \varphi_-(x_1) \varphi_-(x_3) \varphi_+(x_2) + \varphi_-(x_1) \varphi_+(x_2) \varphi_+(x_3) + \varphi_-(x_2) \varphi_-(x_3) \varphi_+(x_1) \\ &\quad + \varphi_-(x_2) \varphi_+(x_1) \varphi_+(x_3) + \varphi_-(x_3) \varphi_+(x_1) \varphi_+(x_2) + \varphi_+(x_1) \varphi_+(x_2) \varphi_+(x_3), \end{aligned}$$

and so on. Then we can show that an arbitrary T product of electron and phonon field operators is always converted to its corresponding N product through a simple relation. For $T[\psi^*(x) \psi(x')]$, it is easy to verify by a direct calculation that

$$T[\psi^*(x) \psi(x')] = \begin{cases} N[\psi^*(x) \psi(x')] + [\psi_+^*(x), \psi_-(x')]_+ & t > t' \\ N[\psi^*(x) \psi(x')] + [\psi_+^*(x), \psi_-(x')]_+ - [\psi_+^*(x), \psi_-(x')]_+ & t < t'. \end{cases} \quad (3.5)$$

For $P[\varphi(x) \varphi(x')]$, we get

$$P[\varphi(x) \varphi(x')] = \begin{cases} N[\varphi(x) \varphi(x')] + [\varphi_+(x), \varphi_-(x')]_- & t > t' \\ N[\varphi(x) \varphi(x')] + [\varphi_+(x'), \varphi_-(x)]_- & t < t'. \end{cases} \quad (3.6)$$

Thus if we define two functions $S(x-x')$ and $D(x-x')$ by

$$S(x-x') = \begin{cases} [\psi_+^*(x), \psi_-(x')]_+ & t > t' \\ [\psi_+^*(x), \psi_-(x')]_+ - [\psi^*(x), \psi(x')]_+ & t < t' \end{cases} \quad (3.7)$$

) Note that there exists a slight difference between our definition of N -product and that used in quantum field theory. This difference arises from the fact that $\psi_+^(x)$ and $\psi_+(x)$ (or $\psi_-^*(x)$ and $\psi_-(x)$) do not necessarily anti-commute with each other, and therefore we have to take care of their order in the product.

$$D(x-x') = \begin{cases} [\varphi_+(x), \varphi_-(x')]_- & t > t' \\ [\varphi_+(x'), \varphi_-(x)]_- & t < t', \end{cases} \quad (3.8)$$

the T -product with two factors is expressed as a sum of the corresponding N -product and S or D function :

$$T[\psi^*(x)\psi(x')] = N[\psi^*(x)\psi(x')] + S(x-x'), \quad (3.9)$$

$$P[\varphi(x)\varphi(x')] = N[\varphi(x)\varphi(x')] + D(x-x'). \quad (3.9b)$$

Trying similar calculations for the T -products with more factors than two, we are led to a conclusion that any T -product can be expressible as a sum of terms, each of which is composed of an N -product multiplied by S or D functions. More correctly, this statement will be mathematically expressed in the following two lemmas:¹⁰⁾

Lemma I For any product of $\varphi(x)$'s denoted by $\mathfrak{F}(\varphi)$, it holds that

$$T[\mathfrak{F}(\varphi)] = N[\mathfrak{F}(\varphi')], \quad (3.10)$$

where $\varphi'(x)$ is defined by

$$\varphi'(x) = \varphi(x) + \int d^4x' D(x-x') \delta/\delta\varphi(x'). \quad (3.11)$$

$\delta/\delta\varphi(x)$ is an operator characterized by the commutation relation,

$$[\delta/\delta\varphi(x), \varphi(x')]_- = \delta(x-x'), \quad (3.12)$$

that is, an operator representing functional differentiation with respect to $\varphi(x)$.

Lemma II For any product of $\psi^*(x)$'s and $\psi(x)$'s, which is denoted by $\mathfrak{G}(\psi^*, \psi)$, it holds that

$$T[\mathfrak{G}(\psi^*, \psi)] = N[\mathfrak{G}(\psi^{*'} \psi')], \quad (3.13)$$

where $\psi^{*'}(x)$ and $\psi'(x)$ are defined by

$$\psi^{*'}(x) = \psi^*(x) + \int d^4x' S(x-x') \delta/\delta\psi(x'), \quad (3.14)$$

$$\psi'(x) = \psi(x) - \int d^4x' S(x'-x) \delta/\delta\psi^*(x'),$$

respectively. $\delta/\delta\psi^*(x)$ and $\delta/\delta\psi(x)$ are operators characterized by the following commutation relations :

$$\begin{aligned} [\delta/\delta\psi(x), \psi(x')]_+ &= [\delta/\delta\psi^*(x), \psi^*(x')]_+ = \delta(x-x'), \\ [\delta/\delta\psi(x), \psi^*(x')]_+ &= [\delta/\delta\psi^*(x), \psi(x')]_+ = 0. \end{aligned} \quad (3.15)$$

A proof of these lemmas will be given in appendix. (3.9) are the special cases of (3.10) or (3.13). It should be noted that the above results are valid for an arbitrary choice of the manner of decomposition (3.3). We can, therefore, decompose the field operators in such a way that the resulting computation rules becomes as simple as possible. A possible good choice will be such as to make the averages of N -products $\langle N[\psi^*(x)\psi(x')] \rangle$ and

$\langle N[\varphi(x)\varphi(x')] \rangle$ vanish :

$$\langle N[\psi^*(x)\psi(x')] \rangle = 0, \quad \langle N[\varphi(x)\varphi(x')] \rangle = 0. \quad (3 \cdot 16)$$

This choice yields for $S(x-x')$ and $D(x-x')$ the results of the form :

$$S(x-x') = \begin{cases} V^{-1} \sum_k f_k e^{-i k \cdot (x-x') + \varepsilon_k (t-t')} & t > t', \\ V^{-1} \sum_k (f_k - 1) e^{-i k \cdot (x-x') + \varepsilon_k (t-t')} & t < t', \end{cases} \quad (3 \cdot 17a)$$

$$D(x-x') = \sum_w (\hbar \omega_s / 2V) [(N_w + 1) e^{i w \cdot (x-x') - \hbar \omega_s |t-t'|} + N_w e^{-i w \cdot (x-x') - \hbar \omega_s |t-t'|}], \quad (3 \cdot 17b)$$

where f_k and N_w represent the average numbers of free electrons with momentum k and of free phonons with momentum w in thermal equilibrium at temperature T , namely

$$f_k = \langle a_k^* a_k \rangle = (e^{\alpha + \beta \varepsilon_k} + 1)^{-1} \\ N_w = \langle b_w^* b_w \rangle = (e^{\beta \hbar \omega_s} - 1)^{-1}. \quad (3 \cdot 18)$$

(3·17) are readily proved with the help of (3·9), (3·16) and (3·20). Although the averages of N -products of higher order do not necessarily vanish, it happens that they are such small quantities that their contributions to the grand partition function can be ignored in the limit of $N \rightarrow \infty$, $V \rightarrow \infty$ (keeping N/V as a constant). To see this fact, we shall consider $\langle T[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle$ as an example. According to the lemma II, we get

$$\begin{aligned} \langle T[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle &= \langle N[\psi^{*'}(x)\psi'(x)\psi^{*'}(x')\psi'(x')] \rangle \\ &= \langle N[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle - S(x-x') \langle N[\psi^*(x')\psi(x')] \rangle \\ &\quad - S(x'-x) \langle N[\psi^*(x)\psi(x')] \rangle + S(x-x) \langle N[\psi^*(x')\psi(x')] \rangle \\ &\quad + S(x'-x') \langle N[\psi^*(x)\psi(x)] \rangle - S(x-x') S(x'-x) + S(x-x) S(x'-x') \\ &= \langle N[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle - S(x-x') S(x'-x) \\ &\quad + S(x-x) S(x'-x'), \end{aligned} \quad (3 \cdot 19)$$

in which the condition $\langle N[\psi^*(x)\psi(x')] \rangle = 0$ has been used. On the other hand, if one compute the left-hand side of (3·19) directly, it will follow that

$$\begin{aligned} \langle T[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle &= \begin{cases} V^{-2} \sum_k \sum_l \sum_m \sum_n \langle a_k^* a_l a_m^* a_n \rangle e^{-i[(k-l)x + (m-n)x'] + (\varepsilon_k - \varepsilon_l)t + (\varepsilon_m - \varepsilon_n)t'} & t > t', \\ V^{-2} \sum_k \sum_l \sum_m \sum_n \langle a_k^* a_l a_m^* a_n \rangle e^{-i[(k-l)x' + (m-n)x] + (\varepsilon_k - \varepsilon_l)t' + (\varepsilon_m - \varepsilon_n)t} & t < t'. \end{cases} \end{aligned} \quad (3 \cdot 20)$$

The average $\langle a_k^* a_l a_m^* a_n \rangle$ vanishes for all values of $(k l m n)$ except for the following three cases :

$$a) \quad k=l \neq m=n,$$

$$\text{b) } k=n \neq l=m,$$

$$\text{c) } k=l=m=n.$$

Referring to the definition of $S(x-x')$ given by (3.17a), one can easily see that case a) gives us the term $S(x-x)S(x'-x')$ in the right hand side of (3.10), and the case b) the term $S(x-x')S(x'-x)$, so that

$$\langle N[\psi^*(x)\psi(x)\psi^*(x')\psi(x')] \rangle = V^{-2} \sum_k \langle a_k^* a_k a_k^* a_k \rangle, \quad (3.21)$$

which is, however, smaller than the other two terms by a factor $1/V$, owing to the contraction of the summation over momenta from double to single. The same reasoning prevails for all the averages of N -products, and the average of an N -product with $2n$ factors is generally shown to be a quantity of the order of $(1/V^{n-1})$ if the c -number terms composed of $S(x-x')$ and $D(x-x')$ alone are regarded as the quantities of the order of unity. Thus, on disregarding all the averages of N -products, we are led to a simple computation rule for $\langle T[H_1(x_1) \cdots H_1(x_n)] \rangle$, that is;

(1) substitute for every field operator $\psi^*(x)$, $\psi(x)$ and $\varphi(x)$ the quantities $\psi^{*'}(x)$, $\psi'(x)$ and $\varphi'(x)$ defined by (3.14) and (3.15) respectively,

(2) perform the operations indicated by $\partial/\partial\psi^*(x)$, $\partial/\partial\psi(x)$ and $\partial/\partial\varphi(x)$ with the help of the commutation relations (3.12) and (3.15),

(3) retain only such terms that do not contain N -product as a factor.

W. can reformulate these results by noting that eqs. (3.11) and (3.14) can be respectively in the form

$$\begin{aligned} e^{\Delta}\varphi(x)e^{-\Delta} &= \varphi'(x), \\ e^{\Sigma}\psi^*(x)e^{-\Sigma} &= \psi^{*'}(x), \\ e^{\Sigma}\psi(x)e^{-\Sigma} &= \psi'(x), \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} \Delta &= \frac{1}{2} \iint dx dy D(x-y) \partial/\partial\varphi(x) \partial/\partial\varphi(y), \\ \Sigma &= \iint dx dy S(x-y) \partial/\partial\psi(y) \partial/\partial\psi^*(x), \end{aligned} \quad (3.23)$$

as can be readily proved by a direct calculation. Let $\mathfrak{U}(\psi^*, \psi, \varphi)$ be any functional of the field operators $\psi^*(x)$, $\psi(x)$ and $\varphi(x)$. then

$$\langle T[\mathfrak{U}(\psi^*, \psi, \varphi)] \rangle = \langle N[e^{\Delta} e^{\Sigma} \mathfrak{U}(\psi^*, \psi, \varphi) e^{-\Delta} e^{-\Sigma}] \rangle. \quad (3.24)$$

In this formalism the grand partition function given by (2.14) can be put into a compact form :

$$\begin{aligned} \Xi/\Xi_0 &= \langle T[\exp - \int H_1(x) dx] \rangle = \langle T[\mathfrak{C}] \rangle \\ &= \langle N[e^{\Delta} e^{\Sigma} \mathfrak{C} e^{-\Delta} e^{-\Sigma}] \rangle \end{aligned}$$

or

$$\Xi/\Xi_0 = \langle N[e^{\Delta} e^{\Sigma} \Theta] \rangle, \quad (3.25)$$

because we can disregard the factor $e^{-\Delta} e^{-\Sigma}$ since there is nothing for it to operate on. Serial expansion of exponential function in powers of $\int H_1(x) dx$ leads us to the formula (2.14b).

In actual calculations dealing with eq. (3.25), it is more convenient to employ the so-called Feynman graphs. Each term in the expansion of the right hand side of (3.25) can be analyzed into various Feynman graphs according to the following rules: On carrying out the rearrangements of operators and retaining only the terms which do not contain N -product, for every factor $D(x-x')$ a dotted (phonon) line is drawn connecting the points x and x' ; for every factor $S(x-x')$ a directed (electron) line is drawn from x to x' . Thus each term in the expansion of the right hand side of (3.25) is composed of a number of Feynman graphs, to each of which a product function of $S(x-x')$ and $D(x-x')$ corresponds. The final result we want is obtained by integrating with respect to all the coordinates involved in each Feynman graphs and by summing up all the terms contributed from possible Feynman graphs.

§ 4. Illustrations

Having established the computation rules, we will apply them to the calculations of ξ_n and C_n with small n for the electron-phonon system. What we are going to calculate is

$$\xi_n = (-1)^n/n! \int \cdots \int \langle T[H_1(x_1) \cdots H_1(x_n)] \rangle dx_1 \cdots dx_n, \quad (4.1)$$

$$H_1(x) = g\psi^*(x)\psi(x)\varphi(x) + g'\varphi(x)\varphi(x).$$

In analyzing ξ_n into Feynman graphs, the following view points are useful: We think that to each $\psi^*(x)$ corresponds an electron line starting from the point x , to each $\psi(x)$ corresponds an electron line entering into the point x , and to each $\varphi(x)$ corresponds a phonon line joining at the point x . Thus $H_1(x)$ represents a point x , at which either three lines, two electron lines and a phonon line, join with strength g , or two phonon lines are connected with strength g' . We can, therefore, carry out the analysis of ξ_n by drawing all the graphs in which n vertices are connected with each other, either by two electron lines outgoing and incoming and a phonon line, or by two phonon lines. We need not consider the term in ξ_n which contain the factors $\varphi(x)$'s of odd numbers. Furthermore, many graphs may be left out of consideration on account of the rule that an electron line is forbidden to join a point to itself. This additional rule comes out from the fact that since a constant factor $S(0)$ corresponds to such an electron line that joins a point to itself, the integration with respect to the coordinate of this vertex is to be reduced to the form $\int D(x) dx$, which evidently vanishes in virtue of (3.17b). In Fig. 1 various Feynman graphs appearing in the lower order terms of ξ_n 's are shown. A comparison of the results from Feynman graph analysis with that of a straightforward application of the computation rules reveals that

the sign of each graph is determined as $+$ or $-$ according to whether the number of closed electron line loops involved in the Feynman graph is odd or even.

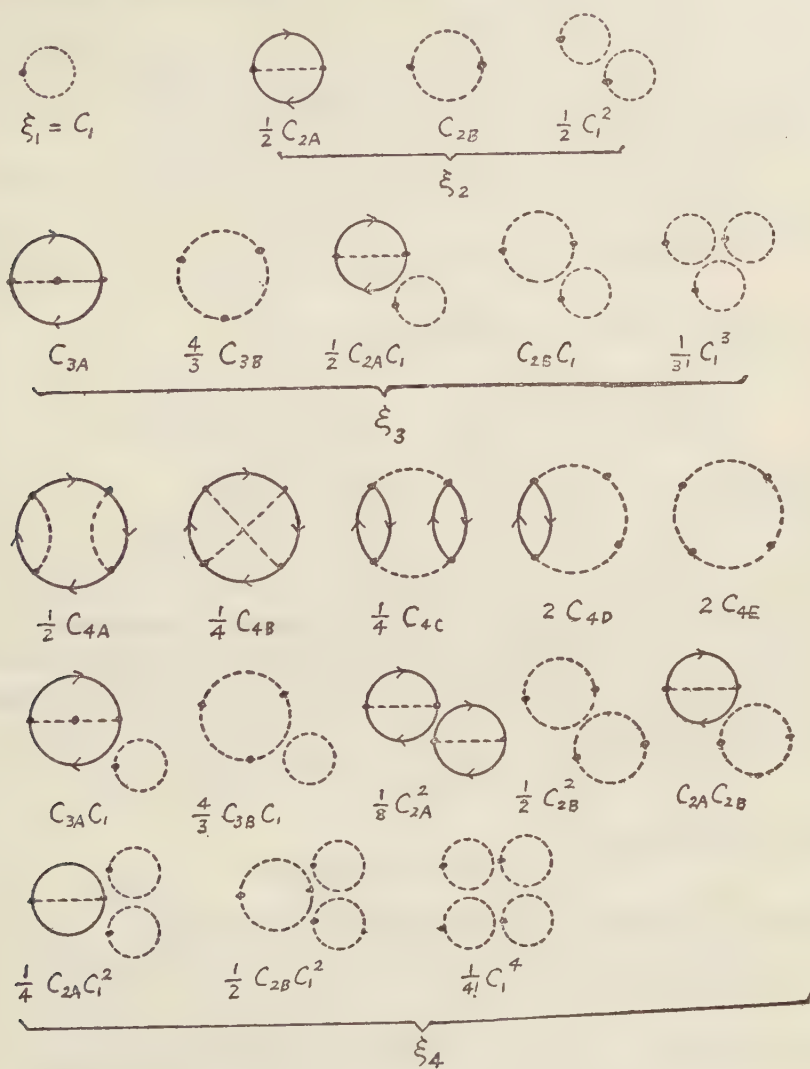


Fig. 1. Various Feynman graphs appearing in ξ_1 , ξ_2 , ξ_3 and ξ_4 .

Referring to the relations between $\hat{\xi}_n$ and C_k given by (2.16), we can see from Fig. 1 that C_k is exclusively constructed of connected Feynman graphs with k vertices. This result is of some importance, because the free energy of the system is given by

$$\begin{aligned} F &= -N\mu - kT \log \Xi \\ &= -N\mu - kT \log \Xi_0 - kT (C_1 + C_2 + C_3 + \dots), \end{aligned} \quad (4.2)$$

which is to be proportional to the whole volume of the system V . For every connected Feynman graphs, the result of the integrations with respect to all the involved coordinates but one turns out to be independent of a remaining coordinate, and the integration with respect to the last coordinate simply gives a factor βV , so that the above requirement that (4.2) is to be proportional to V is always fulfilled if every C_k consist of connected Feynman graphs alone. It should be noted that this proportionality $F \propto V$ is not justified when the averages of N -products disregarded above are take into consideration.

The integrations with respect to the coordinates can be quite easily performed, at least for small n . We shall show below only two lowest order terms in Fig. 1.

$$C_1 = \int D(0) dx = \beta V \sum_w \hbar \omega_s (N_w + 1/2),$$

$$C_{2A} = \iint S(x-x') S(x'-x) D(x-x') dx dx' \quad (4.3)$$

$$= \left[- \sum_k \sum_w \frac{\omega f_k (1 - f_{k+w})}{\epsilon_{k+w} - \epsilon_k + \hbar \omega_s} + \sum_k \sum_w \frac{\omega N(f_{k+w} - f_k)}{\epsilon_{k+w} - \epsilon_k + \hbar \omega_s} \right] \beta V,$$

which are in agreement with those obtained by other authors.⁸⁾

The merits of the present method are, apart from the simplicity of its computation rules, that it enables us to get a deep insight into the structures of the higher order perturbations through the Feynman graphs, and hence to go beyond the usual perturbational calculation. For instance, we can carry out a partial summation of serial terms up to infinite order, by adding certain special Feynman graphs. Thus the so-called renormalization procedures developed in quantum field theory will become available to various degrees. An

modified phonon line

$$\text{-----} = \text{-----} - 2 (\text{-----} \bullet \text{-----}) + 2^2 (\text{-----} \bullet \text{-----} \bullet \text{-----}) \text{-----}$$

modified phonon energy

$$\bigcirc = \bigcirc - (2/2) \bigcirc + (2^2/3) \bigcirc - (2^3/4) \bigcirc + \dots$$

$$C_1 + C_2 + C_3 + C_4 \dots = \text{a sum of all connected Feynman graphs}$$

$$= - \bigcirc - (1/2) \bigcirc - (1/4) \bigcirc - (1/2) \bigcirc - (1/4) \bigcirc - \dots$$

Fig. 2. Elimination of the term $g'\varphi(x)\varphi(x)$ in $H_1(x)$. This is effected by employing a re-defined phonon line in place of -----

example: It is immediately suggested by inspecting Fig. 1 that the effect of the second term in (4.1) can be eliminated by re-defining the phonon line and the phonon energy as illustrated in Fig. 2.

This means in the mathematical formula that $D(x-x')$ is to be replaced by

$$D^*(x-x') = D(x-x') - 2g' \int D(x-x_1) E(x_1-x') dx_1 \\ + (2g')^2 \iint D(x-x_1) D(x_1-x_2) D(x_2-x') dx_1 dx_2 \dots \quad (4.4)$$

The evaluation of $D^*(x-x')$ is not difficult (see appendix B), the result being conveniently expressed in terms of the Fourier component as

$$D^*(\mathbf{k}, t) = 1/g' \int_1^\infty \xi^{-\alpha} D(\xi \mathbf{k}, t) d\xi, \quad (4.5)$$

where

$$D(\mathbf{k}, t) = \int D(\mathbf{x}, t) e^{i\mathbf{k} \cdot \mathbf{x}} d^3\mathbf{x}, \quad (4.6)$$

and

$$\alpha = (1 + 3g')/g'. \quad (4.6)$$

(4.5) and (4.7) show that for small g' $D^*(x-x')$ is nearly equal to $D(x-x')$, whereas for large g' it becomes proportional to $1/g'$. This reduction in magnitude of $D^*(x-x')$, in turn, acts to prevent the sound velocity from suffering large alteration due to electron-phonon interaction. Such a situation remedies a certain difficulty occurred in a perturbational treatment of sound velocity re-normalization.¹¹⁾ But we shall leave this problem for another occasion.

§ 5. Non-perturbational treatment

The treatment described in the preceding sections is essentially an expansion of the grand partition function in powers of the coupling constant, so that, for the case of strong coupling it will not be useful as it stands. It is, however, possible to put forward a method which is free from the serial expansion procedure.¹⁰⁾

We consider a set of functions defined by

$$G_1(x; x') = \langle T[\phi^*(x) \phi(x') \mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle, \\ G_2(xy; x'y') = \langle T[\phi^*(x) \phi^*(y) \phi(y') \phi(x') \mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle, \\ G_3(xyz; x'y'z') = \langle T[\phi^*(x) \phi^*(y) \phi^*(z) \phi(z') \phi(y') \phi(x') \mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle, \quad (5.1)$$

and so on, where

$$\mathcal{E} = \exp \left[- \int H_1(x) dx \right], \quad H_1(x) = g \phi^*(x) \phi(x) \varphi(x). \quad (5.2)$$

(For the sake of simplicity, we shall omit the term $g'\varphi(x)\varphi(x)$ for a while.) Following the terminology used in quantum field theory, we shall call them the Green function of one-electron, of two-electrons and so on. As is evident from the definitions, however, they correspond to the coordinate representation of the reduced density matrices in the grand canonical ensemble, and play a role similar to the molecular distribution function in classical statistical mechanics. For phonon we define in a similar way

$$A_1(x x') = \langle T[\varphi(x)\varphi(x')\mathcal{S}] \rangle / \langle T[\mathcal{S}] \rangle, \quad \text{etc.} \quad (5.3)$$

Now, we will show that these Green functions satisfy a set of coupled integral equations. To do this we apply the equation (3.24) to (5.1) and (5.3). Then, for instance, we obtain

$$G_1(x x') = \langle N[e^\Delta e^\Sigma \psi^*(x)\psi(x')\mathcal{S}] \rangle / \langle T[\mathcal{S}] \rangle. \quad (5.4)$$

Here let us commute e^Σ through $\psi^*(x)$. Referring to the relations

$$\begin{aligned} e^\Sigma \psi^*(x) &= \psi^*(x) e^\Sigma + \int dy S(x-y) \delta / \delta \psi(y) e^\Sigma, \\ e^\Sigma \psi(x) &= \psi(x) e^\Sigma - \int dy S(y-x) \delta / \delta \psi^*(y) e^\Sigma, \end{aligned} \quad (5.5)$$

we see that the result is

$$G_1(x x') = \langle N[e^\Delta \int dy S(x-y) \delta / \delta \psi(y) e^\Sigma \psi(x')\mathcal{S}] \rangle / \langle T[\mathcal{S}] \rangle. \quad (5.6)$$

The term which contains the factor $\psi^*(x)$ standing to the left of e^Σ is omitted, because we are ignoring the average of N -product. Since the operator $\delta / \delta \psi^*(y)$ commutes with Σ , we can perform the indicated differentiation in (5.6) by noting that

$$\begin{aligned} [\delta / \delta \psi(y), \psi(x')]_+ &= \delta(y-x'), \\ \delta / \delta \psi(y) \mathcal{S} &= g \psi^*(y) \varphi(y) \mathcal{S}, \end{aligned} \quad (5.7)$$

and we obtain

$$G_1(x x') = S(x-x') + g \int dy S(x-y) dy \langle N[e^\Delta e^\Sigma \psi^*(y) \psi^*(x') \varphi(y) \mathcal{S}] \rangle / \langle T[\mathcal{S}] \rangle. \quad (5.8)$$

Let us further commute e^Δ through $\psi^*(y)$, $\psi^*(x')$ and $\varphi(y)$, referring to the relations

$$e^\Delta \varphi(y) = \varphi(y) e^\Delta + \int dz D(y-z) \delta / \delta \varphi(z) e^\Delta \quad (5.9)$$

and

$$\delta / \delta \varphi(z) \mathcal{S} = -g \psi^*(z) \psi(z) \mathcal{S}. \quad (5.10)$$

The result is

$$G_1(x x') = S(x-x') - g^2 \iint dy dz S(z-y) D(y-z) \langle T[\psi^*(y) \psi^*(x') \psi^*(z) \psi(z) \mathcal{S}] \rangle / \langle T[\mathcal{S}] \rangle$$

$$=S(x-x')-g^2\iint S(x-y)D(y-z)G_2(yz;x'z)dydz, \quad (5.11)$$

where the use of the definition for $G_2(xy;x'y')$ has been made. Thus $G_1(xx')$ is shown to be coupled with $G_2(xy;x'y')$ through an equation (5.11). Repeating the same procedures, we can easily prove that G_k is directly connected with G_{k-1} and G_{k+1} through an integral equation similar to (5.11). For example

$$\begin{aligned} G_2(xy;x'y') &= S(x-x')G_1(y\gamma') - S(x-\gamma')G_1(yx') \\ &\quad - g^2\int S(x-x_1)D(x_1-\gamma_1)G_3(\gamma x_1\gamma_1;\gamma'\gamma_1)dx_1d\gamma_1. \end{aligned} \quad (5.12)$$

These coupled equations connecting the Green functions of various order bear a resemblance to the integral equations satisfied by the molecular distribution functions of various orders, discovered by Born-Green⁽¹²⁾ and Kirdwood⁽¹³⁾ in classical statistical mechanics. Although it is a very difficult task to solve these equations, one might be able to find an approximation of breaking off the infinite chain of equations into a closed system of few equations.

It is, however, more convenient to handle an equation containing one electron Green function alone, if such an equation exists. In fact, we can derive such an equation by making use of a trick of introducing an auxiliary external field.⁽¹⁴⁾ We define \mathfrak{S} in place of (5.2) by

$$\mathfrak{S} = \exp\left[-\int\{g\psi^*(x)\psi(x)\varphi(x) + \psi^*(x)\phi(x)\psi(x)\}dx\right], \quad (5.13)$$

where $\phi(x)$ is a c-number field which is to be made vanish in the final result. Then following the same procedures as in deriving (5.11), we get

$$G_1(xx') = S(x-x') - \int dy S(x-y) \langle T[\psi(x')\delta/\delta\psi(y)\mathfrak{S}] \rangle / \langle T[\mathfrak{S}] \rangle.$$

Now in view of (5.13) it immediately follows that

$$\delta/\delta\psi(y)\mathfrak{S} = \{g\psi^*(y)\varphi(y) + \phi(y)\psi^*(y)\}\mathfrak{S},$$

and hence

$$\begin{aligned} G_1(xx') &= S(x-x') + \int S(x-y)\phi(y)G_1(yx')dy \\ &\quad + g\int dy S(x-y) \langle N[e^{\Delta}e^{\Sigma}\psi^*(y)\psi(x')\varphi(y)\mathfrak{S}] \rangle / \langle T[\mathfrak{S}] \rangle \\ &= S(x-x') + \int S(x-y)\phi(y)G_1(yx')dy \\ &\quad - g^2\iint S(x-y)D(y-\xi)G_2(\gamma\xi;x'\xi)dyd\xi. \end{aligned} \quad (5.14)$$

The last expression in (5.14) has been attained by commuting e^{Δ} with $\varphi(y)$ and carrying out the differentiation. In order to express G_2 in terms of G_1 , we have to regard $G_1(xx')$

as a functional of the auxiliary field $\phi(x)$. Then we can proceed as follows:

$$\begin{aligned} G_2(x\xi; y\xi) &= \langle T[\psi^*(x)\psi^*(\xi)\phi(\xi)\phi(y)\mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle \\ &= -\langle T[\psi^*(x)\phi(y)\partial/\partial\phi(\xi)\mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle \\ &= -\partial/\partial\phi(\xi) \langle T[\psi^*(x)\phi(y)\mathcal{E}] \rangle / \langle T[\mathcal{E}] \rangle. \end{aligned} \quad (5.15)$$

Inserting (5.15) into (5.14) and carrying out a slight manipulation, we arrive at

$$\begin{aligned} G(x\xi') &= S(x\xi') + \int S(x-y)\phi(y)G(y, \xi')dy \\ &\quad + g^2 \iint S(x-y)D(y-\xi)\partial/\partial\phi(\xi)G(y\xi')d\xi dy. \end{aligned} \quad (5.16)$$

On the other hand, $S(x-x')$ satisfies the following differential equation:

$$\left[\frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \Delta \right] S(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t) = \delta(\mathbf{x}), \quad (5.17)$$

which can be easily proved by a direct operation of $(\partial/\partial t + \hbar^2/2m \cdot \Delta)$ on $S(x)$ defined by (3.17a). Operating $(\partial/\partial t + \hbar^2/2m \cdot \Delta)$ on (5.16) from left, and making use of (5.17), (5.16) is converted to

$$\begin{aligned} \left\{ \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \Delta \right\} G(x\xi') &= \delta(x-x') + \phi(x)G(x\xi') \\ &\quad + g^2 \int D(x-\xi)\partial/\partial\phi(\xi)G(x\xi')d\xi. \end{aligned} \quad (5.18)$$

A similar equation for one-nucleon Green function in meson field was solved by Edwards and Peierls by introducing a special technique of Fourier transformation in functional space.¹⁵⁾ A similar method may be available in the present case.

Another and more tractable method to deal with the grand partition function will be the generalized Hartree approximation. Recently Kinoshita and Nambu have developed a theory of Hartree field for a system composed of a number of particles and a intermediary Bose field.¹⁶⁾ A similar method will be also applicable to the present case, and will be especially useful for the investigation of the cooperative phenomena such as superconductivity.

We shall take again the electron-phonon system as an example. According to the general theory described in § 2 and § 3, the grand partition function is given by

$$\Xi/\Xi_0 = \langle T[\exp\{-\int H_1(x)dx\}] \rangle, \quad (5.19)$$

which is valid for an arbitrary choice of H_0 and H_1 . We assume, therefore, that the free Hamiltonian has, instead of (2.17b), the following form

$$\bar{H} = \sum_k E_k a_k^* a_k + \sum_w W_w (b_w^* b_w + \frac{1}{2}) \quad (5.20)$$

and the interaction Hamiltonian is chosen so as to make it hold that

$$\begin{aligned} \int \bar{H}_1(x) dx = & \int e^{t\bar{H}_0} H_1 e^{-t\bar{H}_0} dt = \frac{1}{2} \iint T[\phi^*(x) \phi(x-x') \phi(x')] dx dx' \\ & + \frac{1}{4} \iint P[\varphi(x) \chi(x-x') \varphi(x')] dx dx' + g \int \phi^*(x) \phi(x) \varphi(x) dx, \end{aligned} \quad (5.21)$$

where $\phi(x-x')$ represents a sort of Hartree field for electron, E_k the energy of electron moving in this Hartree field, $\chi(x-x')$ and W_w are the corresponding quantities for phonon. $\phi^*(x)$, $\phi(x)$ and $\varphi(x)$ are respectively defined by

$$\begin{aligned} \phi^*(x) &= V^{-1/2} \sum_k a_k^* e^{-ik \cdot x + E_k t}, \\ \phi(x) &= V^{-1/2} \sum_k a_k e^{ik \cdot x - E_k t}, \\ \varphi(x) &= \sum_w (W_w/2V)^{1/2} (b_w^* e^{-i w \cdot x + W_w t} + b_w e^{i w \cdot x - W_w t}) \end{aligned} \quad (5.22)$$

in place of (2.20). With the aid of (5.22), each term in the right hand side of (5.21) can be written down as a function of a_k^* , a_k , b_w^* and b_w , the results being

$$\begin{aligned} & \frac{1}{2} \iint T[\phi^*(x) \phi(x-x') \phi(x')] dx dx' \\ &= \int dt e^{t\bar{H}_0} \left[\frac{1}{2} \sum_k \phi(k, -E_k) (a_k^* a_k - a_k a_k^*) \right] e^{-t\bar{H}_0}, \\ & \frac{1}{4} \iint P[\varphi(x) \chi(x-x') \varphi(x')] dx dx' \\ &= \frac{1}{2} \int dt e^{t\bar{H}_0} \left[\sum_w W_w \chi(w, -W_w) (b_w^* b_w + b_w b_w^* + b_w b_{-w}^* + b_w^* b_{-w}^*) \right] e^{-t\bar{H}_0}, \\ & g \int \phi^*(x) \phi(x) \varphi(x) dx \\ &= \int dt e^{t\bar{H}_0} \left[g \sum_k \sum_w (W_w/2V)^{1/2} (a_{k-w}^* a_k b_w^* + a_{k+w}^* a_k b_w) \right] e^{-t\bar{H}_0}, \end{aligned} \quad (5.23)$$

where $\phi(k, E)$ and $\chi(w, W)$ are, respectively, the Fourier-Laplace transform of $\phi(x)$ and $\chi(x)$ defined by

$$\begin{aligned} \phi(k, E) &= \iint \phi(x, t) e^{ik \cdot x - Et} d^3 x dt, \\ \chi(w, W) &= \iint \chi(x, t) e^{i w \cdot x - Wt} d^3 x dt. \end{aligned} \quad (5.24)$$

Combining (5.23) with (5.20) we see that

$$\begin{aligned} \bar{H}_1 = & g \sum_k \sum_w (W_w/2V)^{1/2} (a_{k-w}^* a_k b_w^* + a_{k+w}^* a_k b_w) + \sum_k \phi(k, -E_k) a_k^* a_k - \frac{1}{2} \sum_k \phi(k, -E_k) \\ & + \frac{1}{2} \sum_w W_w \chi(w, -W_w) (b_w^* b_w + b_w b_w^* + b_w^* b_{-w} + b_w b_{-w}^*). \end{aligned}$$

Since the sum of \bar{H}_0 and \bar{H}_1 has to be taken equal to the original total Hamiltonian given

by (2.17), apart from a constant,

$$\bar{H}_0 + \bar{H}_1 = H + C,$$

it follows by comparing (5.25) with (2.17) that

$$\varepsilon_k = E_k + \phi(\mathbf{k}, -E_k), \quad (5.26a)$$

$$\hbar \mathbf{w} s = W_w, \quad (5.26b)$$

$$g' = \chi(\mathbf{w}, -W_w), \quad (5.26c)$$

$$C = -\frac{1}{2} \sum \phi(\mathbf{k}, -E_k). \quad (5.26d)$$

Eqs. (5.26) provide physical meanings for the Hartree field $\phi(x)$ and $\chi(x)$, that is, the Fourier-Laplace component of $\phi(x)$ equals the difference between the energy of a free electron and that of an electron moving in the Hartree field, and the Fourier-Laplace component of $\chi(x)$ gives what we called renormalization constant in § 2. In order to determine $\phi(x)$ and $\chi(x)$ self-consistently, we shall apply the theory of Green function described in the beginning of this section. Define the Green function $G(xx')$ and $\mathcal{A}(xx')$ by

$$\begin{aligned} G(xx') &= \langle T[\psi^*(x) \psi(x') \mathcal{G}] \rangle / \langle T[\mathcal{Z}] \rangle, \\ \mathcal{A}(xx') &= \langle T[\varphi(x) \varphi(x') \mathcal{Z}] \rangle / \langle T[\mathcal{Z}] \rangle \end{aligned} \quad (5.27)$$

with

$$\begin{aligned} \mathcal{G} &= \exp \left[- \{ g \int \psi^{h*}(x) \psi^h(x) \varphi(x) dx + \frac{1}{2} \iint \psi^{h*}(x) \phi(x-x') \psi^h(x') dx dx' \right. \\ &\quad \left. + \frac{1}{4} \iint \varphi(x) \chi(x-x') \varphi(x') dx dx' \} \right]. \end{aligned} \quad (5.28)$$

If we content ourselves with the expressions up to the second order in the coupling constant g , the coupled equations for $G(x-x')$ and $\mathcal{A}(x-x')$, which are derived through the procedures described above, can be easily solved by an iteration procedure. We shall give here only the results;

$$\begin{aligned} G(xx') &= S(x-x') + \frac{1}{2} \iint S(z-y) \phi(x-y) S(z-x') dy dz \\ &\quad + g^2 \iint S(x-y) D(y-z) S(y-z) S(z-x') dy dz, \end{aligned} \quad (5.29)$$

$$\begin{aligned} \mathcal{A}(xx') &= D(x-x') - \frac{1}{2} \iint D(x-y) \chi(y-z) D(z-x') dy dz \\ &\quad + g^2 \iint D(x-y) D(x'-z) S(y-z) S(z-y) dy dz. \end{aligned} \quad (5.30)$$

In this approximation, it will be natural to determine $\phi(x)$ and $\chi(x)$ in such a way that

$$G(xx') = S(x-x'), \quad \mathcal{A}(xx') = D(x-x'), \quad (5.31)$$

because the equations (5.21) state that electrons and phonons behave in the respective

Hartree field as if they are independent of each other. This statement is in accord with the basic assumption in Hartree approximation. From (5.29), (5.30) and (5.31) $\phi(x)$ and $\chi(x)$ are determined as

$$\phi(z-y) = -2g^2 D(y-z) S(y-z), \quad (5.32)$$

$$\chi(y-z) = -2g^2 S(y-z) S(z-y). \quad (5.33)$$

Here we shall briefly discuss the results derived from the equations (5.32) and (5.33), leaving the details to a later publication.

Performing the Fourier-Laplace transformations of (5.32) and (5.33) with the help of (3.17), we readily get

$$\phi(k, -E_k) = \frac{g^2}{V} \sum_w \hbar \omega_s \left[\frac{(N_w + 1)(1 - f_{k-w})}{E_{k-w} - E_k + \hbar \omega_s} + \frac{N_w(1 - f_{k+w})}{E_{k+w} - E_k - \hbar \omega_s} \right], \quad (5.34)$$

$$\chi(w, -W_k) = \frac{2g^2}{V} \sum_k \frac{f_k(1 - f_{k-w})}{E_{k-w} - E_k + W_w}. \quad (5.35)$$

Referring to (5.26), the equation (5.35) gives the change in the sound velocity of phonon caused by the electron-phonon interaction, if the dispersion of sound velocity is ignored. The result obtained here is nearly the same as that calculated by Fröhlich. (5.34) combined with (5.26a), on the other hand, give an equation to determine E_k . On neglecting N_w at very low temperatures, it becomes

$$E_k = \varepsilon_k - \frac{g^2}{V} \sum_s \hbar \omega_s \frac{(1 - f_{k-w})}{E_{k-w} - E_k + \hbar \omega_s}. \quad (5.36)$$

It is interesting to note that the same equation as (5.36) was derived by Bardeen in a quite different way.¹⁷⁾ Although the nature of the solution of the equation (5.36) was already discussed by him, a more careful investigation of this equation has been made by the present author. The result obtained agrees with that given by Bardeen in its essential point. There exists a solution of (5.36) in which one electron energy E_k has a gap at Fermi surface for sufficiently strong coupling constant. The ground state in which all the states of lower energy are occupied by electrons will correspond to a superconducting state at 0°K. Temperature effect on energy spectrum is easily taken into account in the present method, and it is shown that the energy gap, dependent on temperature, becomes to vanish above a certain temperature. Thus a sort of phase transition is expected. A similar result was worked out by Fröhlich and Kuper¹⁸⁾ with one-dimensional model. The basic idea of the present method resembles rather that of Fröhlich's than Bardeen's. Fröhlich has assumed that a cooperative interaction between electrons and phonons produces such a potential for an electron as to give rise to a splitting of the energy spectrum of the electron. In the present theory, we introduced explicitly a possessing a nature which Fröhlich has assumed, and we have proved in a self-consistent manner that this potential actually gives rise to a splitting in one-electron spectrum even in three dimensional case, and hence brings the assembly of electrons into a special state which we want to call the superconducting state.

§ 6. Extension to Other Systems

The electron-phonon system so far considered is a typical example, to which our method can conveniently be applied. In extending our method to other system, it may happen that some modifications in the formalism are required. In this section we shall show that our method is easily extended to a system in which many particles, obeying Fermi or Bose statistics, are interacting with each other through two-body potential.

In the scheme of second quantization we write the Hamiltonian of a system in question as

$$\begin{aligned} H &= H_0 + H_1, \\ H_0 &= \int \psi^*(\mathbf{x}) \mathbf{p}^2 / 2m \psi(\mathbf{x}) d^3\mathbf{x}, \\ H_1 &= \frac{1}{2} \iint \psi^*(\mathbf{x}) \psi^*(\mathbf{x}') J(\mathbf{x} - \mathbf{x}') \psi(\mathbf{x}') \psi(\mathbf{x}) d^3\mathbf{x} d^3\mathbf{x}', \end{aligned} \quad (6.1)$$

where $J(\mathbf{x} - \mathbf{x}')$ represents the interaction potential between two particles located at \mathbf{x} and \mathbf{x}' , \mathbf{p} the momentum operator of a particle. $\psi^*(\mathbf{x})$ and $\psi(\mathbf{x})$ can be expanded into Fourier series:

$$\psi^*(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k}} a_{\mathbf{k}}^* e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad \psi(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (6.2)$$

in which $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ are as usual creation and annihilation operators of a particle with wave vector \mathbf{k} , the commutation relations between them being given by

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}^*]_{\pm} &= \delta_{\mathbf{k}, \mathbf{k}'}, \\ (+ \text{ Fermi, } - \text{ Bose statistics}) \end{aligned} \quad (6.3)$$

The grand partition function Ξ can be written as

$$\begin{aligned} \Xi / \Xi_0 &= 1 + \xi_1 + \xi_2 + \xi_3 + \dots \\ &= \exp(C_1 + C_2 + C_3 + \dots) \end{aligned} \quad (6.4)$$

with

$$\begin{aligned} \xi_n &= (-1)^n / n! \int \dots \int \langle P[H_1(t_1) \dots H_1(t_n)] \rangle dt_1 \dots dt_n, \\ H_1(t) &= e^{tH_0} H_1 e^{-tH_0}, \end{aligned} \quad (6.5)$$

or more formally as

$$\Xi / \Xi_0 = \langle P[\mathcal{S}] \rangle,$$

where

$$\mathcal{S} = \exp \left[- \int H_1(t) dt \right].$$

First we consider the case of Fermi particles. In applying the computation rules derived in § 3 to the present case, a difficulty arises from the interaction term

$$\int H_1(t) dt = \frac{1}{2} \iint \psi^*(\mathbf{x}, t) \psi^*(\mathbf{x}', t) J(\mathbf{x} - \mathbf{x}') \psi(\mathbf{x}', t) \psi(\mathbf{x}, t) d^3\mathbf{x} d^3\mathbf{x}' dt, \quad (6.7)$$

which prevents us from accomplishing in a simple manner the t -ordering indicated in the right hand side of (6.5). The problem can, however, be solved by rewriting (6.7) in the form

$$\begin{aligned} \int H_1(t) dt = & \frac{1}{2} \iint T[\psi^*(\mathbf{x}) \psi(\mathbf{x}) J(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}') \psi(\mathbf{x}')] dx d\mathbf{x}' \\ & - \frac{1}{2} J(0) \int \psi^*(\mathbf{x}) \psi(\mathbf{x}) dx, \end{aligned} \quad (6.8)$$

in which $J(\mathbf{x} - \mathbf{x}')$ is defined by

$$J(\mathbf{x} - \mathbf{x}') = J(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (6.9)$$

That (6.8) equals (6.7) is easily verified by a short calculation. Then, noting that for arbitrary T -products $T[A]$, $T[B]$, ...

$$[T[A], T[B], \dots] = T[A, B, \dots],$$

we obtain, in place of (6.6),

$$\Xi / \Xi_0 = \langle T[\mathfrak{S}] \rangle \quad (6.10)$$

with

$$\begin{aligned} \mathfrak{S} = & \exp \left[-\frac{1}{2} \iint \psi^*(\mathbf{x}) \psi(\mathbf{x}) J(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}') \psi(\mathbf{x}') dx d\mathbf{x}' \right. \\ & \left. + \frac{1}{2} J(0) \int \psi^*(\mathbf{x}) \psi(\mathbf{x}) dx \right], \end{aligned}$$

to which all the rules established in § 3 are now applicable.

We have no trouble with the case of Bose particles. We do not want to repeat here a long analysis, so that we give below only the lemma II modified so as to hold for both statistics.

Lemma III For any product of $\psi^*(x)$ and $\psi(x)$ denoted by $\mathfrak{S}(\psi^*, \psi)$, it holds that

$$T[\mathfrak{S}(\psi^*, \psi)] = N[\mathfrak{S}(\psi^{*'}, \psi')],$$

in which $T = (\mp 1)^P$ and $\psi^{*'}(x)$ and $\psi'(x)$ are defined by

$$\psi^{*'}(x) = \psi^*(x) + \int dx' S(x - x') \delta / \delta \psi(x'),$$

$$\psi'(x) = \psi(x) \mp \int dx' S(x' - x) \delta / \delta \psi^*(x'),$$

respectively. $\delta / \delta \psi^*(x)$ and $\delta / \delta \psi(x)$ satisfy the following commutation relations:

$$[\delta / \delta \psi(x), \psi(x')]_{\pm} = [\delta / \delta \psi^*(x), \psi^*(x')]_{\pm} = \delta(x - x'),$$

$$[\delta / \delta \psi(x), \psi^*(x')]_{\pm} = [\delta / \delta \psi^*(x), \psi(x')]_{\pm} = 0.$$

$S(x-x')$ is given by

$$S(x-x') = \begin{cases} V^{-1} \sum_k f_k e^{-ik \cdot (x-x') + \varepsilon_k (t-t')} & t > t' \\ V^{-1} \sum_k (f_k \mp 1) e^{-ik \cdot (x-x') + \varepsilon_k (t-t')} & t < t' \end{cases}$$

where

$$f_k = (e^{\alpha + \beta \varepsilon_k} \pm 1)^{-1}.$$

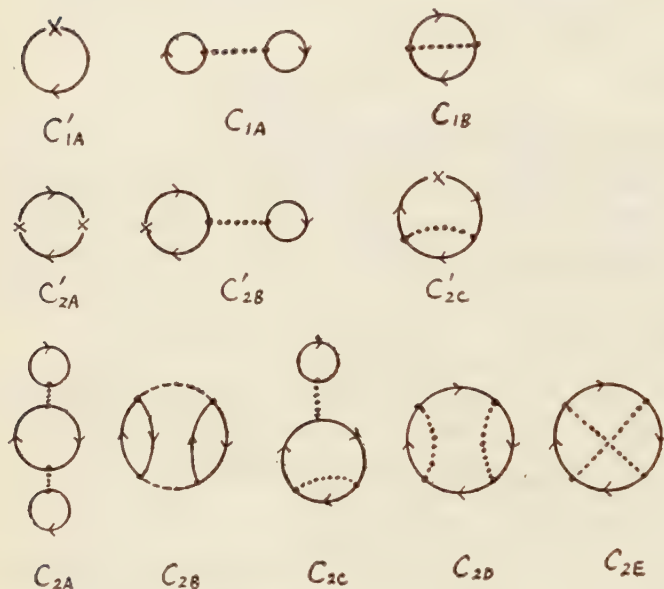


Fig. 3. Various Feynman graphs appearing in C_1 and C_2 . Dotted line $x \cdots x'$ corresponds to $J(x-x')$. Directed line $x \rightarrow x'$ corresponds to $S(x-x)$. \times represents self energy $J(0)$.

that of Ichimura's calculation. In the notation adopted here, the results for C_k 's obtained by Ichimura by his own method are written as follows:⁽⁶⁾

$$C_1 = -\frac{1}{2} \beta V^{-1} \sum_{k \neq l} (J_0 \mp J_{k-l}) f_k f_l \quad (6 \cdot 11a)$$

$$C_2 = \frac{1}{2} \beta^2 V^{-2} \sum_{k \neq l} \sum_{m \neq n} f_k (1 \mp f_k) f_l f_m (J_0 \mp J_{k-l}) (J_0 \mp J_{k-m}) \\ + \frac{1}{4} \beta V^{-2} \sum_{k+l=m+n} \sum_{l \neq m} \sum_{n \neq m} \frac{(J_{k-m} \mp J_{k-n})^2}{\varepsilon_k + \varepsilon_l - \varepsilon_m - \varepsilon_n} (1 \mp f_k) (1 \mp f_l) f_m f_n, \quad (6 \cdot 11b)$$

where J_k is the Fourier component of $J(x)$ defined by

$$J_k = \int J(x) e^{ik \cdot x} d^3x.$$

We will show that (6.11) are in complete agreement with our results. Inspecting Fig. 3,

All the upper signs of doubled signatures correspond to Fermi case and lower signs to Bose case.

We can analyze ξ_n or C_k into Feynman graphs by drawing a directed (particle) line for every factor $S(x-x')$ from x to x' , and a dotted line connecting the points x and x' for every factor $J(x-x')$. In the present case a particle line may join a point to itself. In Fig. 3 the Feynman graphs appearing in the lower order terms of C_k 's are shown.

It will be worth while to compare the results derived from Fig. 3 with

our expression for C_1 is readily written down as

$$C_1 = \frac{1}{2} J(0) \int S(0) dx - \frac{1}{2} \iint J(x_1 - x_2) \{S(0)\}^2 dx_1 dx_2 \\ \pm \frac{1}{2} \iint J(x_1 - x_2) S(x_1 - x_2) S(x_2 - x_1) dx_1 dx_2, \quad (6 \cdot 12)$$

Noting that

$$S(0) = V^{-1} \sum_k f_k \\ S(x - x') S(x' - x) = -V^{-2} \sum_k \sum_{k'} f_k (1 \mp f_{k'}) e^{i(k - k') \cdot (x - x') - (e_{k'} - e_k) |t - t'|}, \\ J(0) = V^{-1} \sum_k J_k, \quad \int J(x - x') dx = J_0,$$

(6.12) can be reduced to

$$C_1 / \beta V = -\frac{1}{2} V^{-2} \sum_k \sum_{k'} \{ -J_k f_{k'} + J_0 f_k f_{k'} \mp J_{k-k'} f_k (f_{k'} \mp 1) \} \\ = -\frac{1}{2} V^{-2} \sum_k \sum_{k'} (J_0 \mp J_{k-k'}) f_k f_{k'},$$

which is just equal to (6.11a). It is seen from the above calculation that the term $J(0) \int \psi^*(x) \psi(x) dx$ in \mathfrak{S} so behaves as to subtract the self-energy from the final results. We may, therefore, disregard this term hereafter, provide we keep in mind that the self-energy parts are always to be subtracted. Then C_2 is shown to consist of five integrals:

$$C_2 = \mp 4C_{2A} + 2C_{2B} + 8C_{2C} \mp 4C_{2D} \mp 2C_{2E}, \\ C_{2A} = \frac{1}{4} \int \cdots \int \{S(0)\}^2 J(x_1 - x_2) S(x_2 - x_3) S(x_3 - x_2) J(x_3 - x_4) dx_1 \cdots dx_4 \\ = \frac{1}{4!} \beta^2 V^{-2} J_0^2 \sum_k \sum_l \sum_m f_k (f_k \mp 1) f_l f_m, \\ C_{2B} = \frac{1}{4} \int \cdots \int S(x_1 - x_2) S(x_2 - x_1) S(x_3 - x_4) S(x_4 - x_3) J(x_1 - x_3) J(x_2 - x_4) dx_1 \cdots dx_4 \\ = \frac{1}{2!} \beta V^{-2} \sum_{\substack{k+l=m+n \\ k+l=m+n}} \sum_l \sum_m \sum_n \frac{(J_{k-m})^2}{\varepsilon_k + \varepsilon_l - \varepsilon_m - \varepsilon_n} (f_k \mp 1) (f_l \mp 1) f_m f_n, \\ C_{2C} = \frac{1}{4} \int \cdots \int S(0) J(x_1 - x_2) S(x_2 - x_3) S(x_3 - x_4) S(x_4 - x_2) J(x_3 - x_4) dx_1 \cdots dx_4 \\ = \frac{1}{4!} \beta^2 V^{-2} \sum_k \sum_l \sum_m J_0 J_{k-l} f_k (f_k \mp 1) f_l f_m, \\ C_{2D} = \frac{1}{4} \int \cdots \int S(x_1 - x_2) S(x_2 - x_3) S(x_3 - x_4) S(x_4 - x_1) J(x_1 - x_2) J(x_3 - x_4) dx \cdots dx_4 \\ = \frac{1}{4!} \beta^2 V^{-2} \sum_k \sum_l \sum_m J_{k-l} J_{k-m} f_k (f_k \mp 1) f_l f_m,$$

$$C_{2E} = \frac{1}{4} \int \cdots \int S(x_1 - x_2) S(x_2 - x_3) S(x_3 - x_4) S(x_4 - x_1) J(x_1 - x_3) J(x_2 - x_4) dx_1 \cdots dx_4$$

$$= \frac{1}{2} \beta V^{-2} \sum_{\substack{k \\ k+l=m+n}} \sum_l \sum_m \sum_n \frac{J_{k-m} J_{k-n}}{\epsilon_k + \epsilon_l - \epsilon_m - \epsilon_n} (f_k \mp 1) (f_l \mp 1) f_m f_n.$$

As to the evaluation of the above integrals, see appendix B. The summation of these five integrals with given numerical coefficients immediately leads us to (6.11b), apart from the self-energy parts which can be cancelled out by the integrals corresponding to the graphs C'_{2A} , C'_{2B} and C'_{2C} in Fig. 3.

An application of the method of partial summation over certain special Feynman graphs, which was proved useful in § 4, gives rise to an interesting result for the electrons interacting with coulomb potential. It is suggested from Feynman graph analysis that the interaction potential $J(x-x')$ has better to be replaced by

$$J^*(x-x') = J(x-x') + 2 \iint J(x-x_1) S(x_1-x_2) S(x_2-x_1) J(x_2-x') dx_1 dx_2$$

$$+ 2^2 \int \cdots \int J(x-x_1) S(x_1-x_2) S(x_2-x_1) J(x_2-x_3) S(x_3-x_1) S(x_4-x_3) J(x_3-x') dx \cdots dx_4 + \cdots \quad (6.13)$$

For coulomb potential $J(x-x') = e^2/|\mathbf{x}-\mathbf{x}'| \delta(t-t')$, the Fourier transform of (6.13), integrated over t , becomes

$$J^*(\mathbf{k}) = \frac{4\pi e^2}{k^2} \left\{ 1 + \frac{4\pi e^2}{k^2} A(\mathbf{k}) + \left(\frac{4\pi e^2}{k^2} A(\mathbf{k}) \right)^2 + \cdots \right\}$$

$$= \frac{4\pi e^2}{k^2 - 4\pi e^2 I(\mathbf{k})},$$

$$A(\mathbf{k}) = \frac{2}{V} \sum_{k'} \int S(\mathbf{k}'; -t) S(\mathbf{k}' - \mathbf{k}; t) dt = \frac{2}{V} \sum_{k'} \left(\frac{f_{k'} - f_{k'-k}}{\epsilon_{k'} - \epsilon_{k'-k}} \right). \quad (6.14)$$

This result shows us that the coulomb interaction between electrons is to be screened (as $1/r \rightarrow e^{-\lambda r}/r$), the screening constant λ being roughly estimated as

$$\lambda^2 = -4\pi e^2 A(0) = -\frac{8\pi e^2}{V} \sum_k \left(\frac{\partial f_k}{\partial \epsilon_k} \right) \cong \frac{4\pi e^2 m}{h^2} K_m. \quad (6.15)$$

(K_m = the magnitude of wave vector of electron with Fermi energy).

The same result as (6.15) for the screening effect was derived by Macke by a variational calculation.¹⁹⁾

Appendix A

A Proof of Lemma I

We shall prove the validity of lemma I by a mathematical induction. Assume that

it is true for some T -product $T[\mathfrak{F}(\varphi)]$, i.e. that

$$T[\mathfrak{F}(\varphi)] = N[\mathfrak{F}(\varphi')]. \quad (\text{A} \cdot 1)$$

Then if we can prove it for $T[\varphi(x)\mathfrak{F}(\varphi)]$, we can conclude that the statement of lemma I is valid, since for $\mathfrak{F}(\varphi) = 1$ and for $\mathfrak{F}(\varphi) = \varphi$ (A.1) is trivially true. Without a loss of generality we can take $\mathfrak{F}(\varphi)$ as a product of n factors; $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$. When

$$t_1 > t_2 > \cdots > t_n$$

$T[\mathfrak{F}(\varphi)]$ can be written as

$$T[\mathfrak{F}(\varphi)] = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n). \quad (\text{A} \cdot 2)$$

Now let us assume that

$$t_1 > t_2 > \cdots > t_r > t > t_{r+1} > \cdots > t_n.$$

Then

$$\begin{aligned} T[\varphi(x)\mathfrak{F}(\varphi)] &= \varphi(x_1)\cdots\varphi(x_r)\varphi(x)\varphi(x_{r+1})\cdots\varphi(x_n) \\ &\equiv X(\varphi)\varphi(x)Y(\varphi). \end{aligned} \quad (\text{A} \cdot 3)$$

Here we decompose $\varphi(x)$ into two parts according to

$$\varphi(x) = \varphi_+(x) + \varphi_-(x),$$

and transfer $\varphi_-(x)$ to the left through $X(\varphi)$, and $\varphi_+(x)$ to the right through $Y(\varphi)$. The result is conveniently expressed as

$$X(\varphi)\varphi(x)Y(\varphi) = \varphi_-(x)XY + XY\varphi_+(x) + \int dx' D_1(xx') \partial/\partial\varphi(x') (XY) \quad (\text{A} \cdot 4)$$

where

$$D_1(xx') = \begin{cases} [\varphi_+(x), \varphi(x')]_- & t > t' \\ -[\varphi_-(x), \varphi(x')]_- & t < t'. \end{cases} \quad (\text{A} \cdot 5)$$

In order to obtain the N -product of $X\varphi(x)Y$ from (A.4), we have to bring back $\varphi_-(x)$ to the right and $\varphi_+(x)$ to the left, because $\varphi_-(x)$ may stand to the left of $\varphi_-(x_1)$, $\varphi_-(x_2)$, \cdots $\varphi_-(x_r)$ and $\varphi_+(x)$ may stand to the right of $\varphi_+(x_{r+1})$, \cdots $\varphi_+(x_n)$. Carrying out rearrangements needed for getting N -product, we can express the result in the form

$$\varphi_-(x)XY = N[\varphi_-(x)XY] + \int dx' [\varphi_-(x), \varphi_-(x')]_- \partial X/\partial\varphi_-(x') Y, \quad (\text{A} \cdot 6)$$

$$XY\varphi_+(x) = N[XY\varphi_+(x)] - \int dx' [\varphi_+(x), \varphi_+(x')]_- X \partial Y/\partial\varphi_+(x'),$$

where operators $\partial/\partial\varphi_+(x)$ and $\partial/\partial\varphi_-(x)$ are defined through

$$\begin{aligned} [\partial/\partial\varphi_+(x), \varphi_+(x')]_- &= [\partial/\partial\varphi_-(x), \varphi_-(x')]_- = \delta(x-x'), \\ [\partial/\partial\varphi_+(x), \varphi_-(x')]_- &= [\partial/\partial\varphi_-(x), \varphi_+(x')]_- = 0, \end{aligned} \quad (\text{A} \cdot 7)$$

namely they represent functional differentiation. Since X and Y contain $\varphi_+(x)$ and $\varphi_-(x)$ as factors through the combined form $\varphi(x) = \varphi_+(x) + \varphi_-(x)$, it follows that

$$\partial X / \partial \varphi_-(x) = \partial X / \partial \varphi(x), \quad \partial Y / \partial \varphi_+(x) = \partial Y / \partial \varphi(x). \quad (\text{A} \cdot 8)$$

If we define further

$$\Delta_2(x, x') = \begin{cases} [\varphi_+(x), \varphi_+(x')]_- & t > t' \\ -[\varphi_-(x), \varphi_-(x')]_- & t < t', \end{cases} \quad (\text{A} \cdot 8)$$

then from (A.3) — (A.8) we obtain

$$X\varphi(x)Y = N[\{\varphi_+(x) + \varphi_-(x)\}XY] + \int dx' [\Delta_1(x, x') - \Delta_2(x, x')] \partial(XY) / \partial \varphi(x'). \quad (\text{A} \cdot 10)$$

In view of the first assumption (A.1) and a property of N -product

$$N[\varphi' N[\tilde{\gamma}(\varphi')]] = N[\varphi' \tilde{\gamma}(\varphi')]$$

(A.10) tells us that

$$T[\varphi(x) \tilde{\gamma}(\varphi)] = N[\varphi'(x) \tilde{\gamma}(\varphi')]$$

where

$$\varphi(x) = \varphi(x) + \int dx' \{\Delta_1(x, x') - \Delta_2(x, x')\} \partial / \partial \varphi(x').$$

With the help of (A.5) and (A.9) it is easy to show that

$$\begin{aligned} \Delta_1(x, x') - \Delta_2(x, x') \\ = \begin{cases} [\varphi_+(x), \varphi_+(x')]_- - [\varphi_+(x), \varphi_+(x')]_- = [\varphi_+(x), \varphi_-(x')]_- & t > t' \\ [\varphi_-(x), \varphi_-(x')]_- - [\varphi_-(x), \varphi_-(x')]_- = [\varphi_-(x'), \varphi_-(x)]_- & t < t' \end{cases} \\ = D(x - x'), \end{aligned}$$

which is identical with the definition of $D(x - x')$ given by (3.8). Thus our proof is completed. The proof of lemma II can be achieved in quite a similar manner, so that it will be unnecessary to repeat here the similar procedure.

Appendix B

First we shall derive the formula (4.5). For simplicity, we employ hereafter such an unit as to make $\hbar = 1$ and $\hbar = 1$. Fourier component of $D(x)$ is then expressed as

$$D(k, t) = \frac{1}{2} k \{ (N_k + 1) e^{-k|t|} + N_k e^{k|t|} \}. \quad (\text{B} \cdot 1)$$

Now the Fourier transform of (4.4) is given by

$$D^*(k, t - t') = D(k, t - t') - 2g' \int_0^{\beta} D(k, t - t_1) D(k, t_1 - t') dt_1$$

$$+ (2g')^2 \int_0^\beta \int_0^\beta D(\mathbf{k}, t-t_1) D(\mathbf{k}, t_1-t_2) D(\mathbf{k}, t_2-t') dt_1 dt_2 \dots \quad (\text{B} \cdot 2)$$

Although $D^*(\mathbf{k}, t)$ can be evaluated by solving an integral equation

$$D^*(\mathbf{k}, t) = D(\mathbf{k}, t) - 2g' \int_0^\beta D(\mathbf{k}, t-s) D^*(\mathbf{k}, s) ds, \quad (\text{B} \cdot 3)$$

we shall follow a more direct and elementary method. A short calculation yields that

$$\int_0^\beta D(\mathbf{k}, t-t_1) D(\mathbf{k}', t_1-t') dt_1 = \frac{k^2}{k^2 - k'^2} D(\mathbf{k}, t-t') - \frac{k'^2}{k'^2 - k^2} D(\mathbf{k}', t-t') \quad (\text{B} \cdot 4)$$

which is reduced in the limit of $\mathbf{k} \rightarrow \mathbf{k}'$ to

$$\int_0^\beta D(\mathbf{k}, t-t_1) D(\mathbf{k}, t_1-t') dt_1 = (1 - \frac{1}{2} k d/dk) D(\mathbf{k}, t-t') = \Delta_k D(\mathbf{k}, t-t') \quad (\text{B} \cdot 5)$$

where

$$\Delta_k = (1 - \frac{1}{2} k d/dk). \quad (\text{B} \cdot 6)$$

Using (B.5) in a repeated manner, we can put (B.2) into the form

$$\begin{aligned} D^*(\mathbf{k}, t) &= (1 - 2g' \Delta_k + (2g' \Delta_k^2 - \dots) D(\mathbf{k}, t) \\ &= \frac{1}{1 + 2g' \Delta_k} D(\mathbf{k}, t), \end{aligned}$$

or

$$(1 + 2g' \Delta_k) D^*(\mathbf{k}, t) = D(\mathbf{k}, t). \quad (\text{B} \cdot 7)$$

It is easy to solve this inhomogeneous linear differential equation of first order with a condition $\lim_{k \rightarrow \infty} D^*(\mathbf{k}, t) = 0$. The result is

$$\begin{aligned} D^*(\mathbf{k}, t) &= 1/g' \int_k^\infty k^{\alpha-1} D(\mathbf{k}', t) k'^{-\alpha} dk' \\ &= 1/g' \int_1^\infty \xi^{-\alpha} D(\xi \mathbf{k}, t) d\xi \end{aligned} \quad (\text{B} \cdot 8)$$

where

$$\alpha = (1 + 3g')/g'.$$

Thus (4.5) is proved. For the Fourier component of $S(x)$, the following equations are easily proved:

$$\int_0^\beta S(\mathbf{k}, t-t_1) S(\mathbf{k}', t_1-t') dt_1 = \frac{S(\mathbf{k}, t-t') - S(\mathbf{k}', t-t')}{\varepsilon_k - \varepsilon_{k'}}, \quad (\text{B} \cdot 9)$$

$$\begin{aligned} &\int_0^\beta S(\mathbf{k}, t-t_1) S(\mathbf{l}, t_1-t) S(\mathbf{m}, t_1-t') S(\mathbf{n}, t'-t_1) dt_1 \\ &= (f_m - f_n) / (\varepsilon_l + \varepsilon_m - \varepsilon_k - \varepsilon_n) S(\mathbf{k}, t-t') S(\mathbf{l}, t'-t) \end{aligned} \quad (\text{B} \cdot 10)$$

$$-(f_k - f_l) / (\varepsilon_l + \varepsilon_m - \varepsilon_k - \varepsilon_n) S(\mathbf{m}, t - t') S(\mathbf{n}, t' - t).$$

In particular, in the limit of $\mathbf{k} \rightarrow \mathbf{k}'$, $t \rightarrow t'$ (B·9) becomes

$$\int_0^\beta S(\mathbf{k}, t - t_1) S(\mathbf{k}, t_1 - t) dt_1 = \frac{\partial}{\partial \varepsilon_k} S(\mathbf{k}, 0) = -\beta f_k (1 \mp f_k). \quad (\text{B} \cdot 11)$$

(B·9), (B·10) and (B·11) were used in evaluating the integrals C_{2A} — C_{2F} in § 6.

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On the Extension of the Casimir Trick, I

— On the Definition of Density Matrix and its Reduction —

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When a Fermion is described by a plane wave, quantities which are to be compared with experimental data can be computed easily by using the well-known Casimir trick. But, when the influence of interaction of the Fermion with another field is taken into account, the calculation becomes very difficult. As the Casimir trick can not be used, complicate calculations have hitherto been necessary.

It is the purpose of this paper to give a method of calculation in which the density matrix, instead of the Casimir operator, is used to avoid the intricate method above mentioned in the case of a Fermion moving in an arbitrary central field. This density matrix will be represented as a sum of direct products of ρ - and σ -operators with numerical coefficients. It is also verified that, for a free Fermion, the density matrix defined in this paper is reduced exactly to the usual Casimir operator (projection operator).

The method of calculation proposed in this paper will be not only favourable to treat such various problems as β -decay, Bremsstrahlung and so on where the influence of a central potential must be considered, but also useful to solve the problem for free Fermions by the method of partial waves.

§ 1. Introduction

When a Fermion interacts with another field, such results as the transition probability and the cross section which are to be compared with experiments are obtained from the following quantities :

$$\sum_i \sum_f | \int dx_1 \cdots dx_1' \psi_f^*(x_1) M(x_1 \cdots x_1') \psi_i(x_1') |^2. \quad (1.1)$$

Here ψ_i and ψ_f are Fermion partial eigenfunctions for initial and final states, respectively. \sum_i means the average over possible initial states and \sum_f is the sum over possible final states, keeping the sign of energy fixed. $M(x_1 \cdots x_1')$ is a quantity which depends on the other fields and, if necessary, intermediate states. This calculation is straightforward, but somewhat tedious. In order to avoid this complicate calculation, we will utilize the fact that, for example, the quantity,

$$\sum_f \psi_f(x_2) \psi_f^*(x_1), \quad (1.2)$$

usually appears in (1.1).

When a Fermion is described by a plane wave, Casimir succeeded to develop a very elegant method of this calculation, named by the so-called Casimir trick¹⁾. As is well

known, the main points are that, in the momentum representation, these sums appeared in (1.1) can be replaced by a trace of Dirac operators by introducing Casimir operators (projection operators) instead of (1.2) and only using commutation relations between operators, without using the practical matrix representations of eigenfunctions and Dirac operators. In this paper, these Casimir operators are defined as follows:*

$$\begin{aligned} A_+(+p) &= (W + H(+p))/2W \\ &= (W + (\alpha \cdot p) + \beta m)/2W, \end{aligned} \quad (1.3)$$

$$\begin{aligned} A_-(-p) &= (W - H(-p))/2W \\ &= (W + (\alpha \cdot p) - \beta m)/2W, \end{aligned} \quad (1.4)$$

where

$$W = \sqrt{p^2 + m^2}. \quad (1.5)$$

$A_+(+p)$ is a projection operator selecting states of free Fermions with positive energy $+W$ and momentum $+p$. $A_-(-p)$ is an operator for the negative-energy Fermion (energy $(-W)$ and momentum $(-p)$), whose absence stands for an anti-Fermion.**

But when the Fermion is not described by such a plane wave, this trick can not be applied at all, and so the complicate method has been used. This method is to multiply each matrix element by corresponding component of the eigenfunction.

It is the purpose of this paper to develop technics of calculation, by which we can avoid this mistakable method even when the Fermion moves in an arbitrary central field. In order to calculate in accordance with the similar idea to the Casimir trick, we introduce the density matrix;

$$D = \sum \phi \phi^*. \quad (1.6)$$

The sum (Σ) is taken over possible states for the Fermion which has the constant energy. ϕ is the eigenfunction of Dirac equation with the potential (V) for one of such states. The formalism which is favourably suited for solving this equation is of the angular momentum representation. In this representation, for a single partial wave, $\phi'_{\kappa, \mu}(x)$ is the eigenfunction of a Fermion with quantum numbers κ and μ and energy $E = +W > 0$, and $\phi''_{\kappa, \mu}(x)$ is that of a Fermion with energy $E = -W < 0$ and momentum $-p$, so that $+p$ is the momentum of the anti-Fermion, if the negative energy Fermion is absent.*** Then Fermion field quantity $\Psi(x)$ is quantized as a series of these partial waves. In this angular momentum representation, the density matrices, corresponding to the Casimir operators $A_+(+p)$ and $A_-(-p)$, are respectively written as follows:

* Throughout this paper, we set $\hbar = c = 1$.

** We thus deal with an anti-Fermion of exactly the same energy and momentum as the Fermion, though the old non-covariant treatment of projection operators is developed for two Fermion states with the same momentum $+p$ but one with energy $E > 0$ and the other with energy $E < 0$).

*** The meanings of the quantum numbers κ and μ will be explained in Appendix A. In the subsequent formalism, all quantities concerning the negative energy are distinguished by primes from the corresponding quantities with respect to the positive energy.

$$D(x_1, x_2; \alpha) = \sum_{\kappa, \mu} \psi'_{\kappa, \mu}(x_1) \psi_{\kappa, \mu}^*(x_2), \quad (1.7a)$$

$$D'(x_1, x_2; -\alpha) = \sum_{\kappa, \mu} \psi'_{\kappa, \mu}(x_1) \psi'_{\kappa, \mu}^*(x_2). \quad (1.7b)$$

Here α 's appeared in these definitions are parameters concerning the potential $(V)^*$.

The idea of this density matrix was already used by de Groot and Tolhoek in order to obtain shape factors for the allowed β -transition.³⁾ They represented this density matrix by (4×4) matrix. It is possible to deal with their formalism only in such special case as the orbital angular momentum of the Fermion being zero, but it seems to be very hard in other cases. Thereupon, the reduction of density matrix to the more convenient form will be considered first (§ 3). It is suitable for this reduction to use the property of Dirac operators (I') which can be represented by the direct products of ρ -operators and σ -operators. ρ 's operate in the Dirac (ρ) space which is introduced since Fermions have positive and negative energy states, and σ 's are operators in the Pauli spin (σ) space. Of course, these ρ 's and σ 's are able to be represented by (2×2) matrices familiar with us.** As such a formalism is used, it is necessary to write the eigenfunction $\psi'_{\kappa, \mu}(x)$ of the Dirac equation in the form corresponding to the direct product of ρ -space and σ -space. Such form of the eigenfunction will be listed in § 2. In this formalism, the density matrix which is usually a (4×4) matrix will be represented as a sum of direct products of ρ 's and σ 's with numerical coefficients. In § 3, this representation of density matrix will be shown for a Fermion moving in an arbitrary central field. When the central field is removed, this density matrix must be, of course, reduced to the Casimir operator ((1.3) or (1.4)). This fact will be shown in § 4. Therefore, density matrices ((1.7a) and (1.7b)) introduced in this paper will be convenient not only for the calculation in the case of the Fermion in the central field, but for the partial wave analysis in the case of the free Fermion.

§ 2. Eigenfunction of Dirac equation

As stated in § 1, it is useful to start our discussion with the formalism, in which each Dirac operator (I') is represented as the direct product of ρ -operator and σ -operator. Therefore, the Dirac equation for a Fermion interacting with a central field (V) is written as follows:

$$H\psi(x) = E\psi(x), \quad (2.1)$$

* The fact that the parameter $(-a)$ is used in definition (D') (1.7b) does not mean that the eigenfunction ψ' is a charge-conjugate wave function. The reason why this notation is used will be cleared up in § 3.

** We list the matrix representations of ρ 's and σ 's used in this paper, though they are well known.

$$\begin{aligned} \rho_0 = I_\sigma &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \rho_3 = \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha = \rho_1 \sigma \quad \text{and} \quad \beta = \rho_3 I_\sigma = \rho_3. \end{aligned} \quad (1.8)$$

$$H = -i\rho_1(\boldsymbol{\sigma} \cdot \boldsymbol{p}) + m\rho_3 + V(r). \quad (2.2)$$

The Fermion eigenfunction (ψ) of this equation in the angular momentum representation has already been listed in the convenient form by Rose, Biedenharn and Arfken.⁽⁴⁾ But, as our definition of Hamiltonian (2.2) differs from theirs, we shall summarize here the results with their notation.* (This eigenfunction will be systematically derived in Appendix A, for convenience.)

We shall consider the Fermion with the energy E and quantum numbers κ and μ . Here, κ is a non-zero integer and gives both the total angular momentum j and the parity $(-1)^{l(\kappa)}$ for the Fermion state:

$$j = |\kappa| - 1/2, \quad (2.4)$$

and

$$l(\kappa) = |\kappa| + 1/2 \cdot (S_\kappa - 1), \quad (2.5)$$

where

$$S_\kappa = \begin{cases} +1 & \text{for } \kappa > 0, \\ -1 & \text{for } \kappa < 0. \end{cases} \quad (2.6)$$

j can also be written as follows:

$$j = l(\kappa) - (1/2)S_\kappa. \quad (2.7)$$

The eigenvalue of J_z is μ , the magnetic quantum number.

In order to obtain an eigenfunction for this state in the angular momentum representation, we introduce the spin eigenfunction $\chi_{1/2}^\tau$, which has eigenvalues $2\tau = \pm 1$ for σ_z -operator:

$$\sigma_z \chi_{1/2}^\tau = 2\tau \chi_{1/2}^\tau. \quad (2.8)$$

Then, the eigenfunction may be represented formally by two components corresponding to the (2×2) matrix in ρ -space.

$$\psi_{\kappa, \mu}(x) = \begin{pmatrix} \psi_{\kappa, \mu}^{(1)}(x) \\ \psi_{\kappa, \mu}^{(2)}(x) \end{pmatrix}. \quad (2.9)$$

Of course, each component ($\psi^{(1)}$ or $\psi^{(2)}$) including the spin function $\chi_{1/2}^\tau$ is itself a matrix with two rows and one column in σ -space.

For the positive energy state ($E = +W > 0$), the Fermion eigenfunction is given as follows:

* Rose, Biedenharn and Arfken⁽⁴⁾ used the following definition of Hamiltonian⁽⁵⁾:

$$H = i\rho_1(\boldsymbol{\sigma} \cdot \boldsymbol{p}) - m\rho_3 + V(r). \quad (2.3)$$

Thereupon, the third and fourth components ($\psi^{(3)}$ (2.9) of eigenfunction obtained by them are the so-called large components. In order to make easier to treat a free Fermion in a covariant formalism, we use here the different definition from theirs. As will be shown, the first and second components ($\psi^{(1)}$) of eigenfunction in this paper are large components.

$$\psi_{\kappa,\mu} = \begin{pmatrix} g_{\kappa}(r) \chi_{\kappa}^{\mu}(\theta, \varphi) \\ if_{\kappa}(r) \chi_{-\kappa}^{\mu}(\theta, \varphi) \end{pmatrix}. \quad (2.10)$$

Radial eigenfunctions f_{κ} and g_{κ} are real and determined by the radial equation with energy $E = +W$:

$$\begin{cases} (E - V - m)g_{\kappa} = (-\partial/\partial r + (\kappa - 1/r))f_{\kappa}, \\ (E - V + m)f_{\kappa} = ((\partial/\partial r) + (\kappa + 1/r))g_{\kappa}. \end{cases} \quad (2.11)$$

The radial variable (r), the parameter concerning the central potential (V) and the energy (E) are included only in these f_{κ} and g_{κ} . For example, in the case of the electron in the nuclear field ($V = -Ze^2/r$), they are given in the convenient forms by Rose⁶⁾ for the continuous energy state ($E = W > m$) and by Bethe⁷⁾ for the discrete state ($E = +W < m$). It is cleared from (2.11) that the first column of the eigenfunction (2.10), $\psi^{(1)}$, is the so-called large component. The spin-angular eigenfunction χ_{κ}^{μ} is defined as follows:

$$\chi_{\kappa}^{\mu} = \sum_{\tau=-1/2}^{1/2} C(l(\kappa) 1/2 j; \mu - \tau \tau \mu) Y_{l(\kappa)}^{\mu-\tau}(\theta, \varphi) \chi_{1/2}^{\tau}, \quad (2.12)$$

where the C -coefficient is the usual vector addition coefficient:⁸⁾

$$C(j_1 j_2 j_3; m_1 m_2 m_3) = (j_1 j_2 m_1 m_2 | j_1 j_2 j_3 m_3). \quad (2.13)$$

This alternative notation is adopted in the interest of brevity. The spherical harmonics are defined according to Condon and Shortley.⁸⁾⁹⁾

For the negative energy state ($E = -W < 0$), the spin-angular parts of the Fermion eigenfunction are the same as ones for the positive energy state. Therefore, as Rose has done,⁶⁾ the radial eigenfunction can be obtained from the equations (2.11) with energy $E = -W < 0$. But it is convenient for the further development to obtain the eigenfunction by using the charge conjugate operator defined by Furry¹⁰⁾:

$$C = -\rho_2 \sigma_y. \quad (2.14)$$

That is, using the Hamiltonian in which only the sign before the potential differs from (2.2),

$$H = -i\rho_1(\sigma \cdot \nabla) + m\rho_3 - V(r), \quad (2.15)$$

we shall obtain the eigenfunction of the following equation:

$$H\varphi_{\kappa,\mu}(x) = W\varphi_{\kappa,\mu}(x). \quad (2.16)$$

The influence of the central potential appears only in equations (2.11) for the radial part of the eigenfunction. Therefore, the so-called charge conjugate eigenfunction (φ) which is of an anti-Fermion with the same energy and momentum as those of a positive energy Fermion can be written in the following form:

$$\varphi_{\kappa,\mu}(x) = \begin{pmatrix} g'_{\kappa}(r) \chi_{\kappa}^{\mu}(\theta, \varphi) \\ if'_{\kappa}(r) \chi_{-\kappa}^{\mu}(\theta, \varphi) \end{pmatrix}, \quad (2.17)$$

where f'_κ and g'_κ are obtained respectively from the radial eigenfunctions f_κ and g_κ of a Fermion with the positive energy by changing only the signs before a parameter concerning the potential (V).

Thus, the negative-energy Fermion whose absence is the anti-Fermion has an eigenfunction,

$$\psi'_{\kappa,\mu}(x) = C\varphi_{\kappa,\mu}^+(x), \quad (2.18)$$

where φ^+ is φ with complex conjugate elements but not transposed. By using the following relation,

$$\sigma_y \chi_{\kappa}^{\mu+} = (-i)(-1)^{l(\kappa)+\mu+1-j} \chi_{\kappa}^{-\mu}, \quad (2.19)$$

the eigenfunction of the negative-energy Fermion is of the form

$$\psi'_{\kappa,\mu}(x) = (-1)^{l(\kappa)+\mu+1-j} \begin{pmatrix} -if'_\kappa(r) \chi_{\kappa}^{-\mu}(\theta, \varphi) \\ g'_\kappa(r) \chi_{\kappa}^{-\mu}(\theta, \varphi) \end{pmatrix}. \quad (2.20)$$

§ 3. Density matrix for a Fermion interacting with an arbitrary central field

The density matrix for a positive energy Fermion defined in (1.7a) can be represented as follows by making use of the eigenfunction (2.9):

$$D(x_1, x_2; \alpha) = \sum_{\kappa, \mu} \begin{pmatrix} \psi_{\kappa, \mu}^{(1)}(x_1) \psi_{\kappa, \mu}^{(1)*}(x_2), & \psi_{\kappa, \mu}^{(1)}(x_1) \psi_{\kappa, \mu}^{(2)*}(x_2) \\ \psi_{\kappa, \mu}^{(2)}(x_1) \psi_{\kappa, \mu}^{(1)*}(x_2), & \psi_{\kappa, \mu}^{(2)}(x_1) \psi_{\kappa, \mu}^{(2)*}(x_2) \end{pmatrix}. \quad (3.1)$$

It is needless to say that this matrix corresponds to a matrix in μ -space. We will first express this matrix in the form of a sum of four μ -operators including the unit operator (ρ_0) in μ -space (1.8),

$$D(x_1, x_2; \alpha) = \rho_0 D_0 + \rho_1 D_1 + \rho_2 D_2 + \rho_3 D_3, \quad (3.2)$$

where coefficients D 's, of course, correspond to the (2×2) matrices in σ -space. As it is convenient to reduce (3.2) in a form in which the density matrix has spin-angular parts as a common factor, we introduce the following notation,

$$\chi_{\varepsilon(u;\kappa)}^{\mu} = \begin{cases} \chi_{\kappa}^{\mu} & \text{for } u=0, \\ \chi_{\kappa}^{-\mu} & \text{for } u=1, \end{cases} \quad (3.3)$$

where

$$\varepsilon(u) = \begin{cases} +1 & \text{for } u=0, \\ -1 & \text{for } u=1. \end{cases} \quad (3.4)$$

Furthermore, the subscript κ appeared in each element of (3.1) can be replaced by $(-\kappa)$, because the summations are over all positive and negative integers of κ , except zero, and over all possible values of μ which are $(|\kappa|-1/2)$ to $(-|\kappa|+1/2)$ in integral steps. Thus,

$$D(x_1, x_2; \alpha) = \sum_{\kappa, \mu} \sum_{u=0}^1 \{ R_{\kappa, \mu}(r_1, r_2; \alpha) \rho_u + R_{\kappa, 3-u}(r_1, r_2; \alpha) \rho_{3-u} \} \times \quad (3.5)$$

$$\times \chi_{\varepsilon(u)\kappa}^{\mu}(\theta_1, \varphi_1) \chi_{\kappa}^{\mu*}(\theta_2, \varphi_2),$$

where numerical coefficients R 's are defined as follows:

$$\begin{aligned} R_{\kappa, 0}(r_1, r_2; \alpha) &= 1/2 \cdot (g_{\kappa}(r_1)g_{\kappa}(r_2) + f_{-\kappa}(r_1)f_{-\kappa}(r_2)), \\ R_{\kappa, 3}(r_1, r_2; \alpha) &= 1/2 \cdot (g_{\kappa}(r_1)g_{\kappa}(r_2) - f_{-\kappa}(r_1)f_{-\kappa}(r_2)), \\ R_{\kappa, 2}(r_1, r_2; \alpha) &= 1/2 \cdot (f_{\kappa}(r_1)g_{\kappa}(r_2) + g_{-\kappa}(r_1)f_{-\kappa}(r_2)), \\ R_{\kappa, 1}(r_1, r_2; \alpha) &= i/2 \cdot (f_{\kappa}(r_1)g_{\kappa}(r_2) - g_{-\kappa}(r_1)f_{-\kappa}(r_2)). \end{aligned} \quad (3.6)$$

The radial variable (r), the energy (W) and the parameter (α) concerning the potential are included only in these R 's.

In order to show apparently the fact that the spin-angular part, $\chi(\theta_1, \varphi_1) \chi^*(\theta_2, \varphi_2)$, is a (2×2) matrix in σ -space, we will try to represent this part as a sum of four σ -operators with numerical coefficients. Since the summation over μ in (3.5) is independent of R 's and ρ 's, it is sufficient to consider the reduction of the following expression:

$$\begin{aligned} & \sum_{\mu} \chi_{\varepsilon(u)\kappa}^{\mu}(\theta_1, \varphi_1) \chi_{\kappa}^{\mu*}(\theta_2, \varphi_2) \\ &= \sum_{\mu} \sum_{\tau, \tau'} C(l' \ 1/2 \ j; \ \mu - \tau \ \tau \ \mu) C(l \ 1/2 \ j; \ \mu - \tau' \ \tau' \ \mu) \times \\ & \quad \times Y_{l'}^{\mu - \tau}(\theta_1, \varphi_1) Y_{l'}^{\mu - \tau'}(\theta_2, \varphi_2) \chi_{1/2}^{\tau} \chi_{1/2}^{\tau'*}, \end{aligned} \quad (3.7a)$$

where

$$\begin{aligned} l &= l(\kappa) = |\kappa| + 1/2 \cdot (S_{\kappa} - 1), \\ l' &= l(\varepsilon(u)\kappa) = |\kappa| + 1/2 \cdot (S_{\varepsilon(u)\kappa} - 1). \end{aligned} \quad (3.7b)$$

It is useful for this reduction to perform a Racah recoupling according to the prescription:

$$\begin{aligned} & C(j_1 j_2 j; \ m_1 m_2 m) C(j \ j_3 j_4; \ m \ m_3 m_4) \\ &= \sum_v \sqrt{(2v+1)(2j+1)} C(j_2 j_3 v; \ m_2 m_3 m_2 + m_3) \times \\ & \quad \times C(j_1 v j_4; \ m_1 m_2 + m_3 m_4) W(j_1 j_2 j_3 j; \ j \ v), \end{aligned} \quad (3.8)$$

where $W(j_1 j_2 j_3 j; \ j \ v)$ is a Racah coefficient.¹¹⁾ The sums over τ' and μ in (3.7) are replaced by the sums over the new variables $t (= \tau - \tau')$ and $m (= \mu - \tau)$, respectively. Moreover, the following relations are used:

$$\sum_{\tau} (-1)^{\tau - t + (1/2)} C(1/2 \ 1/2 \ v; \ \tau \ t - \tau \ t) \chi_{1/2}^{\tau} \chi_{1/2}^{\tau - t*} = \sqrt{2\pi / (2v+1)} \mathcal{Y}_v^t(\sigma), \quad (3.9)$$

where $\mathcal{Y}_v^t(\sigma)$ is one of the $(2v+1)$ solid harmonics of degree v . (The correspondences of these $\mathcal{Y}_v^t(\sigma)$'s with the usual representations of σ -operators (1.8) will be given in Appendix B(B.7)). Here v means merely two numerical values (1 and 0) according to the property of vector addition coefficients:

$$C(1/2 \ 1/2 \ v; \tau \ t - \tau \ t) = (-1)^{(1/2-\tau)(1-v)} \sqrt{(1+2\tau vt)/2}. \quad (3 \cdot 10)$$

By substituting these values for C -coefficients, the relation (3.9) can easily be proved. Thus, our expected results are obtained:

$$\begin{aligned} & \sum_{\mu} \chi_{\varepsilon(u)\kappa}^{\mu}(\theta_1, \varphi_1) \chi_{\kappa}^{\mu*}(\theta_2, \varphi_2) \\ &= \sum_m \sum_{v, l} S_{-\kappa}(2j+1) \sqrt{2\pi/2l+1} C(l' \ v \ l; \ m \ t \ m+t) \times \\ & \quad \times W(l' \ 1/2 \ l \ 1/2; \ j \ v) Y_l^m(\theta_1, \varphi_1) Y_{l'-t}^{m-t*}(\theta_2, \varphi_2) \mathcal{Y}_v^t(\sigma). \end{aligned} \quad (3 \cdot 11)$$

The density matrix (3.1) can be represented as the sum of direct products of ρ -operators (3.5) and σ -operators (3.11) with numerical coefficients by use of definitions of l and l' (3.7b):

$$\begin{aligned} D(x_1, x_2; \alpha) &= \sum S_{-\kappa}(2j+1) \sqrt{2\pi/2l+1} C(l' \ v \ l; \ m \ t \ m+t) \times \\ & \quad \times W(l' \ 1/2 \ l \ 1/2; \ j \ v) Y_l^m(\theta_1, \varphi_1) Y_{l'-t}^{m+t*}(\theta_2, \varphi_2) \times \\ & \quad \times \{R_{\kappa, u} \rho_u + R_{\kappa, 3-u} \rho_{3-u}\} \mathcal{Y}_v^t(\sigma). \end{aligned} \quad (3 \cdot 12)$$

Here the summations are over κ, m, v, t and u . This is the general expression of the density matrix for a positive energy Fermion moving in an arbitrary central field. If the potential is really given, the density matrix can be obtained easily by introducing the radial eigenfunctions f_{κ} and g_{κ} solved in this case into definitions of R 's (3.6).

The density matrix for a negative energy Fermion defined in (1.7b) is expressed by making use of the eigenfunction (2.20) and reduced according to the similar procedure as for the positive energy Fermion:

$$\begin{aligned} D'(x_1, x_2; -\alpha) &= \sum_{\kappa, \mu} \sum_{u=0}^1 \{R'_{\kappa, u}(r_1, r_2; -\alpha) \rho_u + R'_{\kappa, 3-u}(r_1, r_2; -\alpha) \rho_{3-u}\} \times \\ & \quad \times \chi_{\varepsilon(u)\kappa}^{\mu}(\theta_1, \varphi_1) \chi_{\kappa}^{\mu*}(\theta_2, \varphi_2), \end{aligned} \quad (3 \cdot 13)$$

where

$$\begin{aligned} R'_{\kappa, 0}(r_1, r_2; -\alpha) &= 1/2 \cdot (g'_{\kappa}(r_1) g'_{\kappa}(r_2) + f'_{-\kappa}(r_1) f'_{-\kappa}(r_2)), \\ R'_{\kappa, 3}(r_1, r_2; -\alpha) &= -1/2 \cdot (g'_{\kappa}(r_1) g'_{\kappa}(r_2) - f'_{-\kappa}(r_1) f'_{-\kappa}(r_2)), \\ R'_{\kappa, 2}(r_1, r_2; -\alpha) &= 1/2 \cdot (f'_{\kappa}(r_1) g'_{\kappa}(r_2) + g'_{-\kappa}(r_1) f'_{-\kappa}(r_2)), \\ R'_{\kappa, 1}(r_1, r_2; -\alpha) &= -1/2 \cdot (f'_{\kappa}(r_1) g'_{\kappa}(r_2) - g'_{-\kappa}(r_1) f'_{-\kappa}(r_2)). \end{aligned} \quad (3 \cdot 14)$$

The reason why the parameter $(-\alpha)$ is used in the definition of density matrix for the negative energy Fermion is that radial eigenfunctions f'_{κ} and g'_{κ} are followed from positive energy eigenfunctions f_{κ} and g_{κ} by changing the sign before the potential parameter (α) , as it was stated in § 2. Therefore, if these definitions of $R'(-\alpha)$'s for the negative energy Fermion are compared with $R(+\alpha)$'s (3.6) for the positive energy Fermion, the following simple relations will be found:

$$\begin{aligned}
R'_{\kappa,0}(+\alpha) &= R_{\kappa,0}(+\alpha), \\
R'_{\kappa,3}(+\alpha) &= -R_{\kappa,3}(+\alpha), \\
R'_{\kappa,2}(+\alpha) &= R_{\kappa,2}(+\alpha), \\
R'_{\kappa,1}(+\alpha) &= -R_{\kappa,1}(+\alpha).
\end{aligned}
\tag{3.15}$$

The spin-angular part of this density matrix (3.13) is the same as for the positive energy Fermion, so the same reduction of this part (3.11), can be used again. Thus, the density matrix (3.13) for the negative energy Fermion is the same as the one (3.12) for the positive energy Fermion, except definitions of R 's. That is, the signs of parameter (α) are different in each case and, at the same time, the signs of $R'_1(-\alpha)$ and $R'_3(-\alpha)$ are different from corresponding $R_1(+\alpha)$ and $R_3(+\alpha)$, as shown in the relations (3.15). In other words, noting that

$$R'_{\kappa,3}(+\alpha)\rho_3 = -R_{\kappa,3}(+\alpha)\rho_3, \tag{3.16}$$

and

$$R'_{\kappa,1}(+\alpha)\rho_1 = -R_{\kappa,1}(+\alpha)\rho_1,$$

we may say that merely the signs before ρ_1 , ρ_3 and parameter α are changed into minus signs.

§ 4. Density matrix for a free Fermion

In the preceding section, the general expressions of density matrix were obtained for a Fermion moving in an arbitrary central field. Such a density matrix must include one for a free Fermion as a special case. Furthermore, the density matrices obtained in § 3 must be reduced to Casimir operators when the central potential (V) is removed, that is, in the limiting case where the parameter (α) concerning the potential is equal to zero. We should like to verify this fact.

When the central potential is equal to zero, only the radial parts f_κ and g_κ of the Fermion eigenfunction are different from those in the case of $V \neq 0$. Therefore, in the expression ((3.12) or (3.13)) for the density matrix, the spin-angular part given $\chi(\theta_1, \varphi_1)\chi^*(\theta_2, \varphi_2)$ (3.11) is the same as one for $V \neq 0$, but the radial parts R 's are given by the following simpler expressions in the case of the free Fermion with the positive energy*:

* In the case of the free Fermion, the radial part of eigenfunction which is normalized per unit energy interval may be given as follows:

$$\begin{aligned}
f_\kappa &= [p(W-m)/\pi]^{1/2} j_{\kappa}(-\kappa)(pr), \\
g_\kappa &= [p(W+m)/\pi]^{1/2} S_\kappa j_{\kappa}(\kappa)(pr),
\end{aligned}
\tag{4.2}$$

where $j_\kappa(pr)$ is a spherical Bessel function. This is also reduced from the eigenfunction of Dirac equation with the central potential (V) by putting V equal to zero. For example, when V is the Coulomb potential ($V = Ze^2/r$), the eigenfunctions given by eq. (4.2) have only opposite signs to ones which are reduced from the Coulomb field eigenfunctions obtained by Rose⁽⁶⁾.

$$\begin{aligned}
R_{\kappa,0}(r_1, r_2; 0) &= (Wp/\pi) j_{l(\kappa)}(pr_1) j_{l(\kappa)}(pr_2), \\
R_{\kappa,3}(r_1, r_2; 0) &= (mp/\pi) j_{l(\kappa)}(pr_1) j_{l(\kappa)}(pr_2), \\
R_{\kappa,2}(r_1, r_2; 0) &= 0, \\
R_{\kappa,1}(r_1, r_2; 0) &= iS_{\kappa}(p^2/\pi) j_{l(-\kappa)}(pr_1) j_{l(\kappa)}(pr_2).
\end{aligned} \tag{4.1}$$

All quantities concerning the free particle are distinguished from the corresponding ones for the particle moving in the central field by putting the parameter (α) equal to zero. In the case of a Fermion with the negative energy, the radial eigenfunctions f_{κ}' and g_{κ}' are the same ones as f_{κ} and g_{κ} (4.2), but, as is known from the general discussion (3.15), the definitions of $R_{\kappa}'(0)$'s are different:

$$\begin{aligned}
R'_{\kappa,0}(r_1, r_2; 0) &= R_{\kappa,0}(r_1, r_2; 0), \\
R'_{\kappa,3}(r_1, r_2; 0) &= -R_{\kappa,3}(r_1, r_2; 0), \\
R'_{\kappa,2}(r_1, r_2; 0) &= 0, \\
R'_{\kappa,1}(r_1, r_2; 0) &= -R_{\kappa,1}(r_1, r_2; 0).
\end{aligned} \tag{4.3}$$

It is also apparent from the definition that $R_3(0)$ is also equal to zero, if the Fermion has not its mass:

$$R_{\kappa,3}(r_1, r_2; 0) = R'_{\kappa,3}(r_1, r_2; 0) = 0. \tag{4.4}$$

Thus, it is clear that the density matrix for a free Fermion can be expressed as the same form with eq. (3.12) merely by introducing the free Fermion eigenfunction into R 's. But, in this case, as a result of these simple expressions for R 's ((4.1) or (4.3)), two summations over u and v appeared in (3.12) can be reduced to a summation over v . The fact that the possible values (0 or 1) of u are always equal to each corresponding value (0 or 1) of v is verified by using the following relation:

$$\begin{aligned}
&W(l(\varepsilon(u)\kappa) \ 1/2 \ l(\kappa) \ 1/2; j \ v) \\
&= (S_{-\kappa})^{(1-v)(1-u)} \left[\frac{(2j+1+2v S_{\kappa})(1-u) + vu(2l(\kappa)+1)}{(2-u)(2v+1)(2j+1)(2l(\kappa)+1)} \right]^{1/2}
\end{aligned} \tag{4.5}$$

When $v=0$, it is clear that $u=0$, because, if $u=1$, W -coefficient is equal to zero. In the case of $v=1$ and $u=0$, the summation over positive κ cancels out the sum over negative κ , because R 's have the simple dependence on κ for the free Fermion. Furthermore, using expressions for vector addition coefficients,

$$C(l(\kappa) \ 0 \ l(\kappa); \ 0 \ 0 \ 0) = 1, \tag{4.6}$$

$$C(l(\kappa) \ 1 \ l(-\kappa); \ 0 \ 0 \ 0) = S_{-\kappa} \sqrt{(2j+1)/2(2l(\kappa)+1)},$$

we may write the density matrix for a free Fermion as follows:

$$\begin{aligned}
&D(x_1, x_2; 0) \\
&= \sum_{\kappa, m} \sum_{v, l} (1/2)^{1-v} \sqrt{4\pi/2v+1} \ C(l(\varepsilon(v)\kappa) \ v \ l(\kappa); \ m \ t \ m+t) \times
\end{aligned}$$

$$\begin{aligned} & \times C(l(\kappa) v \ l(\varepsilon(v)\kappa); \ 0 \ 0 \ 0) Y_{l(\varepsilon(v)\kappa)}^m(\theta_1, \varphi_1) Y_{l(\kappa)}^{m+t*}(\theta_2, \varphi_2) \\ & \times \{R_{\kappa, v}(r_1, r_2; \ 0) \rho_v + R_{\kappa, 3-v}(r_1, r_2; \ 0) \rho_{3-v}\} \mathcal{J}_v^t(\sigma). \end{aligned} \quad (4.7)$$

This expression is still in a very unwieldy form to assure that this corresponds to the Casimir operator. Therefore, the summation over κ is replaced by the sum over the azimuthal quantum number (l), which is familiar with us,* and Rayleigh's relation is used:

$$e^{ipx} = 4\pi \sum_{l, m} i^l j_l(pr) Y_l^{m*}(\theta, \phi) Y_l^m(\theta, \varphi). \quad (4.8)$$

θ and ϕ are angular variables with respect to the direction of motion of the plane wave. Also we use the well-known formula:

$$\begin{aligned} & Y_l^m(\theta, \varphi) Y_l^t(\theta, \varphi) \\ & = (-1)^v \sum_L \sqrt{(2v+1)/4\pi} C(lvL; m \ t \ m+t) C(LvL; \ 000) Y_L^{m+t}(\theta, \varphi). \end{aligned} \quad (4.9)$$

Then, by using the definitions of R 's for the positive energy Fermion, the density matrix (4.7) can be reduced to the following expression:

$$D(x_1, x_2; \ 0) = \frac{pW}{(2\pi)^3} \int d\Omega \ A_+ (+\mathbf{p}) e^{ipx_1 - ipx_2}. \quad (4.10)$$

Here $d\Omega$ is a solid angle of θ and ϕ , and $A_+ (+\mathbf{p})$ is a Casimir operator defined by (1.3). In the momentum representation, this is expressed as follows,**

$$(\mathbf{p}_1 | D(x_1, x_2; \ 0) | \mathbf{p}_2) = A_+ (+\mathbf{p}) \delta(\mathbf{p}_1 - \mathbf{p}_2) \delta(W - W_1). \quad (4.11)$$

Thus, when a Fermion has a positive continuous energy ($E=W>m$), it has been verified the density matrix (3.12) corresponds to the Casimir operator. Is the same situation kept in the case of a Fermion having a negative energy ($E=-W<-m$)? This question is confirmed easily. The density matrix for a negative energy Fermion is the same as one for the Fermion in the positive energy state, except signs before ρ_1 and ρ'_3 , since the definitions (4.3) of $R'_1(0)$ and $R'_3(0)$ differ from those of $R_1(0)$ and $R_3(0)$. The negative Fermion energy whose absence is an anti-Fermion has momentum ($-\mathbf{p}$) in contrast with momentum ($+\mathbf{p}$) for the positive energy Fermion. Therefore, it is clear from (4.10) that the density matrix for a free Fermion in the negative energy state corresponds to the Casimir operator $A_-(-\mathbf{p})$;

$$D'(x_1, x_2; \ 0) = \frac{pW}{(2\pi)^3} \int d\Omega A_-(-\mathbf{p}) e^{-ipx_1 + ipx_2}. \quad (4.13)$$

* As it will be cleared in Appendix A ((A.16) and (A.17)), the relation between l and κ may be given as follows:

$$\begin{aligned} \kappa = l > 0, \ l(\kappa) = l, \quad \kappa = -(l+1) < 0, \ l(\kappa) = l, \\ l(-\kappa) = l-1, \quad l(-\kappa) = l+1. \end{aligned} \quad (4.12)$$

** The author wishes to express his appreciation to Professors T. Miyazima and N. Fukuda for their discussions on this point.

At last, for the sake of the applications of these technics, we should like to tell of the density matrix (d) for a neutrino. In this case, as the central field (V) can always be set equal to zero, the density matrix with $R_{\kappa,2}=0$ must be used as shown in (4.1). Furthermore, if its mass is ignored, ($W \cdot \frac{1}{2}m$) must be replaced by the energy of the neutrino (q) and $R_{\kappa,3}$ has to be put equal to zero, according to (4.4).^{*} Thus the density matrix for a neutrino is obtained as follows:

$$\begin{aligned} d(x_1, x_2; 0) = & \sum_{\kappa, m} \sum_{v, l} \sum_u S_{-\kappa}(2j+1) \sqrt{2\pi/2l+1} C(l'vl; m \ t \ m+t) \times \\ & \times W(l' \ 1/2 \ l \ 1/2; j \ v) Y_{l'}^m(\theta_1, \varphi_1) Y_l^{m+t*}(\theta_2, \varphi_2) \times \\ & \times R_{\kappa, u}(r_1, r_2; 0) \rho_u \mathcal{J}_l^{t'}(\sigma), \end{aligned} \quad (4.14)$$

where $l=l(\kappa)$ and $l'=l(\varepsilon(u)\kappa)$,

$$\begin{aligned} R_{\kappa,0}(r_1, r_2; 0) &= (q^2/\pi) j_{l(\kappa)}(qr_1) j_{l(\kappa)}(qr_2), \\ R_{\kappa,1}(r_1, r_2; 0) &= iS_{\kappa}(q^2/\pi) j_{l(-\kappa)}(qr_1) j_{l(\kappa)}(qr_2). \end{aligned} \quad (4.15)$$

We may, of course, set the running subscript u equal to v , as it was shown in (4.5) to (4.7). If the neutrino is in the negative energy state, the density matrix ($d'(x_1, x_2; 0)$) is obtained from (4.14) by changing the sign of definition of R_l :

$$R'_{\kappa,1}(r_1, r_2; 0) = -iS_{\kappa}(q^2/\pi) j_{l(-\kappa)}(qr_1) j_{l(\kappa)}(qr_2). \quad (4.16)$$

It is important, in the case of the neutrino, that merely ρ_0 and ρ_1 appear in the density matrix (4.14) among four ρ -operators (ρ_0, ρ_1, ρ_2 and ρ_3). This will be shown in the other paper.

§ 5. Discussions

As expected in § 1, the density matrix could be expressed as the sum of direct products of ρ - and σ -operators with numerical coefficients. Therefore, we can replace the complicate method which has been used till now by a trace of ρ - and σ -operators even when a Fermion moves in an arbitrary central field. This trace is separated into two traces of only ρ -operators and of σ -operators, because all ρ 's are commutable with σ 's. The trace with respect to ρ 's can be evaluated more easily than l 's. But, as the trace of all products of σ -operators is generally included in the form of vector σ , this trace is slightly difficult. Then the following relation being used**

$$\begin{aligned} \mathcal{J}_v^{t'}(\sigma) \mathcal{J}_u^s(\sigma) = & \sum_{\alpha} \frac{(2v+1)(2u+1)}{\sqrt{2\pi(2\alpha+1)}} C(uv\alpha; s \ t \ s+t) \times \\ & \times W(1/2 \ u \ 1/2 \ v; 1/2 \ \alpha) \mathcal{J}_{\alpha}^{t+s}(\sigma), \end{aligned} \quad (5.1)$$

* To avoid confusion, the momentum and the energy for the neutrino will subsequently be denoted by q 's, instead of p and W .

** This relation is easily proved by making use of the relation (3.9) between $\mathcal{J}_v^{t'}(\sigma)$ and $\chi_{1/2}^{\tau} \chi_{1/2}^{\tau-t*}$ and of the prescription (3.8).

all numbers of σ -operators are reduced and at last a trace of $\mathcal{Y}_\alpha^\nu(\sigma)$ remains;

$$\text{Tr}[\mathcal{Y}_\alpha^\nu(\sigma)] = 2\delta_{\alpha,0} \delta_{\nu,0} \cdot 1/\sqrt{4\pi}. \quad (5.2)$$

Thus, the trace of σ -operator is expressed as products of C - and W - coefficients. These coefficients are considered with other coefficients appeared in the definition of the density matrix (3.12) and reduced by using the sum rule (3.8) and the orthonormal properties of these coefficients.

The technics of calculation proposed in this paper will be useful to treat such various problems as β -decay, Bremsstrahlung, pair creation, photoelectric effect and so on, when the influence of a central potential must be considered.

Especially in the case of the β -decay process, the well-known quantities, which have been written by notations L, M, N, P, Q and R after the complicated calculation, play an important role in giving the energy spectrum for the forbidden transition. They are merely given by R 's (3.6) except additional numerical coefficients.^{12) 13) 14)} Therefore, it is clear that the energy spectrum is easily derived. This will be shown in the other paper. Also, the angular correlation between electrons and anti-neutrinos (or positrons and neutrinos) for any degree of forbidden transition can be obtained even when the influence of the Coulomb field on the electron is taken into a account. This problem has been treated as a kind of scattering proposed by Greuling and Meeks,¹⁵⁾ in which an incident negative energy neutrino is transformed by the Fermi interaction into a scattered electron in a positive energy state. The direction of motion of the Fermion is easily introduced into our definition of density matrix, as shown by (4.10). Therefore, we can obtain the angular correlation according to the usual perturbation formalism as well as for a free particle.

In addition, the technics proposed in this paper is not only favourable when we treat the Fermion moving in a central field, but also useful when we try to solve the problem for the free Fermion by the method of partial waves. The application on the latter problem will be reported.

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Appendix A.

On the eigenfunction of Dirac equation

As stated in § 2, the convenient form of the eigenfunction of Dirac equation was given by Rose, Biedenharn and Arfken.⁴⁾ But, the definition of Hamiltonian (2.2) used in this paper differs from theirs and they did not apparently show the derivation to obtain their results. Therefore, we would like to show this procedure for convenience.

As is well known, we have j and μ as quantum numbers which are eigenvalues of the following two operators:

$$(\mathbf{J})^2 = (\mathbf{L} + \boldsymbol{\sigma}/2)^2 \text{ and } J_z, \quad (\text{A.1})$$

both of which, of course, commute with Hamiltonian (2.2). Following Dirac⁵⁾, we introduce an additional operator which commutes also with H and which will distinguish between states with σ parallel or antiparallel to \mathbf{J} .*

$$K = -\rho_3((\sigma \cdot L) + 1). \quad (\text{A} \cdot 2)$$

As there is the following relation between \mathbf{J} and K ,

$$\mathbf{J}^2 = K^2 - (1/4), \quad (\text{A} \cdot 4)$$

K -operator has all positive and negative integer eigenvalues (κ) except zero and

$$j = |\kappa| - (1/2). \quad (\text{A} \cdot 5)$$

The K -operator does not include the quantities depending on the radial variables (r). Therefore, the angular and spin parts of the eigenfunction are fixed by the requirement that ψ is an eigenfunction of the K operator:

$$K\psi = \kappa\psi. \quad (\text{A} \cdot 5)$$

When this eigenvalue problem is considered, the Dirac equation (2.1) can be reduced to the well-known equation;

$$(-i\rho_1\sigma_r(\partial/\partial r + 1/r + \rho_3\kappa/r) + m\rho_3 + V(r))\psi = E\psi, \quad (\text{A} \cdot 6)$$

where

$$\sigma_r = (\sigma \cdot \mathbf{r})/r = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix}, \quad (\text{A} \cdot 7)$$

$$\sigma_r^2 = 1. \quad (\text{A} \cdot 8)$$

As stated in (2.9), we assume that the eigenfunction is formally represented by a matrix with two rows and one column corresponding to the ρ -space:

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}. \quad (\text{A} \cdot 9)$$

Then, the eigenvalue problem (A.5) is rewritten as follows:

$$((\sigma \cdot L) + 1)\psi^{(1)} = -\kappa\psi^{(1)}, \quad (\text{A} \cdot 9)$$

$$((\sigma \cdot L) + 1)\psi^{(2)} = \kappa\psi^{(2)}. \quad (\text{A} \cdot 10)$$

These components including the spin function $\chi_{l,2}^{\pm}$ (2.8) have two components, respectively, one of which we write, for example,

* If we define K as follows,

$$K = \rho_3((\sigma \cdot L) + 1), \quad (\text{A} \cdot 3)$$

as Dyson has done¹⁶⁾, and if we consider the same eigenvalue problem as (A.5), every signs of κ appeared in the eigenfunction must be changed. For example, the definition of $l(\kappa)$ (2.5) is replaced by $l(\kappa) = \kappa - 1/2(S_{\kappa} + 1)$ and the state with $\kappa = +1$ may correspond to the non-relativistic state with $l=0$ (S -state). On the other hand, according to our definition (A. 2) used here, the non-relativistic state with $l=0$ may be derived from the state with $\kappa=-1$.

$$\phi^{(1)} = \sum_{\tau} A_{\tau} \chi_{1/2}^{\tau} = \begin{pmatrix} A \\ B \end{pmatrix}, \quad (\text{A} \cdot 11)$$

where $A_{1/2} = A$ and $A_{-1/2} = B$. Substitution of (A·11) into the eigenvalue problem (A·9) gives us the equations depending merely on angular variables (θ and φ);

$$L_0 A + \sqrt{2} L_{-1} B = -(\kappa + 1) A, \quad (\text{A} \cdot 12a)$$

$$\sqrt{2} L_{+1} A + L_0 B = +(\kappa + 1) B. \quad (\text{A} \cdot 12b)$$

Here representations of L_{μ} -operators in the spherical coordinate are defined and listed in Appendix B(B·2). Using the commutation relation between two L_{μ} -operators (B·3) and the definition of L^2 -operator (B·4), we obtain the following equation,

$$(L^2 - \kappa(\kappa + 1)) A = 0. \quad (\text{A} \cdot 13)$$

The eigenfunctions of this equation are easily obtained according to the well-known eigenvalue problem (B·5):

$$A_k = a_k g_k(r) Y_k^m(\theta, \varphi) \quad \text{for } \kappa = k > 0, \quad (\text{A} \cdot 14a)$$

or

$$A_{-k} = a_{-k} g_{-k}(r) Y_{k-1}^m(\theta, \varphi) \quad \text{for } \kappa = -k < 0. \quad (\text{A} \cdot 14b)$$

Here a 's are constants independent of the variables (r , θ and φ) and $g(r)$ is a function of only the radial variable (r). The Fermion state with quantum numbers κ and μ is now considered and the eigenfunction belongs to the state in which the z -component of spin angular momentum is $\tau = 1/2$. So, the z -component of orbital angular momentum is

$$m = \mu - 1/2. \quad (\text{A} \cdot 15)$$

Following Rose, Biedenharn and Arfken⁴⁾ in order to represent formally the above two solutions (A·14) by a notation, we use the convenient notation $l(\kappa) = |\kappa| + (S_x - 1)/2$ (2·5). According to the definition of j , (2·4),

$$j = l(\kappa) - (S_x/2) = \begin{cases} l(k) - 1/2 & \text{for } \kappa > 0, \\ l(-k) + 1/2 & \text{for } \kappa < 0. \end{cases} \quad (2 \cdot 7)$$

It will be shown from radial equations (2·11) that the first row ($\phi^{(1)}$) of ϕ (2·9) is the so-called large component. Therefore, we may be able to obtain the correspondence to the non-relativistic results by taking

$$l(\kappa) = l. \quad (\text{A} \cdot 16)$$

This fact is equivalent to assuming the relations,

$$\kappa = \begin{cases} l & \text{for } \kappa > 0, \\ -(l+1) & \text{for } \kappa < 0. \end{cases} \quad (\text{A} \cdot 17)$$

This is clearly derived from the definition of $l(\kappa)$ (2·5). Using these notations, we can write two solutions (A·14) by a notation as follows:

$$A_{\kappa} = a_{\kappa} g_{\kappa}(r) Y_{l(\kappa)}^{\mu-1/2}(\theta, \varphi). \quad (\text{A} \cdot 18)$$

As L_{-1} is a step-down operator, the other solution of (A·12) for $\tau = -1/2$ may be assumed to take the form,

$$B_{\kappa} = b_{\kappa} g_{\kappa}(r) Y_{l(\kappa)}^{\mu+1/2}(\theta, \varphi). \quad (\text{A} \cdot 19)$$

One assumes the normalization condition as follows :

$$\int |\psi^{(1)}|^2 \sin \theta \, d\theta \, d\varphi = |g_{\kappa}(r)|^2, \quad (\text{A} \cdot 20)$$

that is,

$$a_{\kappa}^2 + b_{\kappa}^2 = 1. \quad (\text{A} \cdot 21)$$

Substitution of these eigenfunctions (A·18) and (A·19) into equations (A·12) gives us relations for determining these numerical constant factors a 's and b 's. Thus, in the case of $\kappa > 0$,

$$\psi_{k,\mu}^{(1)} = \pm g_k(r) \sum_{\tau} C(l(k) \, 1/2 \, l(k) - 1/2; \mu - \tau \, \tau \, \mu) Y_{l(k)}^{\mu-\tau} \chi_{1/2}^{\tau}, \quad (\text{A} \cdot 22)$$

and in the case of $\kappa < 0$,

$$\psi_{-k,\mu}^{(2)} = \pm g_{-k}(r) \sum_{\tau} C(l(-k) \, 1/2 \, l(-k) + 1/2; \mu - \tau \, \tau \, \mu) Y_{l(-k)}^{\mu-\tau} \chi_{1/2}^{\tau}. \quad (\text{A} \cdot 23)$$

One of these eigenfunctions with either of the two signs can be used in each case. Following the definition (2·12) by Rose, Biedenharn and Arfken⁴⁾, we may write

$$\psi_{\kappa,\mu}^{(1)} = g_{\kappa}(r) \chi_{\kappa}^{\mu}(\theta, \varphi), \quad (\text{A} \cdot 24)$$

instead of (A·22) and (A·23). This also demonstrates the following relation given by Rose et al.⁴⁾

$$((\sigma \cdot \mathbf{L}) + 1) \chi_{\kappa}^{\mu} = -\kappa \chi_{\kappa}^{\mu}. \quad (\text{A} \cdot 25)$$

We can easily obtain the second column $\psi^{(2)}$ of the eigenfunction of (A·9b) by making use of this relation (A·25). That is,

$$\psi_{\kappa,\mu}^{(2)} = C f_{\kappa}(r) \chi_{-\kappa}^{\mu}(\theta, \varphi), \quad (\text{A} \cdot 26)$$

where C is a phase factor and $f_{\kappa}(r)$ is a radial eigenfunction. If we substitute these eigenfunctions into the Dirac equation (A·7) and use the relation $\sigma_r \chi_{\kappa}^{\mu} = -\chi_{-\kappa}^{\mu}$ given by Rose et al. too¹⁾, we may put the phase factor (C) equal to pure imaginary (i) and obtain the radial equations (2·11), by which radial (real) eigenfunctions are determined.

Appendix B.

Relations in the spherical representation

Throughout this paper, the spherical representation has been used. Though various relations are given by Rose¹⁷⁾, we will list some of them for convenience. The spherical components of any vector \mathbf{A} are A_{μ} which are given by

$$A_{\pm 1} = \mp (A_x \pm i A_y) / \sqrt{2} \quad \text{and} \quad A_0 = A_z. \quad (\text{B} \cdot 1)$$

For example, components of the angular momentum operator L are given as follows :

$$\begin{aligned} L_{+1} &= -e^{i\varphi} [\partial/\partial\theta + i \cot\theta \partial/\partial\varphi] / \sqrt{2}, \\ L_0 &= 1/i \cdot \partial/\partial\varphi, \\ L_{-1} &= e^{-i\varphi} [-\partial/\partial\theta + i \cot\theta \partial/\partial\varphi] / \sqrt{2}. \end{aligned} \quad (\text{B} \cdot 2)$$

The commutation rule takes the form

$$L_\mu L_\nu - L_\nu L_\mu = L_{\mu+\nu} \text{ for } \mu < \nu. \quad (\text{B} \cdot 3)$$

The total angular momentum operator is

$$\begin{aligned} L^2 &= -L_{+1} L_{-1} + L_0^2 - L_{-1} L_{+1} \\ &= -\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right). \end{aligned} \quad (\text{B} \cdot 4)$$

If we denote by Y_l^m any simultaneous eigenfunction of L^2 and L_z with eigenvalues $l(l+1)$ and m , respectively, the following relations are held :

$$(L^2 - l(l+1)) Y_l^m = 0, \quad (\text{B} \cdot 5)$$

$$L_\mu Y_l^m = \sqrt{l(l+1)} C(l \ 1 \ l; \ m \ \mu \ m + \mu) Y_l^{m+\mu}. \quad (\text{B} \cdot 6)$$

The correspondences of $\mathcal{Y}_i^m(\sigma)$ to the usual representation of σ -operators are as follows :

$$\begin{aligned} \sqrt{4\pi} \mathcal{Y}_0^0(\sigma) &= I_\sigma = \sum_i \chi_{1/2}^i \chi_{1/2}^{i*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sqrt{4\pi/3} \mathcal{Y}_1^1(\sigma) &= \sigma_{+1} = -1/\sqrt{2} \cdot (\sigma_x + i\sigma_y) = -\sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \sqrt{4\pi/3} \mathcal{Y}_1^0(\sigma) &= \sigma_0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sqrt{4\pi/3} \mathcal{Y}_1^{-1}(\sigma) &= \sigma_{-1} = 1/\sqrt{2} \cdot (\sigma_x - i\sigma_y) = \sqrt{2} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (\text{B} \cdot 7)$$

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Letters to the Editor

Note on the Non-Perturbation- Approach to Quantum Field Theory

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Recently, there have been some attempts¹⁾²⁾³⁾ to investigate the quantum field theory without using any perturbation expansion. These stand on the fundamental assumptions which are introduced from the physical point of view. Moreover, commutability between several operations such as integration, differentiation and limit is assumed in their analysis. (This assumption will be called (A) in this note.)

Under these assumptions, Källén investigated Quantum Electrodynamics and obtained several interesting results. However, we shall show that an undesirable result will be derived if we follow their line of thought.

According to Källén's arguments,¹⁾ we find the following relation,

$$\langle 0 | [j_\mu(x), j_\nu(x')] | 0 \rangle = -i \int_0^\infty d\pi(-a) \times \left(\square \partial_{\mu\nu} - \frac{\partial^2}{\partial x_\mu \partial x_\nu} \right) \mathcal{A}(x-x', a) \quad (1)$$

where $|0\rangle$ is a true vacuum state vector, $j_\mu(x)$ a current operator in Heisenberg representation and $\pi(-a)$ a positive definite function introduced by Källén. This property of $\pi(-a)$ is one of the essential points in his discussion. From the relation (1), using the assumption (A) we obtain

$$\begin{aligned} \langle 0 | [j_i(x), j_i(x')] | 0 \rangle_{t=t'} \\ = \frac{\partial}{\partial x_i} \delta^3(x-x') \int_0^\infty \pi(-a) da \quad i=1, 2, 3. \end{aligned} \quad (2)$$

On the other hand, using the canonical commutation relation we see that

$$[j_i(x, t), j_i(x', t)] = 0.$$

Therefore, in view of their relationship, we can see that the equation

$$\int_0^\infty \pi(-a) da = 0 \quad (3)$$

must hold. From the equation (3) and the positive definiteness of $\pi(-a)$, it is easily seen that

$$\pi(-a) = 0. \quad (4)$$

Thus we come to the following conclusion

$$\begin{aligned} \langle 0 | [A_\mu(x), A_\nu(x')] | 0 \rangle &= -i \partial_{\mu\nu} D(x-x') \\ L &= 0. \end{aligned} \quad (5)$$

where A is a vector potential operator of the electromagnetic field in Heisenberg representation and L a charge renormalization constant. The same result as (5) can be derived also from the arguments of Gell-Mann and Low.²⁾ This is an undesirable result because it means the assumptions are too severe to reach a non-trivial solution; that is, a solution with the exception of the case of free field.

It seems to us that there may be two reasons for this result. The one is the assumption on the cancellation of the contributions from scalar and longitudinal photons. This seems to be not yet verified in the representation in which total Hamiltonian is diagonal. The positive definite property of π is based on this assumption. The other is the assumption (A) which is of use in examining the properties of modified propagators.

We shall not discuss the former in this note. The above mentioned undesirable result may be avoided, if this assumption (A) is applied only under some suitable conditions. This means the asymptotic form of $\pi(-a)$ decreases in such a slow rate in the region of large a that the assumption (A) is not permitted in certain cases. For instance if the asymptotic form of $\pi(-a)$ decreases slower than a^{-1} we can not arrive at the equation (2) and therefore the undesirable result (5) does not occur. This fact suggests that the nature of cut-off factor should not be strong if we use a sort of cut-off procedure tentatively instead of the future rational theory. Furthermore, such an asymptotic property of π would prevent us from obtaining another undesirable conclusion³⁾ that the singularity of \mathcal{A}' is not less than that of \mathcal{A} .

We wish to express our gratitude to Prof. R. Utiyama and Prof. G. Källén for their helpful

letters, and also to Prof. H. Yukawa, Dr. S. Nakai and Mr. S. Sunakawa for their valuable discussions.

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A Note on the Static Approximation of the Nucleon Anomalous Magnetic Moment

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Recently some doubts on the static approximation of anomalous magnetic moment of a nucleon have pointed out by E. Yamada¹⁾. He says that it is difficult to perform the charge renormalization, for the matrix elements of the vertices are all proportional to $(e/2m)\sigma \cdot \mathbf{H}$, in the static approximation. The aim of this note is to analyze what parts of the covariant one are calculated in the static approximation.

In the static approximation the interaction Hamiltonians²⁾

$$\begin{aligned} H^e(x) &= -(e/2m) \psi^*(t) (\sigma \cdot \mathbf{H}) \\ &\times \left(\frac{1+\tau_3}{2} \right) \psi(t) + e A_\mu^e T_{\alpha\beta} \varphi_\alpha \nabla \varphi_\beta, \\ H^i(x) &= (g/\mu) \psi^*(t) \tau_\alpha \sigma \cdot \nabla \varphi_\alpha \psi(t), \end{aligned} \quad (1)$$

are used as the approximated form of the covariant one

$$\begin{aligned} H^e(x) &= -ie \bar{\psi}(x) A_\mu^e \gamma_\mu \\ &\times \left(\frac{1+\tau_3}{2} \right) \psi(x) + e A_\mu^e T_{\alpha\beta} \varphi_\alpha \partial_\mu \varphi_\beta, \end{aligned} \quad (2)$$

$$H^i(x) = if \bar{\psi}(x) \gamma_5 \tau_\alpha \psi(x) \varphi_\alpha; \quad (2m/\mu)g = f,$$

where H^e represents the interaction between particles

and the external electromagnetic field and H^i does between the nucleon and the meson field. m and μ are the nucleon and the meson masses, respectively.

a) Meson Contribution



Fig. 1

The matrix elements corresponding to Fig. 1 are³⁾

$$\begin{aligned} A_G^{M+}(p, p') &= A_G^{M+}(p, p') \\ &+ A_G^{M-}(p, p'), \end{aligned} \quad (3)$$

where

$$\begin{aligned} A_G^{M+}(p, p') &= \frac{4ief^2}{(2\pi)^4} \bar{\psi} \tau_3 \int d^4k \\ &\times \frac{\gamma_5 A^+(\mathbf{P}-\mathbf{K}) \gamma_4 \gamma_5 (\mathbf{A} \cdot \mathbf{K})}{(E_{p'-k} - m + k_0) (k_0^2 - \omega_k^2) (k_0^2 - \omega_{k+\Delta p}^2)} \psi, \\ A_G^{M-}(p, p') &= \frac{4ief^2}{(2\pi)^4} \bar{\psi} \tau_3 \int d^4k \\ &\times \frac{-\gamma_5 A^-(\mathbf{P}-\mathbf{K}) \gamma_4 \gamma_5 (\mathbf{A} \cdot \mathbf{K})}{(E_{p'-k} + m - k_0) (k_0^2 - \omega_k^2) (k_0^2 - \omega_{k+\Delta p}^2)} \psi, \end{aligned}$$

in the covariant case, and

$$\begin{aligned} A_S^M(p, p') &= -2e \left(\frac{g}{\mu} \right)^2 \frac{2i}{(2\pi)^4} \psi^* \tau_3 \int d^4k \\ &\times \frac{(\sigma \cdot \mathbf{K} + \Delta \mathbf{P}) (\mathbf{A} \cdot \mathbf{K}) (\sigma \cdot \mathbf{K})}{k_0 (k_0^2 - \omega_k^2) (k_0^2 - \omega_{k+\Delta p}^2)} \psi, \end{aligned} \quad (4)$$

in the static case.

If we introduce the following approximation;

$$\begin{aligned} \langle \mathbf{p} + \mathbf{q}, \varepsilon | \tau_4 \gamma_5 | \mathbf{P}, \varepsilon' \rangle &\simeq \begin{cases} \sigma \mathbf{q} / 2m & \varepsilon = \varepsilon' \\ 1 & \varepsilon = +, \varepsilon' = - \\ -1 & \varepsilon = -, \varepsilon' = +, \end{cases} \\ \langle \mathbf{p} + \mathbf{q}, \varepsilon | \boldsymbol{\alpha} \cdot \mathbf{A} | \mathbf{P}, \varepsilon' \rangle &\simeq \begin{cases} \sigma \mathbf{H} / 2m & \varepsilon = \varepsilon' = + \\ -\sigma \mathbf{H} / 2m & \varepsilon = \varepsilon' = - \\ -\sigma \mathbf{A} & \varepsilon \neq \varepsilon', \end{cases} \end{aligned} \quad (5)$$

$$E_p \simeq m,$$

where $|\mathbf{P}, \varepsilon\rangle$: $\varepsilon = +$ or $-$, represents the states of the nucleon with the momentum \mathbf{P} and the energy εE , we can get the following relation;

$$A_G^{M+}(p, p') \simeq A_S^M(p, p'), \quad A_G^{M-}(p, p') \simeq 0,$$

so that

$$A_G^M(p, p') \simeq A_S^M(p, p'). \quad (6)$$

This shows that the above approximation (5) produces the static expression from the covariant one

exactly, at least as to meson contribution.

In eq. (3) it is possible to set aside the term proportional to $\alpha \cdot A$ as the charge renormalization from the term which contributes to the magnetic moments, i. e. :

$$A_C^{M+}(p, p') = [A_C^{M+}]_{m, m} + [A_C^{M+}(p, p')]_{c. r.}, \quad (7)$$

where

$$[A_C^{M+}(p, p')]_{c. r.} \simeq \frac{4ie f^2}{(2\pi)^4} \bar{\psi} \tau_3 \int d^4k \times \frac{\gamma_5 \{ \alpha(P-K)/2m \} \gamma_4 \gamma_5 (A \cdot K)}{(E_{p'-k}-m+k_0)(k_0^2-\omega_k^2)(k_0^2-\omega_{k+\Delta p}^2)} \psi. \quad (8)$$

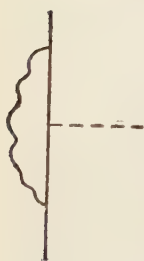


Fig. 2

$[A_C^{M+}]_{c. r.}$ is the charge renormalization term and $[A_C^{M+}]_{m, m}$ is the term which contributes to the magnetic moment. In these separation we do not consider the higher order terms with respect to $(1/m)$.

The same approximation as before shows that

$$[A_C^{M+}(p, p')]_{c. r.} \simeq A_S^M(p, p'), \quad (9)$$

$$[A_C^M(p, p')]_{m, m} \simeq 0. \quad (9)$$

b) Nucleon Contribution

The matrix elements corresponding to Fig. 2 are

$$A_C^N(p, p') = A_C^{N++}(p, p') + A_C^{N+-}(p, p') + A_C^{N-+}(p, p') + A_C^{N--}(p, p'), \quad (10)$$

where

$$A_C^{N++}(p, p') = \frac{ef^2}{2(2\pi)^4} i\pi \bar{\psi} (3-\tau_3) \int d^4k \times \frac{\gamma_5 A^+(P-K) \gamma_4 (\gamma A) A^+(P'-K) \gamma_4 \gamma_5}{\omega_k (E_{p'-k}-m+\omega_k) (E_{p'-k}-m+\omega_k)} \psi, \\ A_C^{N--}(p, p') = \frac{ef^2}{2(2\pi)^4} i\pi \bar{\psi} (3-\tau_3) \int d^4K \times \frac{\gamma_5 A^-(P-K) \gamma_4 (\gamma A) A^-(P'-K) \gamma_4 \gamma_5}{\omega_k (E_{p'-k}+m+\omega_k) (E_{p'-k}+m+\omega_k)} \psi$$

$A_C^{N+-}(p, p')$ and $A_C^{N-+}(p, p')$ have the factors $\gamma_5 A^+(P-K) \gamma_4 (\gamma A) A^-(P'-K) \gamma_4 \gamma_5$ and $\gamma_5 A^-(P-K) \gamma_4 (\gamma A) A^+(P'-K) \gamma_4 \gamma_5$ respectively.

In the static case we have

$$A_S^N(p, p') = -(e/2m)(g/\mu)^2 \frac{1}{4} \frac{1}{(2\pi)^3} \psi^* \int d^4K$$

$$\times \frac{(\sigma K)(\sigma H)(\sigma K)}{\omega_k^3} (3-\tau_3) \phi. \quad (11)$$

The same approximation as the previous subsection shows that $A_C^{N++} \simeq A_S^N$ and $A_C^{N+-} \simeq 0$, $A_C^{N-+} \simeq 0$, but in this case A_C^{N--} has the contribution, which is the same order as A_C^{N++} with respect to $(1/m)$.

A_C^{N++} can be separated into two parts as before

$$A_C^{N++}(p, p') = [A_C^{N++}(p, p')]_{m, m} + [A_C^{N++}(p, p')]_{c. r.}, \quad (12)$$

where

$$[A_C^{N++}]_{c. r.} \simeq \frac{ef^2}{2(2\pi)^4} i\pi \bar{\psi} (3-\tau_3) \int d^4K \times \frac{\gamma_5 \{ \alpha(P-K)/2m \} \gamma_4 (\gamma A) \{ \alpha(P'-K)/2m \} \gamma_4 \gamma_5}{\omega_k (E_{p'-k}-m+\omega_k) (E_{p'-k}-m+\omega_k)} \psi. \quad (12)$$

Here also we consider only the main term with respect to $(1/m)$. The same procedure as before shows;

$$[A_C^{N++}(p, p')]_{c. r.} \simeq A_S^N(p, p'), \\ [A_C^{N++}(p, p')]_{m, m} \simeq 0. \quad (13)$$

From eqs. (9) and (13) we can conclude that in the usual static method we can obtain only the parts of the charge renormalization term, and especially as to the meson current contribution the static one coincides with the charge renormalization term exactly. It will be expected from our results that we will obtain a different result from Friedmans one, if we consider the charge renormalization. So we are now researching for the fourth order correction of A.M.M. in the cut-off meson theory with consideration of the charge renormalization⁴⁾.

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Stripping Reactions and Nuclear Shell Model

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It has been pointed out by many authors^{1,2)} that the angular distributions from certain (d, p) and (d, n) reactions should give a sensitive measure of the accuracy of the nuclear structure in ascribing definite orbital angular momentum states to nucleons in a nucleus. This possibility is due to the facts that these reactions proceed mainly by means of a stripping process and that their angular distributions are characterized by the orbital angular momentum l with which the captured particle can be accepted into the appropriate final state. The theoretical expression of the differential cross section for the deuteron stripping reaction¹⁾ includes the summation over all values of l , allowed by the selection rules. Therefore, each allowed value of l produces a peak in the angular distribution without interfering one another, the peaks corresponding to different l being quite separated. The theoretical angular distribution shows a pronounced peak at small angles, and the maximum resulting from the smallest allowed l is of much larger magnitude than the others. The heights of the peaks decrease rapidly and the peaks move progressively toward the large angles as l increases. The investigations on the validity of the shell model by the deuteron stripping reaction are necessarily restricted to the lower l values. If the experimental angular distribution is characterized by two l values, it shows evidently the deviation from the pure shell model, i.e., the admixture of orbital angular momentum states.

Recently, the necessity³⁾ of the mixing of the configurations in nuclei was pointed out in the studies on the first excited states in the even-even nuclei and on β -decay with anomalous f -values. The mixing of the configurations was applied with much success to explain the deviations of the magnetic moments^{4,5)} from the Schmidt limit in odd- A nuclei, and quadrupole moments,⁶⁾ and the β -decay and γ -transition of the forbidden types.

The purposes of this note are to show that the mixing of the configurations can be applied to the deuteron stripping reactions also with success when

the angular distributions are characterized by two l values, and that we can determine the percentage of the mixing of the configurations from the observed relative heights of the peaks reversely.

We have derived the (d, p) differential cross section in the modified Born approximation. As the result, we have

$$\frac{d\sigma}{d\Omega} = \frac{M_p M_a^*}{2\pi \hbar^4} \frac{k_p}{k_a} \frac{2j_B + 1}{2j_A + 1} G(q)^2 \sum_{j_n l_n} K^2(j_n l_n; \alpha\beta) \cdot |\langle j_n | V | j_n \rangle|^2. \quad (1)$$

We adopt the same notation as that used by Grant⁷⁾. K -factor in eq. (1), which has not appeared explicitly in his formula of the (d, p) cross section, is characteristic of (d, p) reaction including the nuclei with the mixing of the configurations. This factor is relating to the c.f.p. overlapping and mixing parameters (These parameters are defined later on.). The explicit expression of this factor is given in the case of $C^{35}(d, p)C^{36}$.

According to the selection rules, the allowed values of l_n are 0, 2 and 4. The nuclear shell model, however, predicts only $l_n=2$, corresponding to the $d_{3/2}$ -configuration of the captured neutron. But the observed angular distribution⁸⁾ is characterized by two values of $l_n=0$ and 2. This is due to the fact that the ground state of C^{35} has not only the configuration determined by the shell model, but also another configuration which contributes to the peak corresponding to $l_n=0$. In view of the property of the reaction operator, the mixed configuration in the target nucleus C^{35} must be $(d_{3/2})_P, \{s_{1/2}, (d_{3/2})^3\}_N$ and the mixing of the configurations of the residual nucleus does not contribute to the reaction in this case. Hence, the configuration which we are interested in is expressed as

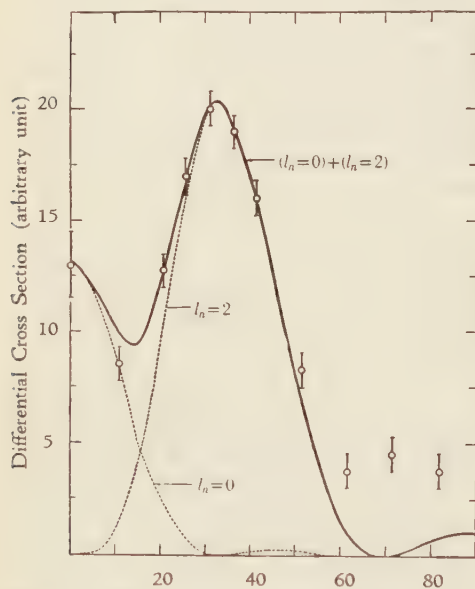
$$|j, \{j_1^{n_1}(0)j_2^{n_2}(0)\}(0); j_{AM}\rangle + \sum_{J\alpha} \alpha_J |j, \{j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)\}(J); j_{AM}\rangle, \quad (2)$$

where J is restricted by the relation $|j_1 - j_2| \leq J \leq j_1 + j_2$. We have neglected the normalization of the wave function (2), because this normalization is independent of the values of l , so that it does not affect the relative height in the angular distribution. The wave function of the residual nucleus is $|j, \{j_1^{n_1}(0)j_2^{n_2+1}(j_2)\}(j_2); j_{BM}\rangle$. Then, the reaction matrix element is given by

$$\begin{aligned} & \langle j_B | V | j_A j_n \rangle \\ &= \sqrt{\frac{2j_2 + 1 - n_2}{2j_2 + 1}} \langle j_2 | V | j_n \rangle \delta_{j_2 j_n} \end{aligned}$$

$$-\sum_J \alpha_J \sqrt{\frac{n_1(2J+1)(2j+1)}{2j_1+1}} W(j_1 j_2 j_1 j_2) \cdot \langle j_1 | V | j_2 \rangle \delta j_1 j_2. \quad (3)$$

The first term and the second one in (3) contribute to the peaks corresponding to the $l_n=2$ and 0, respectively. Thus, the ratio of the absolute squares of the two terms gives the relative height of the $l_n=0$ peak to the $l_n=2$ peak. α_J , to which we refer as the "mixing parameter", can be evaluated by the perturbation theory. If we put ΔE_J as the zero-th order energy difference between the second and the first configuration in (2), α_J is given as



Cl³⁵(d, p)Cl³⁶ E=6.90 Mev Q=6.30 Mev R=5.5 × 10⁻¹³cm.

Fig. 1 The angular distribution of the protons associated with the ground state for the reaction Cl³⁵(d, p)Cl³⁶.

$$\alpha_J = -\langle j, \{j_1^{n_1}(0)j_2^{n_2}(0)\}(0); j_{AM_A} | b | j, \{j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)\}(J); j_{AM_A} \rangle / \Delta E_J, \quad (4)$$

Where b is the inter-nucleonic interaction. If we adopt the δ -function type for this interaction as its short-range limit, the calculation of the α_J is straightforward. We take as the ΔE_J the energy difference 0.84 Mev between the ground state $d_{3/2}$ and first excited state $s_{1/2}$ in ${}_{10}\text{S}_{17}^{33}$, which is approximately equal to the energy difference between the single particle levels j_1 and j_2 . Calculated mixing parameters

are the following: $\alpha_1=0.061$, $\alpha_2=0.389$. Comparison of the calculated angular distribution with the experimental one is shown in Fig. 1.

Besides the example above-mentioned, the following reactions have been calculated in terms of the mixing of the configurations: F¹⁹(d, p)F²⁰, Mg²⁵(d, p)Mg²⁶,* P³¹(d, p)P³², Ti⁴⁷(d, p)Ti⁴⁸,* etc. Particularly, in the case of F¹⁹(d, p)F²⁰ the calculated angular distribution agrees quite well with the experimental one, and we can explain the unexpected small reduced width derived from the experimental (d, p) reaction data at least qualitatively. The detailed results of the investigation will appear soon in this journal.

The authors wish to express their sincere thanks to Professors M. Kobayasi and S. Takagi, Mr. A. Arima and other members in the laboratory for their valuable discussion and continual encouragement.

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On Quantum-Mechanical Nuclear Dipole Vibrations

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Since the giant resonance in nuclear photoeffect was first interpreted by M. Goldhaber and E. Teller¹⁾ as the dipole vibration of the nucleus, the phenomenon has been studied by many authors^{2), 3), 4)}. Especially the recent independent particle treatments³⁾ seem to

be successful for interpreting the experimental facts of giant resonance in many details. It would be desirable to resurvey the other extreme models, that is the collective models of photo-reaction, which assume the existence of nuclear dipole vibration, since in most of these treatments the dipole vibration was treated semi-classically and it is not clear how to get the knowledge about the detailed structure of giant resonance. Already M. Ferentz, M. Gell-Mann and D. Pines⁴⁾ treated the dipole vibration quantum-mechanically, but it seems difficult to extend the theory to the problem with the more realistic boundary condition involving nuclear surface.

Recently S. Tomonaga⁵⁾ has given a method to formulate the collective motions. According to his general prescription we can also reformulate the classical dipole vibration as a quantum-mechanical one. First we define the collective coordinate ξ tentatively to be the distance between the centers of gravity of protons and neutrons. In this case the electric dipole interaction can be described only in terms of ξ and does not contain the internal coordinates ζ 's, though it remains to be investigated if exchange forces play an important role or not.⁶⁾ We can see that such a treatment corresponds to the well-known M. Goldhaber and E. Teller's third model¹⁾.

The kinetic energy part of the Hamiltonian can be divided into the collective part $\pi^2/2M$ and the internal part which contains neither ξ nor π , where π satisfies the following commutation relation exactly,

$$[\pi, \xi] = -i\hbar. \quad (1)$$

The potential part of the Hamiltonian which commutes with ξ can be expanded into the power series of ξ according to Tomonaga's prescription, while the other part, V' , which doesn't commute with ξ , the part containing τ_x or τ_y , for example, must be remained without expansion. Thus the total Hamiltonian can be written as

$$H = H_{\text{nucleus}} + H_{\text{radiation field}} + H_{\text{rad.int.}}, \quad (2)$$

where

$$H_{\text{nucleus}} = H^{(0)}(\zeta) + H^{(1)}(\zeta)\xi + \frac{1}{2}H^{(2)}(\zeta)\xi^2 + \dots + \frac{1}{2M}\pi^2 + V' \quad (3)$$

and

$$H_{\text{rad.int.}} = ie\omega \cdot NZ/A \cdot \xi. \quad (4)$$

Since the resonance frequency of the dipole vibration is very large, we can reasonably assume that the wave function for the ground state of a nucleus, Ψ_0 , is

$$\Psi_0 \cong \psi_0(\xi) \Phi_0(\zeta). \quad (5)$$

The frequency of the dipole vibration should be determined by the expectation value of $H^{(2)}$ with respect to Ψ_0 (we will write it $\langle H^{(2)} \rangle$), for which we can prove that the semi-classical treatment can also give the proper estimate of $\langle H^{(2)} \rangle$. An attempt to estimate semi-classically the frequency directly from the two-body interaction has been carried out by S. Fujii and S. Takagi⁷⁾ and their results are in pretty good agreement with the experimental resonance frequencies and their A-dependence.

The original Tomonaga's theory states that, when the Hartree approximation is adopted, the expansion such as (3) is significant only if the frequency of the collective motion is very low compared with that of the internal motion. Since it is clear that this condition is not satisfied in our case, we have no theoretical guarantee to cut off the higher order terms of the power series in (3) only leaving first several terms, that is, there is no guarantee for Goldhaber and Teller's model to be valid. Nevertheless it seems to be convenient to study the expansion (3) in order to obtain the physical picture how the nuclear dipole vibration couples with the internal motions.

Since it is impossible to obtain the exact solution of the Hamiltonian (3), we will consider three conventional ways to get over the difficulties: (I) If we use the Hartree approximation with respect to the internal coordinates⁸⁾, we get the model of "movable wells for protons and neutrons"⁷⁾. In this direction we can obtain a unified model to amalgamate the collective aspect such as in Goldhaber and Teller's model and the individual particle aspect as studied by D. H. Wilkinson et al.⁹⁾ (II) Besides the dipole vibration we can separate also the collective coordinates corresponding to the surface vibrations from the internal parts of (3). Phenomenologically this model has been studied by M. Soga, S. Iijima and M. Nogami¹⁰⁾ (III) Other types of correlations such as α -model, or d -model will be taken into account in our theory if the corresponding internal wave function can be written in the form $\Phi_0(\zeta)$.

The couplings between the dipole vibration and the internal coordinates come from the terms $H^{(1)}(\zeta)\xi$, etc. and V' . Though the diagonal parts of $H^{(1)}$ should vanish as we expect from the classical analogy, the non-diagonal parts of $H^{(1)}$ lead to the internal excitations after cancelling the dipole vibration. V' that doesn't commute with ξ will probably make

the picture of the dipole vibration lose its proper sense, if the contribution from V' is large. The study of the coupling terms is now being in progress.

The wave function of a nucleus in the excited state corresponding to the giant resonance, Ψ_g , is also to be paid attention to, which can be realized approximately only if the coupling terms with internal coordinates are small. Omitting the constant factor,

$$\Psi_g \sim \xi \Psi_0. \quad (6)$$

This equation clarifies the relation to the idea of configuration mixing, such as previously discussed by D. C. Peaslee¹⁰⁾.

Full note will be published in the near future.

The author wishes to express his sincere thanks to Professors S. Tomonaga, M. Nogami, T. Yamano-uchi and S. Nakamura for their valuable suggestions and discussions.

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On the Nuclear Photo-Reaction, I

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As is well known, excitation curves for the photo-disintegration of nuclei exhibit the behavior of "giant resonance" at about 20 Mev (18.1 Mev for Cu^{63}). It is usually assumed that the electric dipole transition predominates in photo-reaction at such high energies. To calculate the matrix element for the dipole transition from the ground state to an excited state, we must know wave functions of the both states. Unfortunately, little is known of the wave functions for excited states of heavy nuclei at high energies, though the shell model has succeeded in explaining the nuclear structure at lower energies.

Levinger and Kent¹⁾ modified Bethe-Levinger's sum rule formula and got reasonable values for the harmonic mean energies of absorbed photons taking the independent particle model (IPM) for nuclei and antisymmetrizing the nuclear wave functions. Burkhardt²⁾ applied IPM to the calculation of the electric dipole transition for neutrons in Cu^{63} , and found that the cross section of photon absorption has a sharp maximum. But the value of the resonance energy thus obtained was about 9 Mev which is much smaller than the observed one. Reifman³⁾ proposed a model of modified IPM to explain the general feature of the giant resonance, but did not find satisfactory results.

On the other hand, Goldhaber and Teller⁴⁾ postulated a collective motion of nucleus for photo-effect. It was assumed that γ -rays excite a motion in which the bulk of protons moves in one direction while the neutrons in the opposite direction. In spite of the semi-qualitative and rough treatment, the integrated cross sections predicted have reasonable values, and the absolute values of the resonance energies $\hbar\omega_r$ show excellent fits to the experimental results for various nuclei. This may suggest that the postulated collective motion plays an essential role in photo-nuclear interactions.

In this connection, we wish to re-examine Goldhaber and Teller's (abbreviated as G-T's) collective motion from a standpoint of different models, which

have succeeded in many fields of nuclear physics, and moreover, to make an assumption as to the manner in which the dipole vibration couples with other modes of nuclear motion. The success of G-T's collective model suggests that the nuclear Hamiltonian will be reasonably transformed into

$$H_{\text{dip}} + H_n + H_{\text{dip:n}}, \quad (1)$$

where H_{dip} is the Hamiltonian of the nuclear dipole vibration, H_n other nuclear modes, and $H_{\text{dip:n}}$ the coupling term between the dipole vibration and other modes. The interaction between electromagnetic field and the nucleus is given by

$$H' = \frac{ie\hbar}{Mc} \mathbf{A}(\mathbf{x}_p) \text{grad } x_p. \quad (2)$$

\mathbf{x}_p is the position vector of the average center of mass of protons and \mathbf{A} is the electromagnetic vector potential.

One of our main assumptions is that the nuclear photo-interaction proceeds through the first excited state of dipole vibration.

We wish to construct this mode of nuclear dipole vibration from a standpoint of nuclear structure as follows:

It will probably be not unreasonable to take the nuclear wave function of the ground state as

$$\Psi = \frac{1}{\sqrt{A!}} \sum_P \prod_i^A (-1)^P \psi_{n_i}(\mathbf{r}_i, s_i, v_i), \quad (3)$$

where $\psi_n(\mathbf{r}, s, v)$ is one nucleon wave function. Then the excited state, in which all protons in the nucleus are collectively and adiabatically displaced relative to neutrons by an infinitesimal quantity ξ , can be described by the wave function of type (3), where we have replaced $\psi_n(\mathbf{r}, s, v)$ by $\psi_n(\mathbf{r} \pm \xi/2, s, v)$ for $\tau_z \psi_n = \pm \psi_n$ assuming $N=Z$ for the sake of simplicity. τ_z is the z -component of the isotopic spin τ . We evaluate the variation of the expected value of the nuclear potential corresponding to this collective displacement and take it as the potential of nuclear dipole vibration. The variation $\Delta \bar{V}(\xi)$ will be expressed as

$$\Delta \bar{V}(\xi) = \frac{1}{2} K \xi^2 + (\text{terms of higher order}).$$

Because K depends on the wave functions of the ground state, we adopt the following nuclear models for the ground state. (As regards nuclear potential, many body force and the tensor force are neglected and it is assumed for the two body force to have exchange character as follows:

$$V_{ij} = W(r_{ij}) [A_1 + A_2 P_B + A_3 P_M + A_4 P_H].$$

(1) a uniform density model

We shall neglect the effects due to the surface of the nucleus, and use the model of a uniformly dense nucleus of radius R ,

$$\begin{aligned} |\psi_n|^2 &= 1/(4/3)\pi R^3 & |\mathbf{r}| < R, \\ &= 0 & |\mathbf{r}| > R. \end{aligned}$$

(In this case we obtain the same result even if $N=Z$ is not assumed.)

(2) nuclear lattice model

In contrast to the former, let us next consider a nucleus made up of sub-units, such as alpha-particles. This nuclear lattice model is an extreme idealization and the nucleon wave function for the ground state can be approximated by

$$\psi_n = \frac{1}{(2\pi)^{3/2} Q^{1/2}} \int_{(|\mathbf{k}| < k_m)} e^{i(\mathbf{r}-\mathbf{r}_n)\cdot\mathbf{k}} d\mathbf{k},$$

where

$$Q = (4\pi/3) k_m^3.$$

The cut-off parameter k_m depends on the lattice

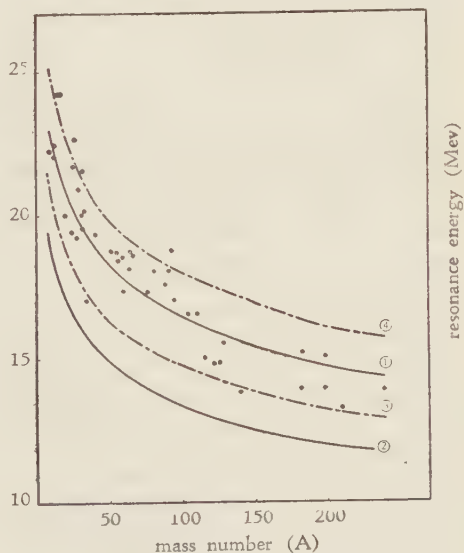


Fig. 1 resonance energy.

- : experimental data⁶⁾
- ①, ② : uniform density model
 $r_0 = 1.35 \times 10^{-13} \text{cm}$, $r_0 = 1.5 \times 10^{-13} \text{cm}$
- ③ : nuclear lattice model (square well)
 $r_0 = 1.35 \times 10^{-13} \text{cm}$
- ④ : " (exponential)
 $r_0 = 1.2 \times 10^{-13} \text{cm}$

spacing through the orthogonality of the function system $[\psi_m]$, and one to one correspondence of the lattice point to four times the volume allotted to a single nucleon, therefore, demand $k_m = \sqrt[3]{9\pi/8} \cdot (1/r_0)$.

When we take for the Hamiltonian of the dipole vibration mode

$$H_{\text{dip}} = -(\hbar^2/2\mu) \nabla^2 + (1/2) K \xi^2, \quad (4)$$

we get $\hbar\sqrt{K/\mu}$ for the resonance energy in nuclear photo-effect, where $\mu = NZ/A \cdot M$.

To compare our results with the experimental data, we use the square well potential as V_{ij} in model (1) and both the square well and exponential type potential in model (2). The parameters contained in these two body potentials are determined from the analysis of the nucleon-nucleon scattering and deuteron problem⁵⁾, where Serber's exchange character is taken.

Exponential well ($W(r) = e^{-r/a}$)

$$a = 0.7088 \times 10^{-13} \text{ cm},$$

$$A_1 = A_3 = -71.25 \text{ Mev},$$

$$A_2 = A_4 = -15.75 \text{ Mev}.$$

Square well ($W(r) = 1 \ r < b$, $W(r) = 0 \ r > b$)

$$b = 2.3 \times 10^{-13} \text{ cm},$$

$$A_1 = A_3 = -11.75 \text{ Mev},$$

$$A_2 = A_4 = -2.75 \text{ Mev}.$$

The numerical results for these two different models, given in Fig. 1 and Table 1, exhibit almost the same behavior. Both of them have reasonable agreements with the experimental data⁶⁾, even for light nuclei. The results obtained depend on the nuclear radii sensitively. The best fit is obtained for $r_0 \cong 1.35 \times 10^{-13} \text{ cm}$ in the uniform density model and for $r_0 \cong 1.25 \sim 1.3 \times 10^{-13} \text{ cm}$ in the nuclear lattice model.

But we cannot get detailed information on the radial dependence of the inter-nucleon potential and on the choice of nuclear models for the ground state from these results, because the experimental data contain considerable errors.

Also in Table 1, the amplitudes of the nuclear dipole vibration are given. They are smaller than one-sixth of the lattice spacing for $A \gtrsim 30$.

Thus it may be concluded that the experimental data support our assumption of dipole vibration, at least referring to the resonance energy in nuclear photo-effect.

The breadth of the resonance will be discussed in a forthcoming letter.

Table 1. Resonance energy $\hbar\omega_r$

mass num- ber	resonance energy (Mev)					vibration amplitude	
	uniform density model		nuclear lattice model			nuclear lattice model	
	square well potential		exponential		square well	$r_0 = 1.35 \times 10^{-13} \text{ cm}$	
	$r_0 = 1.35 \times 10^{-13} \text{ cm}$	$1.5 \times 10^{-13} \text{ cm}$	$1.2 \times 10^{-13} \text{ cm}$	$1.35 \times 10^{-13} \text{ cm}$	$1.35 \times 10^{-13} \text{ cm}$	ground state	excited state
8	23.1	19.1	25.1	19.9	21.3	$7.0 \times 10^{-14} \text{ cm}$	$12.0 \times 10^{-14} \text{ cm}$
27	20.0	16.3	21.6	16.9	17.9	4.1	7.2
64	17.7	14.4	19.2	15.0	15.7	2.9	4.9
125	15.9	12.9	17.6	13.7	14.3	2.1	3.7
216	14.6	11.8	15.9	12.0	13.0	1.7	3.0

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On the Nuclear Photo-Reaction, II

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Recent experiments on the cross section and the energy dependence of the nuclear photo-effect indicate the appearance of fine structures and anomalously narrow widths of the giant resonance for elements with neutron numbers in the vicinity of the magic number.

In Goldhaber and Teller's collective model¹⁾, the width has only qualitatively been considered as due to the dissipation of energy of the ordered vibration by a process analogous to damping by friction. Though the collective model has succeeded in interpreting the absorption process, it can not indicate anything for the decay process. The decaying process may proceed following statistical theory or proceed more rapidly by some other mechanism. Calculation of the transition probability demands knowledges of the coupling between the dipole vibration and other nuclear modes.

In our preceding paper²⁾ (denoted as I), it was shown how a mode of nuclear dipole vibration can be constructed from the nuclear Hamiltonian. If our fundamental assumption is justified to be a good approximation, the coupling scheme may be obtained as follows. The collective displacement of the protons in a nucleus relative to the neutrons by an infinitesimal quantity ξ corresponds to the replacement of $\psi_n(\mathbf{r})$ to $\psi_n(\mathbf{r} \pm \xi/2)$ for $\tau_z \psi_n = \pm \psi_n$ respectively, where τ_z is the z -component of the isotopic spin τ and ψ_n is one nucleon wave function of the ground state. Thus the dipole vibration is accompanied by the variation of the wave functions of nucleons. In other words, it perturbs each nucleons of the nucleus, when we assume the independent particle model for a nucleus. Indeed this perturbation $H_{\text{dip:n}}$ corresponds to the desired coupling and is given by $S^{-1}H_0S - H_0$ where $H_0 = P^2/2M + U(\mathbf{r})$ is the original Hamiltonian of the nucleon. S is a transformation operator defined so that $\psi_n(\mathbf{r}) = S\psi_n(\mathbf{r} \pm \xi/2)$ for $\tau_z \psi_n = \pm \psi_n$ and can be written explicitly as

$$S = 1 - \frac{\tau_z}{2} \xi \text{grad}_r + \dots, \quad (1)$$

When we take a square well potential $U(\mathbf{r})$ of radius R with a finite depth W , we find in the first approximation

$$\begin{aligned} H_{\text{dip:n}} &\cong \left[H_0, \left(-\frac{\tau_z}{2} \xi \text{grad}_r \right) \right] \\ &= \frac{\tau_z}{2} \frac{W}{R} (\xi r) \delta(r-R). \end{aligned} \quad (2)$$

Thus we cannot separate perfectly the mode of the dipole vibration from other nuclear modes and thus we obtain a great number of nuclear levels each of which contains to some extent the dipole vibration. If, however, the coupling is weak, only such levels will contain considerable contribution from the first excited state of the dipole vibration, whose energy does not differ appreciably from the energy of this first excited state. Thus a great number of nuclear levels actually contribute to the γ -absorption, but they all cluster around the first excited state of the dipole vibration. In the following we will separate a mode of the dipole vibration from other nuclear modes and treat the coupling between them as a perturbation of the first order. Following our assumption that the nuclear photo-interaction proceeds through the first excited state of the dipole vibration as an intermediate state, the photo-absorption cross section $\sigma_a(E_\gamma)$ becomes in the dipole approximation

$$\sigma_a(E_\gamma) = 2\pi (NZ/A) \frac{e^2 b E_0}{Mc E_\gamma} \frac{(\Gamma/2)}{(E_\gamma - E_0)^2 + (\Gamma/2)^2}, \quad (3)$$

where E_0 is the level spacing of the nuclear dipole vibration system. Γ corresponding to the half width of the first excited level depends on the transition probability for the process in which one nucleon in the nucleus is excited from the bound level to an excited level in connection with the damping of the dipole vibration. In this case the conservation of energy demands that the excitation energy of one nucleon is equal to the energy of the incident energy E_γ and the half width Γ , therefore, depends on E_γ . At high excitation energy almost all of the nucleons in a nucleus contribute to such a process so long as Pauli's principle does not prevent them to do so. We have calculated the transition probability for two elements (Co^{59} and Nd^{150}) to compare our results for the width of the resonance with the experimental data. They are given in Table 1, where

Γ_{exp} is the experimental value and Γ_n the contribution to the half width from the excitation of neutrons. Similarly we can evaluate also the contribution from the protons Γ_p , taking into consideration the penetration factor of protons. Thus we have for the total half width $\Gamma = \Gamma_n + \Gamma_p$.

$\Gamma \cong 2$ Mev for Co^{59} , $\Gamma \cong 2.2$ Mev for Nd^{150} . For Co^{59} the experimental value³⁾ of the half width is about four times larger than the calculated one.

The comparison suggests that we should have a model where the nucleons in a nucleus couple with the dipole vibration more strongly than discussed above. We can obtain better agreement with the experimental data by considering as a tentative model a nucleus made up of sub-units, such as alpha-particles. Also in such a model, as is shown in I, the

dipole vibration provides a reasonable value of the resonance energy in nuclear photo-effect. We assume that a nucleon in the quasi α -particle is influenced by a local potential of radius $R' = \sqrt[3]{4} r_0$ with a finite depth W . Then the coupling between the nucleon and the dipole vibration is given by

$$H_{\text{dip:n}} = (\tau_z/2) (W/R') (\xi \cdot r) \delta(r-R').$$

Neglecting effects of the nuclear surface and Coulomb barrier, we calculated the photo-absorption cross section for Cu^{63} , where the photo-absorption cross section means a sum of (γ, n) , (γ, p) and (γ, γ) cross sections. The comparison of our results with the experimental data⁴⁾ for $\text{Cu}^{63}(\gamma, n)$ Cu^{62} is shown in Fig. 1.

Application of the similar calculation for other

Table 1. Half widths for Co^{59} and Nd^{150}

element	nuclear radius (10^{-13}cm)	neutron binding energy (Mev)	resonance energy (Mev)	Γ_n (Mev)	Γ_{exp} (Mev)
Co^{59}	4.9	10.0	17.65	1.04	8.4
Nd^{150}	6.83	8.0	15.0	1.88	—

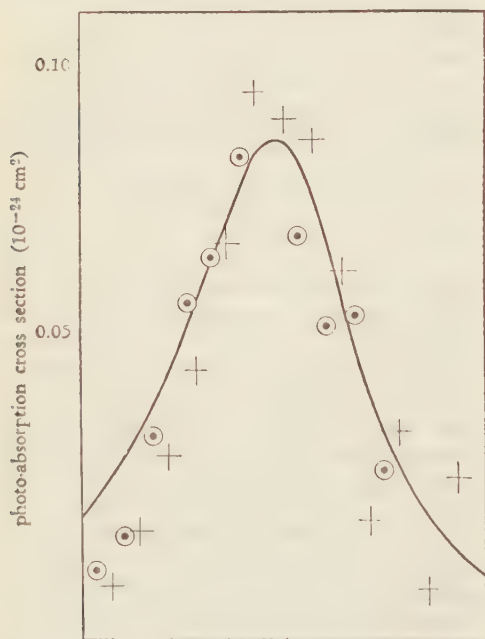


Fig. 1 Total photo-absorption cross section for Cu^{63}
 $\odot, +$: experimental data⁵⁾
 —: theoretical cross section ($r_0 = 1.25 \times 10^{-13}\text{cm}$ $W = 20.1$ Mev)

elements leads to the result that the half width of the giant resonance is broader for heavier elements. For the dependence of the width on the mass numbers, however, the experimental data^{3, 5)} seem to stand against such a behavior for heavy nuclei (neutron number ≥ 20). It may suggest that nuclear structure changes appreciably for elements with neutron numbers in the vicinity of 20.

After we had completed this calculation, a similar calculation was proposed by U. L. Businaro and S. Gallone⁶⁾. But they used the average nuclear potential acting on a nucleon in the nucleus to calculate the restoring force in the dipole vibration. Their results are consistent with ours.

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An Exactly Soluble Example in Quantum Field Theory

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There are numbers of quantized field theories whose consequences can be obtained without approximation; neutral scalar theory with fixed sources, meson pair theory with fixed sources¹⁾, single electron theory interacting with the one-dimensional electromagnetic field²⁾, and Lee's example³⁾. Lee's example is very instructive for studying some features of field theories.

We shall give in this note another exactly soluble example; neutral pseudoscalar meson theory interacting with nucleon field via creation and annihilation of nucleon pair. Since all the known examples do not include vacuum polarization effects, this example may serve as prototype.

Let us consider the interaction between a fermion (nucleon) field and a neutral pseudoscalar boson (pion) field. The Hamiltonian for the free fields is

$$\begin{aligned} H_0 = & \int \psi^* [-i\alpha \nabla + \beta m] \psi d\tau \\ & + \frac{1}{2} \int [\pi^2 + |\nabla \phi|^2 + \mu^2 \phi^2] d\tau \\ = & \sum_p \sum_{i=1}^2 E_p \{ a_p^{(i)*} a_p^{(i)} + b_p^{(i)} b_p^{(i)*} - 2 \} \\ & + \sum_k \omega_k \left\{ \alpha_k^* \alpha_k - \frac{1}{2} \right\}, \end{aligned} \quad (1)$$

where

$$\begin{aligned} \psi_p(\mathbf{x}) = & \mathcal{Q}^{-1/2} \sum_{i=1}^2 [a_p^{(i)} u_p^{(i)}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} \\ & + b_p^{(i)} v_p^{(i)*} e^{-i\mathbf{p}\cdot\mathbf{x}}], \\ \phi(\mathbf{x}) = & \sum_k (2\omega_k \mathcal{Q})^{-1/2} [\alpha_k e^{i\mathbf{k}\cdot\mathbf{x}} + \alpha_k^* e^{-i\mathbf{k}\cdot\mathbf{x}}], \\ E_p = & (m^2 + \mathbf{p}^2)^{1/2}, \quad \omega_k = (\mu^2 + \mathbf{k}^2)^{1/2}, \end{aligned}$$

\mathcal{Q} is the volume of the system, m and μ are "observed" masses of nucleon and pion, $u^{(i)}$ and $v^{(i)}$ are spinors, and $a_p^{(i)}$, $b_p^{(i)}$ and α_k are destruction operators of nucleon, antinucleon and pion respectively. As the interaction Hamiltonian we

adopt*

$$\begin{aligned} H_1 = & ig \sum_p \sum_k \sum_{i=1}^2 (2\omega_k \mathcal{Q})^{-1/2} [a_p^{(i)*} b_{k-p}^{(i)*} \alpha_k \\ & - b_p^{(i)} a_{k-p}^{(i)} \alpha_k^*] + (1/2) \delta \mu^2 \int \phi^2 d\tau, \end{aligned} \quad (2)$$

where $\delta \mu^2$ is used to cancel any change in mass of the pion due to pair creation and annihilation and p_m is the cut-off momentum. There being no coupling between different spin components ($i=1$ and 2) in H_0 and H_1 , we may consider hereafter only one spin component and suitably substitute g by $2g$ without loss of generality.

Upon examining the total Hamiltonian, one sees that this system possesses two simple conservation laws:

$$\begin{aligned} \mathcal{N}_\Pi + \mathcal{N}_+ & = \text{const}, \\ \text{and} \quad \mathcal{N}_\Pi + \mathcal{N}_- & = \text{const}, \end{aligned} \quad (3)$$

with \mathcal{N}_+ , \mathcal{N}_- , \mathcal{N}_Π the total number of nucleon, antinucleon, and pion, respectively. The situation is very similar to that of Lee's example, and our example is also soluble because of this special selection rule.

In the following, we denote the states of a "bare" nucleon, antinucleon and pion with momentum \mathbf{p} by $|N_p^+\rangle$, $|N_p^-\rangle$ and $|\Pi_p\rangle$, while the states of the corresponding "physical" particles are indicated by $|N_p^+\rangle$, $|N_p^-\rangle$ and $|\Pi_p\rangle$. Using (3), one sees that

$$\begin{aligned} |N_p^+\rangle & = |N_p^+\rangle, \\ |N_p^-\rangle & = |N_p^-\rangle, \end{aligned}$$

and

$$|\Pi_0\rangle = \mathcal{Z}_3^{1/2} [|\Pi_0\rangle + g \sum_p f(p) |N_p^+, N_{-p}^-\rangle], \quad (4)$$

where $\mathcal{Z}_3^{1/2}$ is a normalization constant and $f(p)$ is proportional to the probability amplitude for finding a bare nucleon pair with momenta $\pm \mathbf{p}$ in a physical pion at rest.

Applying the total Hamiltonian on $|\Pi_0\rangle$ and requiring that the eigenvalue should be the "observed" mass μ , one finds

$$\delta \mu^2 = 2g^2 \mathcal{Q}^{-1} \sum_p^{p_m} (2E_p - \mu)^{-1}, \quad (5)$$

$$f(p) = -i(2\mu \mathcal{Q})^{-1/2} (2E_p - \mu)^{-1}, \quad (6)$$

* This interaction Hamiltonian identical with a part of the pair creation and annihilation term of the ordinary pseudoscalar coupling, apart from $(p/m)^2$ corrections.

and

$$Z_3^{-1} = 1 + g^2 (2\mu\Omega)^{-1} \sum_p^{p_m} (2E_p - \mu)^{-2}, \quad (7)$$

Remark that the eigenvalues of the Hamiltonian contain logarithmic divergence when p_m is infinity, even after the renormalization of mass and coupling constant. This is because our interaction Hamiltonian is similar to that of meson pair theory in this respect.

Nucleon-antinucleon scattering can also be exactly solved, and the phase shift δ for this process is

$$\tan \delta = -\frac{g_c^2 p}{8\pi\mu(2E_p - \mu)} \left[1 + \frac{g_c^2}{16\pi^3\mu} P \int_0^{p_m} \frac{d^3 q' (2E_{q'} - \mu)}{2(E_p - E_{q'}) (2E_{q'} - \mu)^2} \right]^{-1} \quad (8)$$

where $g_c^2 = g^2 Z_3$, (9)

and p is the momentum of incoming nucleon in the center of gravity system.

Identical conclusions concerning renormalization of both the mass and coupling constant are also obtained by the power series method. One can also obtain the renormalized propagator for pion.

We shall discuss the nature of the coupling constant renormalization briefly. As far as g is held to be real, consistent with the interpretation of the starting Hamiltonian, one sees from (7) and (9) that

$$0 \leq Z_3 \leq 1, \quad (10)$$

and

$$0 \leq g_c^2 \leq g_c^2 \max, \quad (11)$$

where

$$g_c^2 \max = [(2\mu\Omega)^{-1} \sum_p^{p_m} (2E_p - \mu)^{-2}]^{-1}. \quad (12)$$

In reality (10) and (11) are equivalent to each other, since, using (12), one may write

$$Z_3 = 1 - g_c^2 / g_c^2 \max. \quad (13)$$

Eq. (10) is consistent with the physical interpretation that Z_3 be probability. Similar relation to (11), as the effect of vacuum polarization, has also been obtained approximately by Landau et al.⁽⁴⁾ and by Feynman⁽⁵⁾. We may eliminate g^2 from the entire theory using (9) and (13), and if we would regard the numerical value of g_c^2 as arbitrarily chosen, difficulties would appear concerning negative probability ($Z_3 < 0$) and "ghost" states⁽⁶⁾.

Since our interaction Hamiltonian, (2), coincides

with the pair creation and annihilation part of the relativistic γ_5 -coupling neglecting $(p/m)^2$ corrections, and inclusion of isotopic spin does not cause difficulties, we may use the above example as the unperturbed system for pion-nucleon system interacting via charge-independent γ_5 -coupling. This treatment will be reported in a separate paper.

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On the Heisenberg's Non-linear Meson Equation

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In order to account for the multiple meson production in high energy nuclear events, the non-linear meson theory has been proposed by Heisenberg⁽¹⁾. In his theory, the Lagrangian is taken to be of the so-called Born type:

$$L = \int \{ 1 + l^{-2} (\partial_\mu \phi \cdot \partial_\mu \phi + \kappa^2 \phi^2) \}^{1/2}, \quad (1)$$

in which ϕ is the scalar field quantity, κ is the meson mass and, l^{-1} is a small constant. Accordingly, the field equation is given by*

$$[\partial_{\mu\nu} + l^{-2} ((\partial_\lambda \phi \cdot \partial_\lambda \phi + \kappa^2 \phi^2) \delta_{\mu\nu} - \partial_\mu \phi \cdot \partial_\nu \phi)] \partial_\mu \partial_\nu \phi - [1 + l^{-2} (2(\partial_\mu \phi \cdot \partial_\mu \phi) + \kappa^2 \phi^2)] \kappa^2 \phi = 0. \quad (2)$$

The purpose of this note is to investigate the solutions

of the above equation, on the basis of the characteristic theory of partial differential equation.²⁾ In what follows we restrict ourselves to the case of one dimensional motion and the mass κ is put equal to zero.

The equation (2) then becomes

$$(-l^2 + v^2)u_x + uv(u_t - v_x) - (l^2 + u^2)v_t = 0, \quad (3 \cdot a)$$

$$u_t + v_x = 0, \quad (3 \cdot b)$$

in which u and v are given by the equations:

$$\begin{aligned} u &= \partial\phi/\partial x & v &= -\partial\phi/\partial t \\ \dots (4 \cdot a), & & \dots (4 \cdot b). \end{aligned}$$

It can easily be seen that (3·a) is automatically satisfied by the relation:

$$v = mu \pm \sqrt{1 - m^2} l, \quad (5)$$

in which m is an arbitrary constant.

Substituting (5) into (3·b), the original equation reduces to

$$u_t \pm mu_x = 0.$$

Therefore, we can obtain the solutions

$$\begin{aligned} u &= f(x \mp mt), \\ v &= mu \pm \sqrt{1 - m^2} l, \end{aligned} \quad (6)$$

corresponding to the initial conditions:

$$\begin{aligned} u &= f(x), & v &= mf(x) + \sqrt{1 - m^2} l, \\ \text{at } t &= 0. \end{aligned}$$

In the following, it can be shown that these solutions correspond to the simple wave³⁾ in the hydrodynamics. (In fact, the equation (3·a, b) are quite analogous to the equations of the isentropic irrotational steady plane flow.)⁴⁾

Following the procedure similar to that employed in the hydrodynamics, we have the characteristic equations:

$$C^+: \quad t_\alpha - \lambda_+ x_\alpha = 0, \quad (7 \cdot a)$$

$$C^-: \quad t_\beta - \lambda_- x_\beta = 0, \quad (7 \cdot b)$$

$$I^+: \quad u_\alpha - \lambda_- v_\alpha = 0, \quad (8 \cdot a)$$

$$I^-: \quad u_\beta - \lambda_+ v_\beta = 0, \quad (8 \cdot b)$$

$$\lambda^\pm = [uv \pm l(l^2 + u^2 - v^2)^{1/2}] / (v^2 - l^2),$$

which are equivalent to (3·a, b).

Instead of (8·a, b), we can also employ the following equations for the characteristics.⁵⁾

$$\varepsilon^2 (du^2 - dv^2) = (udu - vdv)^2, \quad (9)$$

in which ε is given by $\varepsilon = (l^2 + u^2 - v^2)^{1/2}$.

It follows immediately that the general solution of (9) is given by the straight lines in the $u-v$ plane:

$$v = mu \pm \sqrt{1 - m^2} l, \quad (10)$$

in which m is an arbitrary constant.

The detailed calculations show that $v = mu + \sqrt{1 - m^2} l$ represents I^+ , corresponding to $\sqrt{1 - m^2} \cdot u - lm \geq 0$, respectively; and $v = mu - \sqrt{1 - m^2} l$ represents I^- , corresponding to $\sqrt{1 - m^2} u + lm \geq 0$ respectively. However, the C characteristics are not, in general, straight lines and their geometrical relation to the I characteristics can be given by the equation:

$$t_\alpha v_\beta - x_\alpha u_\beta = 0,$$

which indicates that the angle between I^+ and u -axis and the angle between C^+ and x -axis are complementary to each other and the similar relation holds for I^- and C^- . It is quite obvious from these results that the simple wave solution of (7) and (8), can be given by (6). In non-linear theories, it seems to be rather peculiar that the simple wave propagates keeping its shape constantly with the development of time. In this respect, the meson field under consideration is analogous to the Kármán-Zien's gas in the hydrodynamics.** The author expresses his thanks to Prof. Y. Tanikawa for his kind interest and to Prof. Z. Koba for his discussions.

* Our time coordinate x^0 is real, $c = \hbar = 1$, $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = 1$.

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The Statistical Mechanical Aspect of H-Theorem

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A simple model of diffusion is investigated without the assumption such as the "Stosszahlansatz". We consider a one-dimensional box which is divided into two parts by a wall at its middle point. In the left of the box, at time $t=0$, there are N small particles which collide with each other elastically. If the wall is removed at $t=0$ these particles diffuse into the right half of the box and the number of particles $n(t)$ in the left half of the box changes, decreases generally, as a function of time t . The behavior of functions $n(t)$ in this process, which have widely different properties according as the choice of initial conditions or mechanical trajectories in phase space, are examined and it is shown that if the number of particles is sufficiently large almost all the trajectories show the mean property of $n(t)$ which is characterized by the irreversibility. Further our results obtained here are compared with those calculated by the usual method in which the "Stosszahlansatz" is used as a basic assumption. Finally a statistical mechanical interpretation of H-theorem which seems to be a generalization of our results is presented.

§ 1. Introduction

The statistical nature which is concealed in the irreversibility asserted by H-theorem has been investigated by many authors since the Ehrenfest's conscientious criticism appeared.¹⁾ The contradiction with the kinetic molecular theory which demands the essential reversibility of all processes seems to have been resolved, at first sight, by showing that the irreversibility has its origin in such a statistical assumption as the "Stosszahlansatz" or the process of "coarse graining" used in the derivation of H-theorem. In order to clarify the significance of this assertion, P. and T. Ehrenfest presented two simple examples, i.e., urn model and wind-tree model, which were afterward investigated over again in detail or reformed by several authors²⁾. The above mentioned statistical assumptions used always in such problems may be a kind of qualitative expression of the incompleteness of the specification of states by macroscopic observations. However the physical significance of necessity of "Stosszahlansatz" or process of "coarse graining" is still obscure, because the incompleteness or uncertainty of the macroscopic observation at each instant does not seem to be so fundamental for the system whose mechanical behavior can be described by the classical mechanics. In other words, we can not directly accept the insistence that because of the incompleteness of macroscopic observations the observable properties of the system at given time t are determined only by the coarse grained properties at time $t-\tau$, where τ is a suitably chosen small time interval. The applicability of such a Markoffian random process is not self-evident but to be proved. From this point of view the reconciliation of the phenomenological irreversibility with the reversible character of the law of exact mechanics has not

still been achieved. In order to make this point clear as far as possible, we shall present in this paper an example which can be solved without usual assumption and investigate the irreversible character of a process rather mechanically. We consider a one-dimensional box of length $2l$ which is divided into two equal rooms by a wall in the middle. As the initial condition we assume that N particles, which have equal mass and collide with each other elastically, are contained in the left room of the box. Remove the wall at time $t=0$, then these particles diffuse into the right room and the number of the particles $n(t)$ contained in the left room changes, decreases generally, as a function of time t . The behavior of the function $n(t)$ in this process undoubtedly depends upon the choice of the microscopic initial conditions or the mechanical trajectories in phase space which satisfy the macroscopic initial condition that all particles must be enclosed in the left room of the box at $t=0$.

In the following two sections we investigate the mean properties of function $n(t)$ by somewhat geometrical method. As for the initial conditions two cases are adopted, i.e., the uniform distribution assumed in § 2 and the Boltzmann distribution assumed in § 3. In § 4 the change of distribution for the directions of velocities as the function of time t , which has often been investigated using wind-tree model, is examined in just the same manner. In § 5 we apply the usual method, in which the "Stosszahlansatz" is assumed, to the present model and compare its results with those obtained in the preceding chapters. Lastly we shall give a statistical mechanical interpretation of H-theorem basing on our results, though its validity in general cases can not be proved at the present stage.

§ 2. The uniform distribution for the initial state

In the present model each particle collides elastically and only bimolecular collisions are considered neglecting the collisions of higher order. Then it is evident that any pair of particles just exchange their momentum when they collide with each other in our one-dimensional box. If we consider that these particles exchange not only their momentum but also their assigned numbers by which the particles are distinguished, the particle which has a given assigned number moves as if it passed clear through another particle without collision. Evidently this argument is approximately valid when the size of particles is sufficiently small compared with the dimension of the box. From this point of view, the motion of our system can be treated as a mechanical system of N free particles in one-dimensional box. Furthermore, it is to be remarked that if the collisions are not taken into consideration, the calculations in this paper can also be applied directly to the three-dimensional similar systems, because the number of particles in the left room of the box $n(t)$ at time t depends only upon the values of x -components of velocities v_x 's and initial values of x -coordinates of all particles, if two rooms are connected in x -direction. Thus hereafter we shall restrict our attention only to the mechanical system of N independent free particles in one-dimensional box.

In the beginning, we shall consider a free particle which has the coordinate x_0 ($0 \leq x_0 \leq l$) measured from the left end of the box and the velocity $v_0 = p_0$ (mass $m=1$) at time

$t=0$, and investigate the condition to be satisfied by x_0 and v_0 in order that this particle is just in the right room at time t . Considering the fact that, at the boundaries, particles are reflected elastically without any change of the magnitude of velocity, this condition can readily be obtained and becomes

$$(4m+1)l \leq v_0 t + x_0 \leq (4m+3)l, \quad (1)$$

$$m=0, \pm 1, \pm 2, \dots$$

This relation is indicated schematically in Fig. 1. If the representative point of particle which gives its initial state x_0 and v_0 is in the shaded bands of Fig. 1, then this particle satisfies the condition (1) and will be found in the right room of the box at time t . Of course, the shape of these bands changes as time passes and uniformly they become narrower in width and slower in slope.

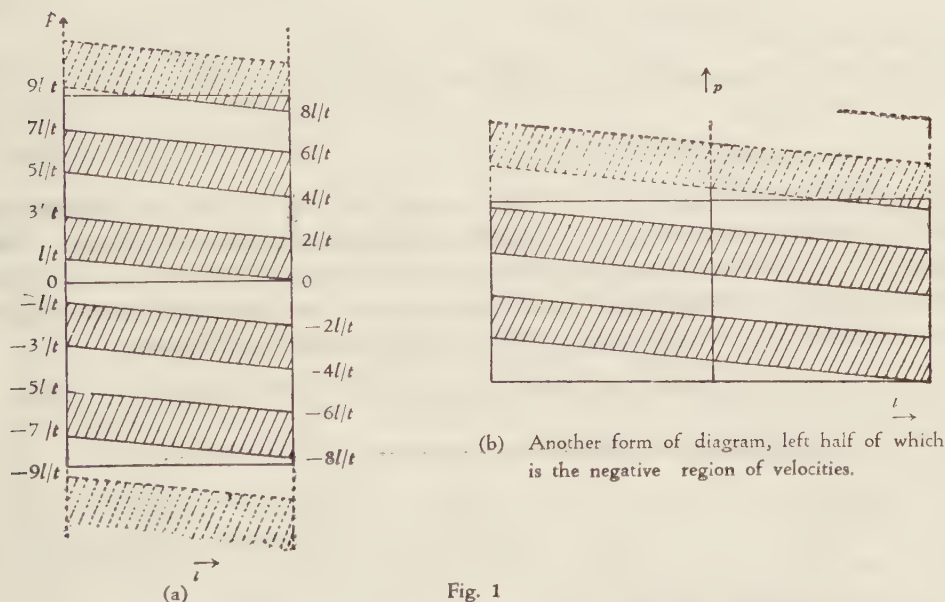


Fig. 1

Using this diagram we can obtain the knowledge about the statistical behavior of N independent free particles started from the left room of the box, making some fundamental assumption on their initial states. In this section, for mathematical convenience, we shall take the uniform distribution for the initial states, namely we assume that any one of these particles can take every point of diagram equally probably as the initial state so long as it lies in the square region in which the conditions $0 \leq x_0 \leq l$ and $|v_0| \leq \sqrt{2E}$ are satisfied. Then, as is well known, the probability that, at time t , n particles are found in the left room of the box, is given by Bernoulli distribution function

$$P_N(n) = N! / n! (N-n)! \cdot r^{n-n} (1-r)^n, \quad (2)$$

where r is a function of time t and gives the ratio of the area of shaded bands stretched over the above mentioned square region and is given by

$$\begin{aligned}
 r(t) &= \frac{2lm}{\sqrt{2E}t} + \frac{\sqrt{2E}t}{4l} \left(1 - \frac{4ml}{\sqrt{2E}t}\right)^2, \\
 &\text{for } 4ml/\sqrt{2E} \leq t \leq (4m+2)l/\sqrt{2E}, \\
 r(t) &= 1 - (2m+1)\frac{l}{\sqrt{2E}t} - \frac{\sqrt{2E}t}{4l} \left(1 - \frac{(4m+2)l}{\sqrt{2E}t}\right)^2, \\
 &\text{for } (4m+2)l/\sqrt{2E} \leq t \leq 4(m+1)l/\sqrt{2E}.
 \end{aligned} \tag{3}$$

Using the probability function (2), the average number of particles in the left room of the box at time t can readily be calculated and we have

$$\begin{aligned}
 \langle n \rangle_N &= N(1-r) = N \left\{ 1 - \frac{m}{2T} - T \left(1 - \frac{m}{T} \right)^2 \right\}, \\
 &\text{for } m \leq T \leq m + \frac{1}{2}, \\
 \langle n \rangle_N &= N \left\{ \frac{(2m+1)}{4T} + T \left(1 - \frac{2m+1}{2T} \right)^2 \right\}, \\
 &\text{for } m + \frac{1}{2} \leq T \leq m + 1,
 \end{aligned} \tag{4}$$

where the dimensionless time $T = t\sqrt{2E}/4l$ is used. In Fig. 2 $\langle n \rangle_N$ as a function of time T and in Fig. 3 the change of probability distribution with time are shown. This probability distribution has a sharp maximum at the most probable value of n , if the total number of particles is sufficiently large, and the behavior of this distribution function about its mean value is approximately represented by Gaussian distribution function

$$P_N(n) = [1/2\pi N(1-r)]^{1/2} \exp[-\{n - N(1-r)\}^2/2N(1-r)]. \tag{5}$$

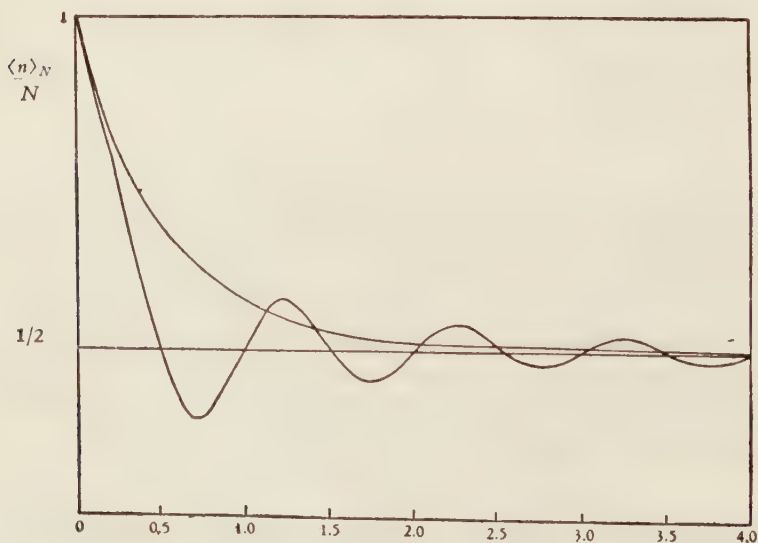


Fig. 2

 $= \sqrt{2E}t/4l$

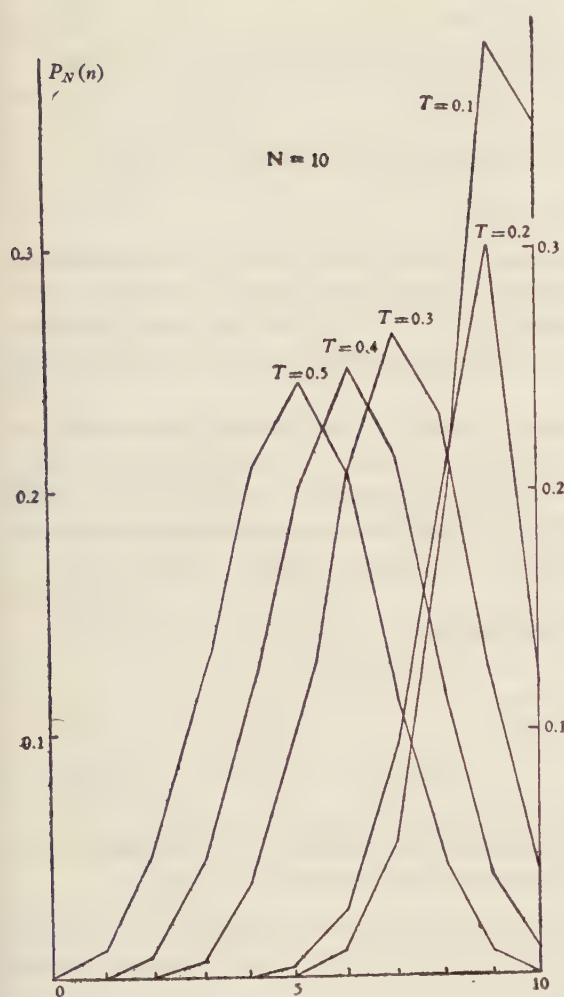


Fig. 3

to be used for any value of the total number of particles N and time t . Then we take only the cases $N=1$, 2 and $N \gg 1$, and from their results we surmise the general properties of this system.

(I) Case $N=1$. In this case the initial states of a particle are restricted on a line in our diagram corresponding to the initial velocity $v_0 = \sqrt{2E}$ and $0 \leq x_0 \leq l$. Then for the probability that this particle can be found in the left room of the box at time t , we have

$$P_1(1) = 1 - (\sqrt{2E}t - 4ml)/2l, \quad 4ml/\sqrt{2E} \leq t \leq (4m+2)l/2E, \quad (6)$$

$$P_1(1) = 1 - \{4(m+1)l - \sqrt{2E}t\}/2l, \quad (4m+2)l/\sqrt{2E} \leq t \leq 4(m+1)l/\sqrt{2E}, \quad (7)$$

Namely, in practice, it is very unlikely for us to encounter such mechanical path that it has the value of $n(t)$ appreciably deviated from its mean value (4). The relaxation time of this system, that is, the time which is required to establish the equilibrium value $n(t) = 1/2$, is $2l/\sqrt{2E}$ and this result is also valid when we assume more generally that at $t=0$, $n(0)$ and $N-n(0)$ particles are contained respectively in the left and right rooms of the box.

§ 3. The Maxwell-Boltzmann distribution for the initial state

The assumption of the uniform distribution for the initial state used in the above argument may be unnatural for the physical system. In this section we shall adopt the Maxwell-Boltzmann distribution for the initial state, in which we assume that all initial states are equally probable under the condition that the total energy $v_1^2/2 + v_2^2/2 + \dots + v_N^2/2$ is constant E and all molecules are contained initially in the left room of the box. However it is somewhat difficult to obtain a general expression of $\langle n(t) \rangle_N$ or distribution function $P_N(n)$

$$P_1(1) = 1 - 2(T - m), \quad m \leq T \leq m + 1/2, \quad (6')$$

$$P_1(1) = 1 - 2(m + 1 - T), \quad m + 1/2 \leq T \leq m + 1, \quad (7')$$

which is shown in Fig. 4 by the curve A.

(II) Case $N=2$. The sum of the energies of the two particles is constant :

$$v_{01}^2/2 + v_{02}^2/2 = E \equiv 2\bar{E}.$$

Under this condition we now wish to obtain the probabilities of finding 2, 1 and 0 particles in the left room at time t , but it is cumbersome to obtain their complete expressions which are applicable to any time interval. However, in order to see only the average properties of system going to the equilibrium state, we can be satisfied by the results only of the following interval of time.

i) $0 \leq t \leq 2l/\sqrt{2E}$. The probability that the particle 1 has a velocity in range v_{01} to $v_{01} + dv_{01}$ and exists in the right half of the box at time t , is simply $v_{01}tdv_{01}/2l\sqrt{2E}$. Then the particle 2 has a velocity $\sqrt{2E - v_{01}^2}$, and the probability of finding this second particle in the right room is $\sqrt{2E - v_{01}^2}t/2l$. Thus for the probability of finding both particles in the right room and no particle in the left room at time t , we have

$$\begin{aligned} P_2(0) &= (t^2/4l^2\sqrt{2E}) \int_0^{\sqrt{2E}} v_{01}\sqrt{2E - v_{01}^2} dv_{01} \\ &= 2Et^2/12l^2 = 2\bar{E}t^2/6l^2 = \frac{8}{3}T^2, \end{aligned} \quad (8)$$

$$T = t\sqrt{2E}/4l.$$

In just the same manner, we can obtain the probabilities of finding only one particle and two particles in the left room of the box, respectively in the forms

$$\begin{aligned} P_2(1) &= (1/4l^2\sqrt{2E}) \int_0^{\sqrt{2E}} (2l - v_{01}t)\sqrt{2E - v_{01}^2} t dv_{01} \\ &= (2 + \pi)/\sqrt{2} \cdot T - (16/3) \cdot T^2, \end{aligned} \quad (9)$$

and

$$\begin{aligned} P_2(2) &= (1/4l^2\sqrt{2E}) \int_0^{\sqrt{2E}} (2l - v_{01}t)(2l - \sqrt{2E - v_{01}^2}t) dv_{01} \\ &= 1 - (2 + \pi)/\sqrt{2} \cdot T + (8/3) \cdot T^2. \end{aligned} \quad (10)$$

Using these results we obtain the average number of particles in the left room at time t :

$$\langle n \rangle_2 = 0P_2(0) + 1P_2(1) + 2P_2(2) = 2 - (2 + \pi)/\sqrt{2} \cdot T, \quad (11)$$

and the average probability of existing in the left room per particle,

$$\langle n \rangle_2/2 = 1 - (2 + \pi)/2\sqrt{2} \cdot T. \quad (12)$$

These results are valid only for the time interval $0 \leq T \leq 1/2\sqrt{2}$, and equation (12)

gives the linear part of curve B in Fig. 4.

Before entering into the next time interval we shall make a remark that in this time interval one can obtain the mean value $\langle n \rangle_N$ for any number of particles N , which can be written in the form $\langle n \rangle_N = 1 - C_N \sqrt{N} T$, where the coefficients C_N have the relations $C_{N+1} = \pi/4 \cdot C_N + 1$ and $C_1 = 2$. Furthermore the probability distributions $P_N(n)$ in this time interval can be derived successively as follows,

$$\begin{aligned} P_3(0) &= 2/3 \cdot (\sqrt{3} T)^3, \\ P_3(1) &= (14 + 3\pi)/9 \cdot (\sqrt{3} T)^2 - 2(\sqrt{3} T)^3, \\ P_3(2) &= (\pi^2 + 2\pi + 8)/8 \cdot (\sqrt{3} T) - (28 + 6\pi)/9 \cdot (\sqrt{3} T)^2 + 2(\sqrt{3} T)^3, \\ P_3(3) &= 1 - (\pi^2 + 2\pi + 8)/8 \cdot (\sqrt{3} T) + (14 + 3\pi)/9 \cdot (\sqrt{3} T)^2 - 2/3 \cdot (\sqrt{3} T)^3, \end{aligned} \quad (13)$$

and $P_4(0) = 4/15 \cdot (2T)^4$,

$$\begin{aligned} P_4(1) &= (56 + 21\pi)/72 \cdot (2T)^3 - 16/15 \cdot (2T)^4, \\ P_4(2) &= (9\pi^2 + 42\pi + 184)/108 \cdot (2T)^2 - (56 + 21\pi)/24 \cdot (2T)^3 + 8/5 \cdot (2T)^4, \\ P_4(3) &= (\pi^3 + 2\pi^2 + 8\pi + 32)/32 \cdot (2T) \\ &\quad - (9\pi^2 + 42\pi + 184)/54 \cdot (2T)^2 + (21\pi + 56)/24 \cdot (2T)^3 \\ &\quad - 16/15 \cdot (2T)^4, \end{aligned} \quad (14)$$

$$\begin{aligned} P_4(4) &= 1 - (\pi^3 + 2\pi^2 + 8\pi + 32)/32 \cdot (2T) \\ &\quad + (9\pi^2 + 42\pi + 184)/108 \cdot (2T)^2 - (21\pi + 56)/72 \cdot (2T)^3 \\ &\quad + 4/15 \cdot (2T)^4. \end{aligned}$$

Using these probability functions we can evaluate the mean values $\langle n \rangle_3$ and $\langle n \rangle_4$ in the above mentioned forms.

ii) $t \geq 2l/\sqrt{2E}$ In this time interval the calculation becomes more complicated because of the appearance of two or more shaded bands in the diagram, so we shall show only their results in the following.

For the time interval $2l/\sqrt{2E} \leq t \leq 4l/\sqrt{2E}$, or $1/2\sqrt{2} \leq T \leq 1/\sqrt{2}$ where $T = t\sqrt{2E}/4l$,

$$\langle n \rangle_{2/2} = \frac{\sqrt{2}}{8T} (1 - \sqrt{8T^2 - 1}) + \frac{\sqrt{2}}{2} T \left(1 - \frac{\pi}{2} + \sin^{-1} \frac{\sqrt{8T^2 - 1}}{2\sqrt{2}T} \right), \quad (15)$$

for $1/\sqrt{2} \leq T \leq 3/2\sqrt{2}$

$$\begin{aligned} \langle n \rangle_{2/2} &= 2 - \frac{\sqrt{2}}{8T} (3 + \sqrt{8T^2 - 1} - 4\sqrt{2T^2 - 1}) \\ &\quad - \frac{\sqrt{2}}{2} T \left(1 + \frac{\pi}{2} - 2\sin^{-1} \frac{\sqrt{8T^2 - 1}}{2\sqrt{2}T} + 2\sin^{-1} \frac{\sqrt{2T^2 - 1}}{\sqrt{2}T} \right), \end{aligned} \quad (16)$$

and for $3/2\sqrt{2} \leq T \leq 2/\sqrt{2}$

$$\begin{aligned} \langle n \rangle_2/2 = & -1 + \frac{\sqrt{2}}{8T} (6 + 4\sqrt{2T^2-1} - \sqrt{8T^2-1} - 3\sqrt{8T^2-9}) \\ & + \frac{\sqrt{2}}{2} T \left(1 - \frac{\pi}{2} - 2 \sin^{-1} \frac{\sqrt{2T^2-1}}{\sqrt{2}T} + 2 \sin^{-1} \frac{\sqrt{8T^2-1}}{2\sqrt{2}T} + 2 \sin^{-1} \frac{\sqrt{8T^2-9}}{2\sqrt{2}T} \right). \end{aligned} \quad (17)$$

In Fig. 4 these values of $\langle n \rangle_2/2$ are plotted by the curve B.

(III) Case $N \gg 1$. In this case, the probability that any one of the particles has the velocity in range v_0 to $v_0 + dv_0$ is given by the Maxwell-Boltzmann distribution function

$$(\beta/2\pi)^{1/2} e^{-\beta v_0^2/2} dv_0. \quad (18)$$

And the probability of finding this particle, whose velocity is v_0 , in the right room at time t is given by $(v_0 t - 4ml)/2l$ for $4ml/t \leq v_0 \leq (4m+2)l/t$ and $\{4(m+1)l - v_0 t\}/2l$ for $(4m+2)l/t \leq v_0 \leq 4(m+1)l/t$ as stated in the case (I). Thus we can obtain the expression of $\langle n \rangle_N/N$ as a function of time t by integrating the product of these two probabilities over the entire range of velocities. Then we have

$$\begin{aligned} \langle n \rangle_N/N = & 1 - (\beta/2\pi)^{1/2} \int_0^\infty e^{-\beta v^2/2} \text{prob (in right room)} dv \\ = & 1 - \frac{1}{2l} \left(\frac{\beta}{2\pi} \right)^{1/2} \sum_{m=0}^\infty \left[\int_{4ml/t}^{(4m+2)l/t} e^{-\beta v^2/2} (vt - 4ml) dv \right. \\ & \left. + \int_{(4m+2)l/t}^{4(m+1)l/t} e^{-\beta v^2/2} \{4(m+1)l - vt\} dv \right] \\ = & 1 - \frac{2T}{\pi^{1/2}} \sum_{m=0}^\infty \left[\int_{m/T}^{(m+1/2)/T} e^{-x^2} \left(x - \frac{m}{T} \right) dx + \int_{(m+1/2)/T}^{(m+1)/T} e^{-x^2} \left\{ \frac{m+1}{T} - x \right\} dx \right] \\ = & 1 - \frac{T}{\pi^{1/2}} \sum_{m=1}^\infty (e^{-m^2/T^2} - 2e^{-(m+1/2)^2/T^2} + e^{-(m+1)^2/T^2}) \\ & + \sum_{m=1}^\infty \left[(m+1) \left\{ \Phi\left(\frac{m+1}{T}\right) - \Phi\left(\frac{m+1/2}{T}\right) \right\} - m \left\{ \Phi\left(\frac{m+1/2}{T}\right) - \Phi\left(\frac{m}{T}\right) \right\} \right], \quad (19) \end{aligned}$$

where $T = \sqrt{2} t / 4l \sqrt{\beta}$ and $\Phi(x)$ is the error function. In this expression of $\langle n \rangle_N/N$ we can show that its value decreases linearly for small value of time T and approaches to the equilibrium value $1/2$ for large value of T , by the use of the expansion formulas of the error function.

Summing up the results obtained hitherto we can draw conclusions that the approach to the equilibrium value of n is the most probable feature of the mechanical trajectories, and if the number of particles is sufficiently large the probability of the realization of such a trajectory that its value of n appreciably deviates from the mean value $\langle n \rangle$ becomes very small.

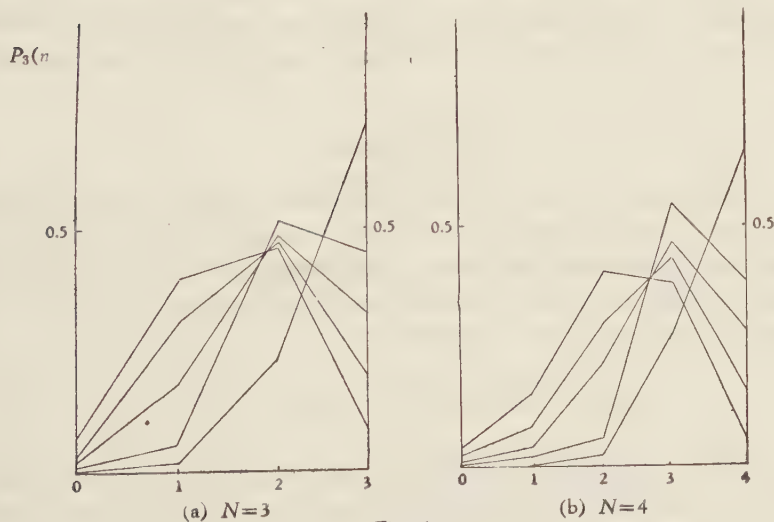
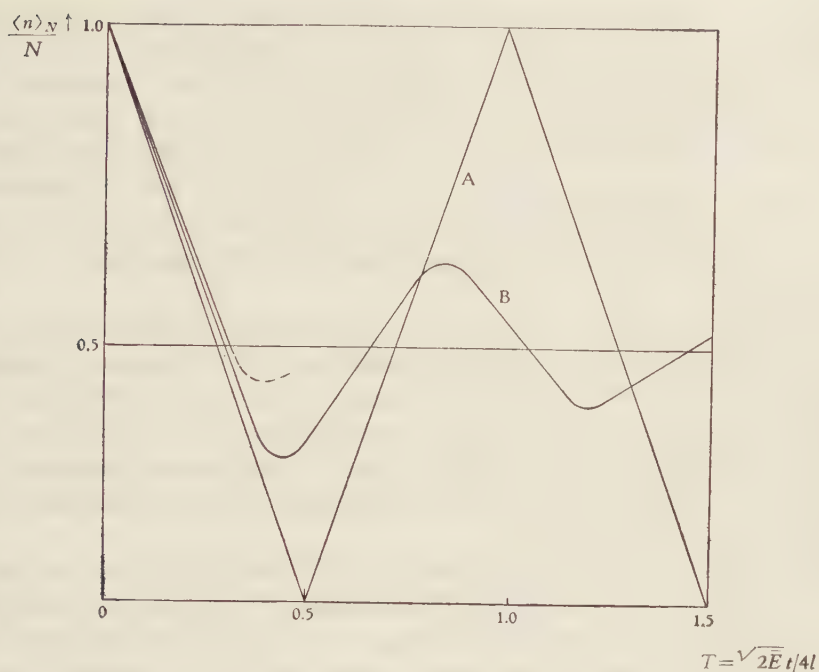


Fig. 5

§ 4. The distribution function of directions of velocities

In just the same manner we can investigate the change of distribution of directions of velocities of particles in a similar system. Let us consider a one-dimensional box of length l , and assume that at time $t=0$ all particles have positive direction of velocity and the initial distribution of the magnitudes of velocities is uniform just as in § 2. Then

the number of particles which have positive velocities at time t , $f(t)$, can be determined by a diagram similar to that of Fig. 1 (b). However the square region of the diagram corresponding to the present problem in which the initial uniform distribution is to be assumed, is given only by positive part of velocity coordinate and the width of the bands is uniformly reduced to half. Thus it is easily understood that the present diagram is equivalent to the reduced one of Fig. 1 (b) by half, in which $2l$ is replaced by l , so the desired results can be obtained directly by the replacement of length $2l$ in the equations in § 2 with length l . Of course the number distribution $n(t)$ corresponds to the function $f(t)$ in this case.

Thus the mean value $\langle f(t) \rangle$ decreases as time passes in the same way as Fig. 2 except the change of time scale, and the relaxation time which is required to establish the equilibrium value $\langle f(t) \rangle = 1/2$ is $l/\sqrt{2E}$, contrary to the value $2l/\sqrt{2E}$ for that of number distribution $\langle n(t) \rangle$. These values of the relaxation time do not change even if we assume that at time $t=0$ there are $n(0)$ and $N-n(0)$ particles respectively in left and right room of the box or $f(0)$ and $N-f(0)$ particles have the velocities respectively of positive and negative directions.

For these values of the relaxation time $2l/\sqrt{2E}$ and $l/\sqrt{2E}$, we can give simple meanings as follows. The mean value of the velocities of N particles is $\sqrt{2E}/2$ in both cases and the mean positions of particles at time $t=0$ are $x=0$ in the former and $x=l/2$ in the latter, because the particles have equally positive and negative velocities in the former contrary to the latter in which only positive velocities are considered. The relaxation time is obtained by the time that a particle which has this average velocity and position reaches the boundary $x=l$.

§ 5. Discussion

Here we shall apply the usual method to the present problem in which the assumption similar to the "Stosszahlansatz" is used. Let us consider that, at time t , n and $N-n$ particles are respectively in the left and right rooms and their velocities are distributed uniformly from $-\sqrt{2E}$ to $\sqrt{2E}$. Then the number of particles which pass out of the left room during the time interval τ is given by

$$\frac{n}{2l\sqrt{2E}} \int_0^{\sqrt{2E}} v\tau dv = \frac{\tau\sqrt{2E}}{4l} n \equiv g\tau n,$$

where $ndv/2l\sqrt{2E}$ is the number of particles in unit length whose velocities are in the range from v to $v+dv$. Similarly the number of particles which come into the left room is $g\tau(N-n)$. Consequently the change of the number of particles in the left room in this time interval becomes

$$dn/dt \doteq \{n(t+\tau) - n(t)\} / \tau = \{g\tau(N-n) - g\tau n\} / \tau = -g(2n-N)$$

and we obtain $n(t) = A \exp(-2gt) + N/2$. If at the initial time $n(0) = N$, then the constant factor $A = N/2$ and the relaxation time is given by $1/2g = 2l/\sqrt{2E}$, which is quite in agreement with that obtained by our method. Similarly for the distribution of

directions of velocities the factor g is given by $\sqrt{2E}/l$, because the number of particles in unit length whose velocities are in the range from v to $v+dv$ and have positive (negative) direction is equal to $fdv/l\sqrt{2E}$ ($(N-f)dv/l\sqrt{2E}$). Thus we obtain the relaxation time $l/\sqrt{2E}$ which also coincides with our result. In these calculations we have assumed that the condition $\sqrt{2E}\tau < l$ is satisfied. This condition may be always possible if we choose the magnitude τ suitably.

However, this agreement seems to be rather accidental. If we start from the initial condition $n(0)=N$, in our treatment no particle goes into the left from the right room during the relaxation process from $t=0$ to $t=2l/\sqrt{2E}$, and the number of particles in the left room $n(t)$ decreases at constant rate. On the other hand, in the treatment adopted in this section, the rate of change $|dn/dt|$ decreases as the time increases, due to the process of coarse graining at each instant; by the redistribution the number of particles which go out of the left room decreases and moreover the particles which come into the left from the right room appear. Actually the assumption of coarse graining can be applied approximately only in such a case that the redistribution seems to be realized sufficiently by some physical mechanism such as collisions of particles in this small time interval τ , so that our model is not suitable for this assumption. Consequently the agreement of the both results does not mean the validity of the assumption of "coarse graining".

Furthermore we shall refer to the recurrence time of the initial state. The initial states of N particles are represented by N points distributed on the diagram Fig. 1. We shall define the recurrence time of initial state T_r as such a time that at the time T_r all points are not in the shaded bands, namely all particles appear in the left room again. The magnitude of this recurrence time in general increases as N increases. As an example, we shall consider Fig. 6 in which the representative

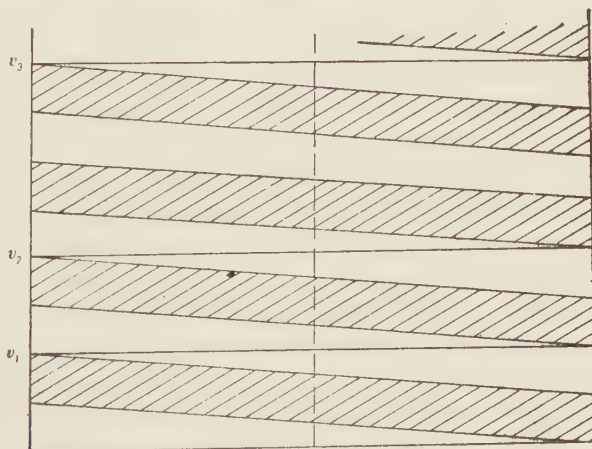


Fig. 6

points of initial state are distributed only on the velocities v_1, v_2, \dots, v_n and uniformly in positions $0 \leq x \leq l$ quite densely. Then in order to realize the state in which all particles are in the left room, it is necessary that the differences of all pair of the initial velocities $v_i - v_j$ must be commensurable with each other, just as seen in Fig. 6. The probability for such a restricted initial condition is zero.

The results obtained in our treatment suggest the possibility of an explanation of the irreversibility in which the process of redistribution of states at each instant is not necessary. Namely, if we observe a kind of quantity of the system and it has not the equilibrium value at the initial time, then its value changes almost always in a way that is characterized

by the irreversibility, in other words almost all the trajectories have the average property which show the tendency toward the equilibrium state. However, such irreversible quantity is not arbitrary but restricted to a certain class of quantities, for example, those expressible as sum functions. Obviously if we complete the general statistical mechanical theory of irreversible processes on this line, it can include directly the usual statistical mechanics in itself.

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Note on the Spin Wave Theory of Antiferromagnetism

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Spins in an antiferromagnet subjected to an external field perpendicular to the preferred axis of antiferromagnetic magnetization are not exactly aligned parallel and antiparallel to this axis. It is shown that a better approximation to the spin wave theory can be worked out, if one takes the equilibrium directions of the spins as the axes of quantization. One obtains the field-dependent resonance frequency of a microwave, which the usual spin wave theory fails to give, and also a higher approximation to the perpendicular susceptibility. The value of the latter in the limit of vanishing field strength agrees with the one obtained by Kubo.

§ 1. Introduction

Since Anderson¹⁾ has shown that the spin wave theory is a good approximate method for the ground state of antiferromagnetics, several authors²⁾ have applied this method to the calculation of thermodynamic properties of these substances. The perpendicular susceptibility χ_{\perp} calculated in the first approximation of the spin wave theory has the same value as that obtained on the basis of the molecular field theory, namely, it depends neither on temperature nor on field strength. However, Kubo³⁾ has shown, by calculating the contribution from terms neglected in the first approximation by the perturbation method, that χ_{\perp} deviates at absolute zero from its classical value and also depends on temperature. On the other hand, Ziman⁴⁾, through a different approach, attained the result that χ_{\perp} in a higher approximation is still constant. Nakamura⁵⁾ discussed antiferromagnetic resonance absorption, and according to his results, for the case of perpendicular field, only one field-independent resonance frequency is obtained under the assumption of a uniaxial anisotropy. The classical theory developed by Nagamiya⁶⁾ and Keffer and Kittel⁷⁾ yielded, however, two resonance frequencies, of which one corresponds to the frequency given by the spin wave theory and the other is a frequency which is field-dependent. Nakamura attributed this discrepancy to an insufficient approximation made in his spin wave theory.

We propose here another method of treating these problems. Kubo⁸⁾ suggests a similar method, but his statement is not so specific as ours. By this method the terms neglected in the usual spin wave theory can be taken into account to a certain extent and accordingly the discrepancy concerning the resonance frequency can be removed. We shall also derive χ_{\perp} and discuss the discrepancy between two conclusions attained by Kubo and Ziman. We are led to believe that Kubo's result is more reasonable. We always confine ourselves to the case of two sublattices and to external field applied perpendicularly to the preferred axis. The spins on one sublattice are, in this case, not exactly antiparallel to the spins on the

other one, but they take new equilibrium directions which are inclined by certain angles $\pm\theta$ to the preferred axis, towards the direction of the external field. We will take these directions as the axis of quantization of the spins on each sublattice, instead of taking the preferred axis, as usual. In other words, we choose the axes of quantization in such a way that the terms of the Hamiltonian which are linear with respect to spin deviations vanish. This should be reasonable because the essential point of the spin wave theory lies in its analogy to the classical theory of vibrating system and thus in making the approximation that the deviations of the spins measured from their equilibrium directions are small and only the quadratic terms of the deviation operators are retained in the Hamiltonian, the equilibrium directions themselves being determined by the condition of disappearance of the linear terms. In § 2 we shall calculate the frequencies of the spin waves and in § 3 discuss the resonance frequencies for an oscillating external magnetic field; in § 4 we derive the perpendicular susceptibility and its dependence on temperature and field strength, and in the last section we discuss the relation between our treatment and the usual one.

§ 2. Hamiltonian for the the spin waves and its eigenvalues.

The Hamiltonian of an antiferromagnetic is written as

$$H = H_{\text{ex}} + H_f + H_{\text{anis}}, \quad (1)$$

where

$$H_{\text{ex}} = 2J \sum_{\langle j, k \rangle} \mathbf{S}_j \cdot \mathbf{S}_k \quad (\text{exchange energy}), \quad (2)$$

$$H_f = g\mu_B H (\sum_j S_{jx} + \sum_k S_{kx}) \quad (\text{Zeeman energy}), \quad (3)$$

$$H_{\text{anis}} = -K (\sum_j S_{jz}^2 + \sum_k S_{kz}^2) \quad (\text{anisotropy energy}). \quad (4)$$

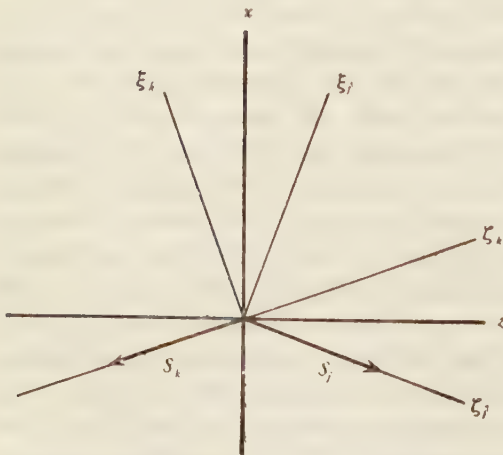


Fig. 1.

\mathbf{S}_j is a spin on one sublattice and \mathbf{S}_k a spin on the other; $\sum_{\langle j, k \rangle}$ means the summation over nearest-neighboring pairs. The x -axis is taken to the direction of the external field, g is the Landé factor and μ_B the Bohr magneton. The anisotropy energy is taken to be uniaxial.

We introduce new coordinate axes (ξ_j, η_j, ζ_j) for the j -sublattice and (ξ_k, η_k, ζ_k) for the k -sublattice. They are obtained by rotating the coordinate axes (x, y, z) through an angle $-\theta$ and $+\theta$, respectively, about the y -axis (Fig. 1). Then we have the following relations:

$$S_{jx} = S_{j\zeta} \cos \theta - S_{j\eta} \sin \theta,$$

$$S_{jx} = S_{jx} \sin \theta + S_{jz} \cos \theta; \quad (5a)$$

$$S_{ky} = S_{ky} \cos \theta + S_{kz} \sin \theta, \quad (5b)$$

$$S_{kx} = -S_{ky} \sin \theta + S_{kz} \cos \theta.$$

Defining S_j^\pm and S_k^\pm by

$$\begin{aligned} S_j^+ &= S_{jx} + iS_{jy}, & S_k^+ &= S_{kx} + iS_{ky}, \\ S_j^- &= S_{jx} - iS_{jy}; & S_k^- &= S_{kx} - iS_{ky} \end{aligned} \quad (6a), (6b)$$

we obtain

$$\begin{aligned} H_{ex} &= 2J \sum_{\langle j, k \rangle} [1/2 \cdot \{S_j^+ S_k^- + S_j^- S_k^+\} \cos^2 \theta - 1/2 \cdot \{S_j^+ S_k^+ + S_j^- S_k^-\} \sin^2 \theta \\ &\quad - 1/2 \cdot \{S_{jz} (S_k^+ + S_k^-) - (S_j^+ + S_j^-) S_{kz}\} \sin 2\theta + S_{jz} S_{kz} \cos 2\theta], \end{aligned} \quad (7)$$

$$\begin{aligned} H_f &= g \mu_B H \left[\sum_j \{1/2 \cdot (S_j^+ + S_j^-) \cos \theta - S_{jz} \sin \theta\} \right. \\ &\quad \left. + \sum_k \{1/2 \cdot (S_k^+ + S_k^-) \cos \theta + S_{kz} \sin \theta\} \right], \end{aligned} \quad (8)$$

$$\begin{aligned} H_{anis} &= -K \sum_j [1/4 \cdot (S_j^+ + S_j^-)^2 \sin^2 \theta + S_{jz}^2 \cos^2 \theta + 1/4 \cdot \{(S_j^+ + S_j^-) S_{jz} \\ &\quad + S_{jz} (S_j^+ + S_j^-)\} \sin 2\theta] \\ &\quad - K \sum_k [1/4 \cdot (S_k^+ + S_k^-)^2 \sin^2 \theta + S_{kz}^2 \cos^2 \theta - 1/4 \cdot \{(S_k^+ + S_k^-) S_{kz} \\ &\quad + S_{kz} (S_k^+ + S_k^-)\} \sin 2\theta]. \end{aligned} \quad (9)$$

To obtain the Hamiltonian of the spin waves, it is convenient to write the spin operators in terms of the operators introduced by Holstein and Primakoff and Kubo⁹⁾:

$$\begin{aligned} S_j^+ &= (2S)^{1/2} (1 - a_j^* a_j / 2S)^{1/2} a_j, & S_k^+ &= (2S)^{1/2} b_k^* (1 - b_k^* b_k / 2S)^{1/2}, \\ S_j^- &= (2S)^{1/2} a_j^* (1 - a_j^* a_j / 2S)^{1/2}, & S_k^- &= (2S)^{1/2} (1 - b_k^* b_k / 2S)^{1/2} b_k, \\ S_{jz} &= S - a_j^* a_j; & S_{kz} &= -S + b_k^* b_k. \end{aligned} \quad (10a), (10b)$$

a_j and b_k satisfy the commutation relations:

$$[a_j, a_{j'}^*] = \delta_{jj'}, \quad [b_k, b_{k'}^*] = \delta_{kk'}. \quad (11)$$

The Hamiltonian of the spin waves is obtained by omitting the terms of higher orders than quadratic of these operators. We obtain from (7), (8), (9) and (10a), (10b) the following expressions:

$$\begin{aligned} H_{ex} &= -NJzS^2 \cos 2\theta - (2S)^{1/2} Jz \sin 2\theta \left\{ \sum_j (a_j + a_j^*) + \sum_k (b_k + b_k^*) \right\} \\ &\quad + 2JzS \cos 2\theta \left(\sum_j a_j^* a_j + \sum_k b_k^* b_k \right) + 2Jz \cos^2 \theta \sum_{\langle j, k \rangle} (a_j b_k + a_j^* b_k^*) \\ &\quad - 2JS \sin^2 \theta \sum_{\langle j, k \rangle} (a_j^* b_k + a_j b_k^*), \end{aligned} \quad (12)$$

$$H_J = -Ng\mu_R SH \sin \theta + (2S)^{1/2}/2 \cdot g\mu_R H \cos \theta \left\{ \sum_j (a_j + a_j^*) + \sum_k (b_k + b_k^*) \right\} \\ + g\mu_R H \sin \theta \left\{ \sum_j a_j^* a_j + \sum_k b_k^* b_k \right\}, \quad (13)$$

$$H_{\text{anis}} = -NKS^2 \cos^2 \theta - (2S)^{1/2}/2 \cdot KS \sin 2\theta \left\{ \sum_j (a_j + a_j^*) + \sum_k (b_k + b_k^*) \right\} \\ + 2KS \cos^2 \theta \left(\sum_j a_j^* a_j + \sum_k b_k^* b_k \right) - 1/2 \cdot KS \sin^2 \theta \\ \times \left\{ \sum_j (a_j^* a_j + a_j a_j^* + a_j^2 + a_j^{*2}) + \sum_k (b_k^* b_k + b_k b_k^* + b_k^2 + b_k^{*2}) \right\}, \quad (14)$$

where z is the number of nearest neighbors. Rearranging the terms, the Hamiltonian can be written

$$H = H_0 + H_1 + H_2, \quad (15)$$

$$H_0 = -NJzS^2 - NKS^2 + 2NJzS^2 \sin^2 \theta - Ng\mu_R SH \sin \theta + NKS^2 \sin^2 \theta, \quad (16)$$

$$H_1 = (2S)^{1/2}/2 \cdot (g\mu_R H \cos \theta - 2JzS \sin 2\theta - KS \sin 2\theta) \left\{ \sum_j (a_j + a_j^*) + \sum_k (b_k + b_k^*) \right\}, \quad (17)$$

$$H_2 = (2JzS \cos 2\theta + 2KS \cos^2 \theta + g\mu_R H \sin \theta) \left(\sum_j a_j^* a_j + \sum_k b_k^* b_k \right) \\ + 2JS \cos^2 \theta \sum_{\langle j, k \rangle} (a_j b_k + a_j^* b_k^*) - 2JS \sin^2 \theta \sum_{\langle j, k \rangle} (a_j^* b_k + a_j b_k^*) \\ - 1/2 \cdot KS \sin^2 \theta \left\{ \sum_j (a_j^* a_j + a_j a_j^* + a_j^2 + a_j^{*2}) + \sum_k (b_k^* b_k + b_k b_k^* + b_k^2 + b_k^{*2}) \right\}. \quad (18)$$

H_0 is the zeroth order Hamiltonian which does not contain the deviation operators. H_1 consists of linear term and H_2 of quadratic term.

Now we determine the value of θ . This can be done in either way: H_1 must vanish at the equilibrium configuration, because it corresponds to torque, or H_0 must be minimum, because it corresponds to the energy at a configuration specified by θ . These two ways yield the same value for θ . We obtain

$$\sin \theta = g\mu_R H / (4Jz + 2K)S. \quad (19)$$

In the molecular field theory χ_{\perp} is given by

$$\chi_{\perp c} = Ng^2 \mu_R^2 / (4Jz + 2K), \quad (20)$$

where the suffix c means 'classical'. Therefore (19) can be written as

$$\sin \theta = \chi_{\perp c} H / Ng\mu_R S. \quad (21)$$

We see from this formula that the value of θ is the same as that which follows classically. H_0 now becomes

$$H_0 = -NJzS^2 - NKS^2 - 1/2 \cdot \chi_{\perp c} H^2. \quad (22)$$

In order to diagonalize H_2 , we introduce the Fourier transforms of a_j , a_j^* and b_k , b_k^* :

$$\begin{aligned} a_j &= (2/N)^{1/2} \sum_j a_j \exp(-i\mathbf{\lambda} \cdot \mathbf{j}), & a_j^* &= (2/N)^{1/2} \sum_j a_j^* \exp(i\mathbf{\lambda} \cdot \mathbf{j}), \\ b_k &= (2/N)^{1/2} \sum_k b_k \exp(i\mathbf{\lambda} \cdot \mathbf{k}), & b_k^* &= (2/N)^{1/2} \sum_k b_k^* \exp(-i\mathbf{\lambda} \cdot \mathbf{k}). \end{aligned} \quad (23)$$

Furthermore, we define new operators $A_{1\lambda}$, $A_{2\lambda}$, $B_{1\lambda}$, $B_{2\lambda}$ as follows :

$$\begin{aligned} A_{1\lambda} &= (a_\lambda + a_{-\lambda}) / \sqrt{2}, & A_{2\lambda} &= (a_\lambda - a_{-\lambda}) / \sqrt{2}, \\ B_{1\lambda} &= (b_\lambda + b_{-\lambda}) / \sqrt{2}, & B_{2\lambda} &= (b_\lambda - b_{-\lambda}) / \sqrt{2}. \end{aligned} \quad (24)$$

Here we limit λ to half the wave vector space. For $\lambda=0$ we define $A_{10}=a_0$, $A_{20}=0$, $B_{10}=b_0$, $B_{20}=0$. Commutation relations for these operators are

$$[A_{i\lambda}, A_{i'\lambda'}^*] = \delta_{ii'} \delta_{\lambda\lambda'}, \quad [B_i, B_{i'\lambda'}^*] = \delta_{ii'} \delta_{\lambda\lambda'} \quad (i=1, 2). \quad (25)$$

By transformation (24), the interactions between λ and $-\lambda$ are eliminated and we obtain the transformed Hamiltonian as

$$\begin{aligned} H &= H_0 + 1/2 \cdot NKS \sin^2 \theta + (2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_R H \sin \theta) \\ &\times \sum_{i,\lambda} (A_{i\lambda}^* A_{i\lambda} + B_{i\lambda}^* B_{i\lambda}) + 2JzS \left[\sum_{i,\lambda} \{ \gamma_\lambda \cos^2 \theta (A_{i\lambda} B_{i\lambda} + A_{i\lambda}^* B_{i\lambda}^*) \right. \\ &\left. + (-1)^i \gamma_\lambda \sin^2 \theta (A_{i\lambda}^* B_{i\lambda} + A_{i\lambda} B_{i\lambda}^*) \right] + 1/2 \cdot KS \sin^2 \theta \sum_{i,\lambda} (-1)^i \\ &\times (A_{i\lambda}^2 + A_{i\lambda}^{*2} + B_{i\lambda}^2 + B_{i\lambda}^{*2}), \end{aligned} \quad (26)$$

where γ_λ is defined by $\gamma_\lambda = 1/z \cdot \sum_p \exp(i\mathbf{\lambda} \cdot \mathbf{\rho})$, ρ being a vector which points to one of the nearest neighbours. Next we introduce real operators $Q_{i\lambda}$, $P_{i\lambda}$, $R_{i\lambda}$, $S_{i\lambda}$ by the relations

$$\begin{aligned} Q_{i\lambda} &= (A_{i\lambda} + A_{i\lambda}^*) / \sqrt{2}, & P_{i\lambda} &= (A_{i\lambda} - A_{i\lambda}^*) / \sqrt{2} i, \\ R_{i\lambda} &= (B_{i\lambda} + B_{i\lambda}^*) / \sqrt{2}, & S_{i\lambda} &= (B_{i\lambda} - B_{i\lambda}^*) / \sqrt{2} i. \end{aligned} \quad (27)$$

Commutation relations for these operators are

$$[Q_{i\lambda}, P_{i\lambda}] = i, \quad [R_{i\lambda}, S_{i\lambda}] = i, \quad (28)$$

other commutators are all equal to zero. The Hamiltonian becomes

$$\begin{aligned} H &= H_0 + 1/2 \cdot NKS \sin^2 \theta - N/2 \cdot (2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_R H \sin \theta) \\ &+ 1/2 \cdot (2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_R H \sin \theta) \sum_{i,\lambda} (Q_{i\lambda}^2 + P_{i\lambda}^2 + R_{i\lambda}^2 + S_{i\lambda}^2) \\ &+ 2JzS \left[\sum_{i,\lambda} \{ \gamma_\lambda \cos^2 \theta (Q_{i\lambda} R_{i\lambda} - P_{i\lambda} S_{i\lambda}) + (-1)^i \gamma_\lambda \sin^2 \theta (Q_{i\lambda} R_{i\lambda} + P_{i\lambda} S_{i\lambda}) \} \right] \\ &+ 1/2 \cdot KS \sin^2 \theta \sum_{i,\lambda} (-1)^i (Q_{i\lambda}^2 + R_{i\lambda}^2 - P_{i\lambda}^2 - S_{i\lambda}^2). \end{aligned} \quad (29)$$

Finally we introduce $q_{i\lambda}$, $p_{i\lambda}$, $r_{i\lambda}$, $s_{i\lambda}$ defined by

$$\begin{aligned} q_{i\lambda} &= (Q_{i\lambda} + R_{i\lambda}) / \sqrt{2}, & r_{i\lambda} &= (Q_{i\lambda} - R_{i\lambda}) / \sqrt{2}, \\ p_{i\lambda} &= (P_{i\lambda} + S_{i\lambda}) / \sqrt{2}, & s_{i\lambda} &= (P_{i\lambda} - S_{i\lambda}) / \sqrt{2}. \end{aligned} \quad (30)$$

$(q_{i\lambda}, p_{i\lambda})$ and $(r_{i\lambda}, s_{i\lambda})$ are pairs of canonically conjugate variables. Through this transformation we obtain the Hamiltonian in the diagonalized form:

$$\begin{aligned}
 H = & H_0 + 1/2 \cdot NKS \sin^2 \theta - N/2 \cdot (2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_B H \sin \theta) \\
 & + \sum_{i,\lambda} [1/2 \cdot \{2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_B H \sin \theta + (-1)^i KS \sin^2 \theta \\
 & \quad + 2Jz\gamma_\lambda (\cos^2 \theta + (-1)^i \sin^2 \theta)\} \cdot q_{i\lambda}^2 \\
 & + 1/2 \cdot \{2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_B H \sin \theta - (-1)^i KS \sin^2 \theta \\
 & \quad - 2Jz\gamma_\lambda (\cos^2 \theta - (-1)^i \sin^2 \theta)\} \cdot p_{i\lambda}^2 \\
 & + 1/2 \cdot \{2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_B H \sin \theta + (-1)^i KS \sin^2 \theta \\
 & \quad - 2Jz\gamma_\lambda (\cos^2 \theta + (-1)^i \sin^2 \theta)\} \cdot r_{i\lambda}^2 \\
 & + 1/2 \cdot \{2JzS \cos 2\theta + 2KS \cos^2 \theta - KS \sin^2 \theta + g\mu_B H \sin \theta - (-1)^i KS \sin^2 \theta \\
 & \quad + 2Jz\gamma_\lambda (\cos^2 \theta - (-1)^i \sin^2 \theta)\} \cdot s_{i\lambda}^2]. \quad (31)
 \end{aligned}$$

The frequency $\omega_{1\lambda}$ of the oscillator $(q_{1\lambda}, p_{1\lambda})$ is the same as that of the oscillator $(r_{2\lambda}, s_{2\lambda})$, and the frequency $\omega_{2\lambda}$ of the oscillator $(q_{2\lambda}, p_{2\lambda})$ the same as that of the oscillator $(r_{1\lambda}, s_{1\lambda})$. Substituting the value of $\sin \theta$ and carrying out some calculations, we obtain the eigenvalues of the Hamiltonian as

$$\begin{aligned}
 H_0 = & -NJz(1+\alpha)S(S+1) - 1/2 \cdot \chi_{\perp c} H^2 + 1/4S \cdot \alpha / (1+\alpha/2) \cdot \chi_{\perp c} H^2 \\
 & + \sum_{\lambda} (n_{1\lambda} + 1/2) (\hbar\omega_{1\lambda}) + \sum_{\lambda} (n_{2\lambda} + 1/2) (\hbar\omega_{2\lambda}), \quad (32)
 \end{aligned}$$

where $\alpha = K/Jz$, $n_{1\lambda}$ and $n_{2\lambda}$ are positive integers, \sum_{λ} means the summation over the whole space of λ , and

$$\hbar\omega_{1\lambda} = (2JzS) [\{1+\alpha-\gamma_\lambda\} \{1+\alpha+\gamma_\lambda - (2\gamma_\lambda+\alpha)\sin^2 \theta\}]^{1/2}, \quad (33)$$

$$\hbar\omega_{2\lambda} = (2JzS) [\{1+\alpha+\gamma_\lambda\} \{1+\alpha-\gamma_\lambda + (2\gamma_\lambda-\alpha)\sin^2 \theta\}]^{1/2}. \quad (34)$$

§ 3. Resonance frequency for microwave absorption

A microwave can excite only those spin waves which correspond to an in-phase motion of all the spins. So the frequencies of the spin waves which belong to $\lambda=0$ give rise to resonance frequencies. We have two frequencies, ω_{10} and ω_{20} , which are given by

$$\hbar\omega_{10} = (2JzS) \{ (2+\alpha)\alpha \}^{1/2} \cos \theta \simeq (2JzS) (2\alpha)^{1/2}, \quad (35)$$

$$\hbar\omega_{20} = (2JzS) [\{2+\alpha\} \{ \alpha + (2-\alpha) \sin^2 \theta \}]^{1/2} \simeq (2JzS) (2\alpha + 4 \sin^2 \theta)^{1/2}. \quad (36)$$

To compare these results with the classical ones, we put $4Jz/Ng^2\mu_B^2 = A$ and $NKS^2 = K_1$, then (35) and (36) become

$$\hbar\omega_{10} \simeq g\mu_B (2AK_1)^{1/2}, \quad \hbar\omega_{20} \simeq g\mu_B (2AK_1 + H^2)^{1/2}. \quad (35'), (36')$$

These agree exactly with the classical ones derived by Nagamiya⁶⁾ and Keffer-Kittel.⁷⁾

The selection rule can be obtained easily. Writing the microwave Hamiltonian in terms of the deviation operators and transforming it, we find that (35') and (36') are resonance frequencies for the microwave fields applied along the x -axis and the y -axis respectively. Nakamura⁵⁾ obtained only the frequency (35'). This corresponds to an oscillation of the total magnetization along the applied static field, so that this frequency is little affected by the field strength. However, the mode of oscillation with frequency (36') is such that the total magnetization oscillates perpendicularly to the static field, so that this frequency will not be obtained, if one does not take account of the magnetization induced by the static field. This point will be discussed further in § 5.

§ 4. The perpendicular susceptibility

The free energy can easily be obtained as

$$F = -NJz(1+\alpha)S(S+1) - 1/2 \cdot \chi_{\perp c} H^2 + 1/4S \cdot \alpha / (1+\alpha/2) \chi_{\perp c} H^2 \\ + N/4 \cdot \{ \langle \hbar\omega_{1\lambda} \rangle_{AV} + \langle \hbar\omega_{2\lambda} \rangle_{AV} \} + N/2\beta \cdot \{ \langle \log \{ 1 - \exp(-\beta\hbar\omega_{1\lambda}) \} \rangle_{AV} \\ + \langle \log \{ 1 - \exp(-\beta\hbar\omega_{2\lambda}) \} \rangle_{AV} \}, \quad (37)$$

where $\beta = 1/kT$ and $\langle \dots \rangle_{AV}$ means the average over all oscillators, that is,

$$\langle \dots \rangle_{AV} = \frac{1}{(2\pi)^D} \iiint \dots d\lambda^D$$

D is the dimension of the system. From the free energy χ_{\perp} is calculated as

$$\chi_{\perp} = -\frac{1}{H} \cdot \frac{\partial F}{\partial H} = \chi_{\perp c} - \chi_{\perp c} / 2S \cdot \alpha / (1+\alpha/2) - N/2 \left\{ \left\langle \frac{\partial(\hbar\omega_{1\lambda})}{\partial(H^2)} \right\rangle_{AV} + \left\langle \frac{\partial(\hbar\omega_{2\lambda})}{\partial(H^2)} \right\rangle_{AV} \right\} \\ - N \left\{ \left\langle \frac{\partial(\hbar\omega_{1\lambda})}{\partial(H^2)} \cdot \frac{1}{\exp(\beta\hbar\omega_{1\lambda}) - 1} \right\rangle_{AV} + \left\langle \frac{\partial(\hbar\omega_{2\lambda})}{\partial(H^2)} \cdot \frac{1}{\exp(\beta\hbar\omega_{2\lambda}) - 1} \right\rangle_{AV} \right\}. \quad (38)$$

Using (33) and (34), we have

$$\frac{\partial(\hbar\omega_{1\lambda})}{\partial(H^2)} = -1/N \cdot \chi_{\perp c} / 2S(1+\alpha/2) \cdot (\gamma_{\lambda} + \alpha/2) (1+\alpha-\gamma_{\lambda})^{1/2} \\ \times \{ 1+\alpha+\gamma_{\lambda} - (2\gamma_{\lambda}+\alpha)\sin^2\theta \}^{-1/2}, \quad (39)$$

$$\frac{\partial(\hbar\omega_{2\lambda})}{\partial(H^2)} = 1/N \cdot \chi_{\perp c} / 2S(1+\alpha/2) \cdot (\gamma_{\lambda} - \alpha/2) (1+\alpha+\gamma_{\lambda})^{1/2} \\ \times \{ 1+\alpha-\gamma_{\lambda} + (2\gamma_{\lambda}-\alpha)\sin^2\theta \}^{-1/2}. \quad (40)$$

We can calculate from these expressions the dependence of χ_{\perp} on temperature and field strength. Also we can calculate the deviation of χ_{\perp} from its classical value at the absolute zero of temperature.

First, we put $\alpha=0$ and calculate χ_{\perp} at $H=0$. From (38), (39) and (40) we obtain

$$\chi_{\perp} = \chi_{\perp 0} - \chi_{\perp 0} / 2S \cdot \langle \gamma_{\lambda}^2 / (1 - \gamma_{\lambda}^2)^{1/2} \rangle_{AV} - \chi_{\perp 0} / S \cdot \left\langle \frac{\gamma_{\lambda}^2}{(1 - \gamma_{\lambda}^2)^{1/2}} \cdot \frac{1}{(\exp \{ (1 - \gamma_{\lambda}^2)^{1/2} / \theta \} - 1)} \right\rangle_{AV}. \quad (41)$$

Here θ is defined, after Kubo,³⁾ as

$$\theta = kT / 2JzS. \quad (42)$$

(41) agrees completely with the result obtained by Kubo by the perturbation method. The second term in the right-hand side of (41) is due to the zero-point-motion of the spins.

In the case of the lattice of CsCl-type we have

$$\langle \gamma_{\lambda}^2 / (1 - \gamma_{\lambda}^2)^{1/2} \rangle_{AV} = 0.192,^* \quad (43)$$

$$\left\langle \frac{\gamma_{\lambda}^2}{(1 - \gamma_{\lambda}^2)^{1/2}} \cdot \frac{1}{(\exp \{ (1 - \gamma_{\lambda}^2)^{1/2} / \theta \} - 1)} \right\rangle_{AV} = 2/3 \cdot \theta^2 + \dots \quad (44)$$

It would be worth noting that we arrived at these results by considering no higher terms than quadratic, while Kubo derived them by calculating the effect of the third and fourth order terms.

In the second place, we see that the temperature dependence of χ_{\perp} is much influenced by the anisotropy energy near the absolute zero of temperature, as was already pointed out by Kubo³⁾ and Eisele-Keffe.¹⁰⁾ The latter authors, in particular, discussed this point in detail. We can directly apply their method to the calculation of χ_{\perp} and find the following expression for the simple cubic lattice:

$$\begin{aligned} \chi_{\perp} &= \chi_{\perp}|_{T=0} - \sqrt{3}/2 \cdot \chi_{\perp 0} / S \cdot \theta^2 \{ (6/\pi^2) (T_{AE}/T) \sum_{n=1}^{\infty} 1/n \cdot K_1(nT_{AE}/T) \} \\ &= \chi_{\perp}|_{T=0} - \sqrt{3}/2 \cdot \chi_{\perp 0} / S \cdot \theta^2 \cdot \varphi(T/T_{AE}). \end{aligned} \quad (45)$$

kT_{AE} is defined by Keffe as

$$kT_{AE} = (2JzS) (2\alpha)^{1/2}.$$

K_1 is the real Hankel function of degree one. The functional form of $\varphi(T/T_{AE})$ is the same as that of $M(T/T_{AE})$ given in Keffe's paper, which represents the temperature-dependent part of the magnitude of the magnetization of each sublattice.

The magnetizations of the two sublattices, inclined with angles $\pm \theta$ to the preferred axis, decrease in magnitude as temperature rises, and this in turn gives rise to a decrease in χ_{\perp} . Considering the same functional form of $\varphi(T/T_{AE})$ and $M(T/T_{AE})$, it can be shown that (45) gives just such decrease in χ_{\perp} . Actually, however, the magnetization vectors averaged over the thermal and zero-point motions of the spins will have different inclinations from $\pm \theta$ to the preferred axis, and this difference will result in a smaller temperature dependence of χ_{\perp} than that given by (45). This effect should, in principle, be calculable by going to a higher approximation, namely by considering the neglected terms in (44) (higher order terms in T) and the higher order terms than quadratic in the Hamiltonian (higher order terms in $1/S$).

* This value was obtained from the table in Kubo's paper.⁸⁾

In the third place, χ_{\perp} depends on the field strength. The field dependence of the frequencies of spin waves is exactly given by our calculation, provided one neglects other terms than the first term of the expansion of the Hamiltonian in inverse powers of S , as we have done in the present treatment.

We shall omit here the temperature dependent part of the susceptibility, since this part can give only a small contribution to the field-dependent part χ_{\perp} . The field dependence of the zero-point energy, which thus remains, can be calculated from (39) and (40). However, we cannot obtain the field dependence of χ_{\perp} in the form of a power series of H , as one sees easily from (39) and (40). Therefore we computed

χ_{\perp} numerically for the lattice of the CsCl type, putting $\alpha=0$. At absolute zero χ_{\perp} is expressible as

$$\chi_{\perp 0} = \chi_{\perp c} - \chi_{\perp c}/S \cdot f(\sin \theta), \quad (46)$$

where the function $f(\sin \theta)$ is plotted in Fig. 2. This result shows that $\chi_{\perp c}$ increases with increasing field strength.

For ordinary antiferromagnetic crystals under a magnetic field of the magnitude obtainable in ordinary laboratory experiments, $\sin \theta$ remains so small that the change in χ_{\perp} with field strength would hardly be observable. For particular antiferromagnetic crystals with a very low Curie point, such as $\text{MnCl}_2 \cdot 4\text{H}_2\text{O}^{(11)}$ whose Curie point is at 1.6°K, $\sin \theta$ may attain a considerable value, say 0.2 or 0.3, under a moderately high field, so that one expects an observable increase in χ_{\perp} , say of a few percent.

§ 5. Discussion

In the preceding sections we have shown that our treatment gives reasonable results for the field-dependent and -independent resonance frequencies, the value of χ_{\perp} at absolute zero, its field dependence, and its temperature dependence. We obtained these results in the first step of the approximation of the spin-wave theory, whereas in the usual theory they appear only in the higher approximation. We shall now make clear the relation between the usual theory and ours. In order to do this, we shall drop the anisotropy term for the sake of simplicity.

Putting $\theta=0$ in (16), (17) and (18), we obtain the spin wave Hamiltonian of the

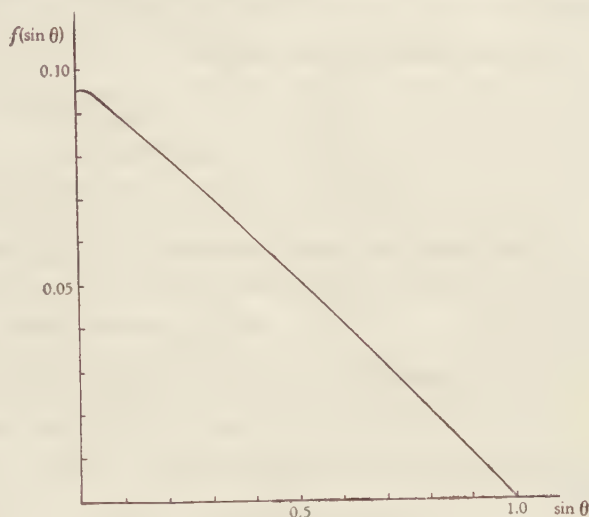


Fig. 2. The curve for $f(\sin \theta)$

usual treatment. Its linear term is

$$(2S)^{1/2} g \mu_B \cdot H \left\{ \sum_j (a_j + a_j^*) + \sum_k (b_k + b_k^*) \right\}, \quad (47)$$

which, after transformations, becomes

$$(NS)^{1/2} g \mu_B H q_{10}. \quad (48)$$

If we shift q_{10} by a further transformation

$$q'_{10} = q_{10} + (NS)^{1/2} g \mu_B H / 4JzS, \quad (49)$$

the linear terms will become eliminated from the Hamiltonian and the field-dependent part of the Hamiltonian will become involved in H_0 in the form

$$-1/2 \cdot \chi_{\perp c} H^2. \quad (50)$$

The unitary transformation S which causes this shift of q_{10} is given by

$$S = \exp \{ i (SN)^{1/2} \cdot g \mu_B H / 4JzS \cdot p_{10} \}. \quad (51)$$

If we go back to the original variables, S will be

$$S = \exp \left[(S/2)^{1/2} g \mu_B H / 4JzS \cdot \left\{ \sum_j (a_j - a_j^*) + \sum_k (b_k - b_k^*) \right\} \right]. \quad (52)$$

This formula means that S causes a uniform shift of the operators a_j , a_j^* , and b_k , b_k^* of an amount

$$\varepsilon = (S/2)^{1/2} g \mu_B H / 4JzS. \quad (53)$$

Now, we attach a prime to the operators for the case in which ξ_j and ξ_k are taken as the axes of quantization. Then, we obtain the following relations between the primed and unprimed operators:

$$\begin{aligned} a'_j &= \exp(i\theta S_{jy}) \cdot a_j \cdot \exp(-i\theta S_{jy}), \\ b'_k &= \exp(-i\theta S_{ky}) \cdot b_k \cdot \exp(i\theta S_{ky}), \end{aligned} \quad (54)$$

with

$$S_{jy} = (2S)^{1/2} / 2i \cdot (f_j a_j - a_j^* f_j), \quad S_{ky} = (2S)^{1/2} / 2i \cdot (b_k^* f_k - b_k f_k). \quad (55)$$

Here $f_j = (1 - a_j^* a_j / 2S)^{1/2}$ and $f_k = (1 - b_k^* b_k / 2S)^{1/2}$. If we replace f_j and f_k by unity, (54) reduces to

$$\begin{aligned} a'_j &= \exp \{ (S/2)^{1/2} \theta (a_j - a_j^*) \} \cdot a_j \cdot \exp \{ - (S/2)^{1/2} \theta (a_j - a_j^*) \}, \\ b'_k &= \exp \{ (S/2)^{1/2} \theta (b_k - b_k^*) \} \cdot b_k \cdot \exp \{ - (S/2)^{1/2} \theta (b_k - b_k^*) \}. \end{aligned} \quad (56)$$

Assuming $\theta = g \mu_B H / 4JzS$, we see that our operators a'_j and b'_k are the operators a_j and b_k shifted by (52).

By applying the transformation S to the usual total Hamiltonian of the form of a power series of a_j , a_j^* , b_k , b_k^* and then retaining linear and quadratic terms of the primed operators, we obtain as a part of these linear and quadratic terms the contributions from

the higher order terms than quadratic in the original usual Hamiltonian. Such linear terms are of the order of H^3 and quadratic terms of the order of H^2 . Kubo's second approximation corresponds to taking these quadratic terms into account. In our treatment, these terms are automatically included in the spin wave Hamiltonian, so that we obtain the same result as Kubo's in the first approximation. Strictly speaking, however, the relations between the spin deviation operators in the present paper and those used by Kubo are not expressible by a mere shift transformation caused by S as (51), since f_j and f_k are not approximated by unity in the transformation given by (54). It is due to this non-linearity of the transformation that our spin wave Hamiltonian, written in terms of the deviation operators, does not contain terms of the higher powers of H than H^2 . Therefore our method can be applied to the case of a high field as well as to the case of a low field, and it gives the correct field-dependence of the thermodynamic properties. The only errors in our treatment come from our limitation to the first term of expansion of the Hamiltonian in powers of $1/S$. To go over to the next approximation, it is necessary to carry out the second order perturbation calculation, not merely the first order one, since the $1/S^2$ term contains both the terms of the third and fourth orders of the spin deviation operators. This can be seen clearly if we write the total Hamiltonian in terms of the deviation operators :

$$\begin{aligned}
 H = & H_0 + 2JzS \left(\sum_j a_j^* a_j + \sum_k b_k^* b_k \right) + Jz \cos 2\theta \sum_{\langle j, l \rangle} (f_j a_j + a_j^* f_j) (b_k^* f_k + f_k b_k) \\
 & + JS \sum_{\langle j, k \rangle} (f_j a_j - a_j^* f_j) (f_k b_k - b_k^* f_k) + J(2S)^{1/2} \sin 2\theta \sum_{\langle j, k \rangle} \{ a_j^* a_j (b_k^* f_k + f_k b_k) \\
 & + b_k^* b_k (f_j a_j + a_j^* f_j) \} - 2J \cos 2\theta \sum_{\langle j, k \rangle} a_j^* a_j b_k^* b_k.
 \end{aligned} \tag{57}$$

Here we used the relation (19) between H and $\sin \theta$.

Through a different approach, Ziman⁴⁾ has come to a conclusion that χ_{\perp} is equal to $\chi_{\perp 0}$ even in the higher approximation. His method is based on the assumption that the operators f_j and f_k can be replaced by their average values calculated in 'a spin wave state'. However, even if we leave the justification of this assumption out of our consideration, we still must be careful enough to notice that the average value of f_j and f_k depends on the field strength, since the average must be taken for the given field strength instead of taking it for vanishing field, as Ziman did, and this will affect χ_{\perp} by an amount of the order of $\chi_{\perp 0}/S$. This does not seem to have been taken into account in his calculation. Moreover, the replacement of f_j and f_k by their average values may cause an error of the order of $\chi_{\perp 0}/S$ in the calculation of χ_{\perp} owing to the non commutability of $a_j^* a_j/S$ with a_j and a_j^* . These errors are just of the same order of magnitude as the deviation of χ_{\perp} from $\chi_{\perp 0}$ calculated by Kubo and in the present paper. Therefore we are led to believe that Kubo's and our result is more reasonable.

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The Self-Energy of the Scalar Nucleon

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The self-energy of the scalar nucleon, with the omission of the effects of the closed loops, interacting with the neutral scalar meson (with vanishing rest mass) field for arbitrary strength of the coupling is computed by the Feynman's variational method extended to the path integrals. It is found that there is no lower limit of the energy eigenvalue even after the logarithmic divergence involved is cut off. However, there does exist a metastable state provided that μ' , a measure of the coupling, is less than about 0.34.

§ 1. Introduction

In the meson theory which has been constructed as an extension of quantum electrodynamics the perturbation approximation has also been adopted in the theory. However, the strength of the coupling inferred from experiments does not seem to be weak enough to justify the use of this method. In view of this the strong coupling approximation has then been introduced. This approximation is not quite satisfactory although some of the experimental findings seem to be accounted for by this method. Recently there has been developed a method which is aimed at treating the case of the intermediate coupling strength. It has also been applied to some problems outside the realm of the meson theory. In most of these approximation methods the recoil of the source is neglected unwillingly. Also in the higher approximations of these methods the calculations involved are practically impossible to perform and we may have infinities and singularities of higher order at the same time.

It seems to be a great disadvantage that we must employ different method according to the strength of the coupling. The most desirable is to invent a method which gives answers for any value of the coupling constant. And it is not impossible to achieve this end when we select particularly simple problem for investigation. In fact, Feynman¹⁾ treated recently the problem of the slow electron in a polar crystal (polaron) by a variational technique extended to path integrals over trajectories for the arbitrary strength of the phonon-electron coupling. He obtained the lowest energy of the system which is as accurate as is known previously. The method of Feynman has, besides the feature of giving answers for arbitrary magnitude of the coupling constant, another advantage in that it avoids the

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objectionable approximation of "neglecting the recoil" of the source particle.

In the present work the self-energy of the scalar nucleon interacting with the neutral scalar meson field is calculated for arbitrary magnitude of the coupling constant by the same variational technique mentioned above. It includes all the effects of the self-action with the exception of those of the closed loops. For the sake of simplicity the rest mass of the meson was assumed to be zero. Although it is necessary to treat nucleons as Dirac particles and to consider, for example, the pseudoscalar symmetric meson theory, such a realistic problem accompanies a greater complexity indicating the necessity of more detailed study of this technique. For this reason the case of the scalar theory is investigated here, in which no further complications pertaining to the operators such as Dirac matrices γ_μ , spin operator σ , and isotopic spin operator τ will occur. This work is meant as a preliminary test of the method to see whether the work needed to extend it to the real problem is justified.

In order to make use of the variational technique it is essential to replace it (t is the physical time) by another variable τ , say. The purpose of this is to arrange that the kernel K will have the asymptotic form $\exp(-ET)$ (for a very large interval T of τ). In this case it is the easiest to determine the lowest energy of the system E . Therefore in Section 2 the scalar nucleon in an external field is treated starting from the equation $-\partial Q/\partial \tau = HQ$ instead of the ordinary one $-\partial Q/\partial t = HQ$.

The Klein-Gordon equation satisfied by the scalar nucleon is not, unfortunately, one of those which permit the direct application of the Lagrangian formulation. Because of this we will introduce a new function of five variables and a first order differential equation satisfied by this function. Then the kernel associated with Klein-Gordon equation can be given in a simple way in terms of that associated with this new function.

In the Lagrangian formalism, instead of the trial wave function in the Hamiltonian scheme, a trial action will be chosen. The kernel associated with the first order equation has the asymptotic form $\exp(-E_0 \Sigma)$ where E_0 is the lowest "energy" corresponding to the "Hamiltonian" of the equation and Σ is a very large interval of the fifth variable. We will estimate the best value of this energy by the variational method. The expression for the energy to be minimized, E , contains a logarithmically divergent integral and it is cut off by the direct analogy to quantum electrodynamics²⁾. The self-energy of the scalar nucleon can be given in terms of E .

It has often been noted that a scalar coupling of third power in the field (such as the $\psi^* \chi \psi$ in this theory) gives no lowest eigenstate due to the fact that the Hamiltonian is not positive definite. This difficulty has nothing to do with cut-off or renormalization procedures. It is found that the present case of the direct coupling scalar meson theory has no solution and no sense because there is no lowest energy corresponding to the Hamiltonian. This is true not only in the case here but also if the meson has a mass or if the nucleons are Dirac particles*.

* The author thanks Professor R. P. Feynman for this information (private communication.)

§ 2. Scalar nucleon in an external field

It is well known that the amplitude for some process in the quantum mechanical field can be obtained using the answer for the same problem in an external field. For this reason we will study in this section the scalar nucleon in a given field in a manner similar to the case of the Dirac electron³⁾ adopting a new fourth variable τ instead of t ($\tau=it$). The answer, for example, the equation of motion and the transition matrix element, etc., is in most cases simply what would result from the conventional expression by replacing t by $-i\tau$, as one expects. Nevertheless it was thought worth while to make sure, at least once, that such a prescription would give the correct result by following necessary steps in detail.

The Hamiltonian for a charged nucleon in an external field is given by ($\hbar=c=1$)

$$H(t) = \int \{ \pi^*(\mathbf{x}') \pi(\mathbf{x}') + \mathbf{p} \psi^*(\mathbf{x}') \cdot \mathbf{p} \psi(\mathbf{x}') + (m^2 + \sqrt{4\pi} g \chi(\mathbf{x}', t)) \psi^*(\mathbf{x}') \psi(\mathbf{x}') \} d\mathbf{x}'$$

with the commutation relations

$$[\pi(\mathbf{x}, t), \psi(\mathbf{x}', t)] = [\pi^*(\mathbf{x}, t), \psi^*(\mathbf{x}', t)] = -i\delta(\mathbf{x} - \mathbf{x}').$$

Here ψ , π , and m are the operator wave function, its canonically conjugate momentum, and the rest mass of the scalar nucleon, respectively, and χ is the given potential. The case of the neutral nucleon is obvious.

The equation we will be interested in is

$$-\partial\Omega/\partial\tau = H\Omega$$

instead of the Schrödinger equation for the state vector Ω describing the motion of the system. It will prove convenient to state at this stage some relations between corresponding quantities in both formalisms in which t or τ is the independent variable (t - or τ - formalism, respectively). If a function of \mathbf{x} and t is of the form $f(\mathbf{x}, t) = F(\mathbf{x}, it)$ we write this $\tilde{f}(\mathbf{x}, \tau)$. In the following the tilde will often be omitted wherever it is apparent that there will be no confusion by doing so. Similarly $f^*(\mathbf{x}, t) = F^*(\mathbf{x}, -it) = \tilde{f}^*(\mathbf{x}, -\tau)$ and we shall call this $\tilde{f}^\dagger(\mathbf{x}, t)$.

For the time derivatives we have

$$\partial f(\mathbf{x}, t)/\partial t = i \partial \tilde{f}(\mathbf{x}, \tau)/\partial \tau, \quad (2.1)$$

$$\partial f^*(\mathbf{x}, t)/\partial t = i \partial \tilde{f}^\dagger(\mathbf{x}, \tau)/\partial \tau. \quad (2.2)$$

The theory of scalar particles was developed by Pauli and Weisskopf⁴⁾, who showed that the oppositely charged particles in this theory can be interpreted as corresponding to the electron and the positron of Dirac's hole theory. Let us call arbitrarily the negatively charged particle "the electron" and the positively charged one "the positron," respectively. We attempt to calculate the transition element of a process in which there is only one "electron" with wave function $g(\mathbf{x})$ at $\tau=T$ if there was only one "electron" with wave function $f(\mathbf{x})$ at $\tau=0$. If the wave functionals representing the initial and the final states are Ω_i and Ω_f the transition element is given by

$$r = (\mathcal{Q}_f^\dagger \exp(-\int_0^T H d\tau) \mathcal{Q}_i) = (\mathcal{Q}_0^\dagger G S F^\dagger \mathcal{Q}_0), \quad (2.3)$$

where $\mathcal{Q}_i = F^\dagger \mathcal{Q}_0$ and $\mathcal{Q}_f = G^\dagger \mathcal{Q}_0$ in terms of the state vector \mathcal{Q}_0 of the vacuum and $S = \exp(-\int_0^T H d\tau)$. To evaluate r we need F^\dagger and G . In finding the expression for F^* we refer to the work of Pauli and Weisskopf who give the creation operator a_k^* of an "electron" as

$$a_k^* = (V)^{-1/2} \{ (2\omega_k)^{-1/2} \{ \pi(\mathbf{x}) + i\omega_k \phi^*(\mathbf{x}) \} \exp(i\mathbf{k}\mathbf{x}) d\mathbf{x},$$

where $\omega_k = \sqrt{m^2 + \mathbf{k}^2}$. Since the wave function $f(\mathbf{x})$ of a free "electron" with energy ω_k varies as $\exp(-i\omega_k t)$ we may regard the coefficient of $\phi^*(\mathbf{x})$, $i\omega_k$, as coming from $-\partial f(\mathbf{x})/\partial t$. Thus making use of the Fourier expansion of a_k^* given above we obtain

$$F^* = \{ \{ \pi(\mathbf{x}_1) f(\mathbf{x}_1) - \phi^*(\mathbf{x}_1) \partial f(\mathbf{x}_1) / \partial t_1 \} d\mathbf{x}_1$$

leaving the factor $(2\omega_k)^{-1/2}$ which can be taken care of as the normalization constant (Cf. footnote of this section). G is given by taking the complex conjugate of G^* .

In τ -formalism the use of (2.1) and (2.2) gives

$$F^\dagger = \{ \{ \pi(\mathbf{x}_1) f(\mathbf{x}_1) - \phi^\dagger(\mathbf{x}_1) i \partial f(\mathbf{x}_1) / \partial \tau_1 \} d\mathbf{x}_1 \quad (2.4)$$

and

$$G = \{ \{ \pi^\dagger(\mathbf{x}_2) g^\dagger(\mathbf{x}_2) - \phi(\mathbf{x}_2) i \partial g^\dagger(\mathbf{x}_2) / \partial \tau_2 \} d\mathbf{x}_2. \quad (2.5)$$

From (2.4) we have

$$\exp(-\int_0^T H d\tau) F^\dagger \exp(+\int_0^T H d\tau) = \{ \{ \pi(\mathbf{x}_1) f(\mathbf{x}_1, \tau) - i\phi^\dagger(\mathbf{x}_1) h(\mathbf{x}_1, \tau) \} d\mathbf{x}_1, \quad (2.6)$$

which defines $f(\mathbf{x}, \tau)$ and $h(\mathbf{x}, \tau)$. Equation (2.6) in turn gives

$$F^\dagger = \{ \{ \pi(\mathbf{x}_1, \tau) f(\mathbf{x}_1, \tau) - i\phi^\dagger(\mathbf{x}_1, \tau) h(\mathbf{x}_1, \tau) \} d\mathbf{x}_1, \quad (2.7)$$

where

$$A(\mathbf{x}, \tau) = \exp(+\int_0^T H d\tau) A(\mathbf{x}) \exp(-\int_0^T H d\tau) \text{ for } A = \phi^\dagger \text{ and } \pi.$$

The first derivative gives

$$\partial \phi^\dagger(\mathbf{x}, \tau) / \partial \tau = -i\pi(\mathbf{x}, \tau); \quad \partial \pi(\mathbf{x}, \tau) / \partial \tau = i\Gamma(\mathbf{x}, \tau) \phi^\dagger(\mathbf{x}, \tau) \quad (2.8)$$

with

$$\Gamma(\mathbf{x}, \tau) = -\mathbf{p}^2 + m^2 + \sqrt{4\pi} g\chi(\mathbf{x}, \tau),$$

while the second gives, when the expression for Γ is used explicitly,

$$\{ \square_\tau^2 - m^2 - \sqrt{4\pi} g\chi(\mathbf{x}, \tau) \} A(\mathbf{x}, \tau) = 0 \quad (2.9)$$

for $A = \phi^\dagger$ and π . Equation (2.9) differs from the conventional Klein-Gordon equation in that \square^2 -operator is replaced by an operator $\square_\tau^2 = \mathbf{p}^2 + \partial^2/\partial \tau^2$ which can formally be obtained from \square^2 by the direct substitution $t = -i\tau$.

Since F^\dagger given by (2.4) is independent of τ this must also be the case for (2.7). The differentiation of the latter and the use of (2.8) yields

$$h(\mathbf{x}, \tau) = \partial f(\mathbf{x}, \tau) / \partial \tau; \quad \partial h(\mathbf{x}, \tau) / \partial \tau = \Gamma(\mathbf{x}, \tau) f(\mathbf{x}, \tau). \quad (2.10 a, b)$$

Combining these two we see that $f(\mathbf{x}, \tau)$ also satisfies the equation (2.9). Substituting (2.10a) into (2.6) and letting $\tau = T$ we have

$$SF^\dagger S^{-1} = F^\dagger, \quad (2.11)$$

where

$$F^\dagger = \int \{ \pi(\mathbf{x}_1) f(\mathbf{x}_1, T) - i \psi^\dagger(\mathbf{x}_1) \partial f(\mathbf{x}_1, T) / \partial T \} d\mathbf{x}_1.$$

From (2.5) (with $\tau_2 = T$) and this expression for F^\dagger we obtain

$$[G, F^\dagger] = - \int \{ g^\dagger(\mathbf{x}_2, T) \partial f(\mathbf{x}_2, T) / \partial T - \partial g^\dagger(\mathbf{x}_2, T) / \partial T \cdot f(\mathbf{x}_2, T) \} d\mathbf{x}_2$$

and the use of (2.11) reduces r to

$$r = - \int \{ g^\dagger(2) \partial f(2) / \partial \tau_2 - \partial g^\dagger(2) / \partial \tau_2 \cdot f(2) \} d\mathbf{x}_2 \cdot C_v + (\Omega_0^\dagger F^\dagger G S \Omega_0), \quad (2.12)$$

where $C_v = (\Omega_0^\dagger S \Omega_0)$ is the amplitude for having a vacuum at $\tau = \tau_2 = T$ if we had one at $\tau = 0$. That the second term of this expression may be dropped can be seen from the discussion in Reference 3.

In the following we will be concerned only with the factor by which C_v is multiplied to give r when the particle is in an external potential.* Consequently the self-energy of the nucleon that will be obtained in the remainder of this work does not include the self-action effects coming from the closed loops.

It is interesting to see the contribution to r of various orders in the perturbation method though we are not going to use it in the following. When the nucleon is in a given potential the wave function at point 2 is given by

$$f(2) = - \int \left\{ \frac{\partial \Xi^{(x)}(2, 1)}{\partial \tau_1} f(1) - \Xi^{(x)}(2, 1) \frac{\partial f(1)}{\partial \tau_1} \right\} d\mathbf{x}_1 \\ + \int \left\{ \frac{\partial \Xi^{(x)}(2, 3)}{\partial \tau_3} f(3) - \Xi^{(x)}(2, 3) \frac{\partial f(3)}{\partial \tau_3} \right\} d\mathbf{x}_3 \quad (2.13)$$

in terms of the kernel $\Xi^{(x)}$ associated with the equation of motion of this particle. Here $\tau_1 < \tau_2 < \tau_3$ and the first and the second terms represent the contributions from the "electron" and the "positron" components, respectively. The solution of the equation

$$\{ \square_{\tau_2}^2 - m^2 - \sqrt{4\pi} g \chi(2) \} \Xi^{(x)}(2, 1) = \delta^4(2, 1)$$

may be expressed as

$$\Xi^{(x)}(2, 1) = \Xi^{(0)}(2, 1) + \int \Xi^{(0)}(2, 3) \sqrt{4\pi} g \chi(3) \Xi^{(0)}(3, 1) d^4\omega_3 \\ + \int \int \Xi^{(0)}(2, 4) \sqrt{4\pi} g \chi(4) \Xi^{(0)}(4, 3) \sqrt{4\pi} g \chi(3) \Xi^{(0)}(3, 1) d^4\omega_4 d^4\omega_3 + \dots,$$

where $\Xi^{(0)}$ is the kernel associated with the free particle.

* In this case the normalization integral becomes

$$- \int \{ f^\dagger \partial f / \partial \tau - \partial f^\dagger / \partial \tau \cdot f \} dx.$$

If the amplitude of the wave function is normalized to 1 so that $f = \exp(i\mathbf{x}\mathbf{k} - \omega_k\tau)$ then the integral above becomes $2\omega_k$ (when it is normalized to unit volume). This is the reason why we have to divide the transition probability per second by $2\omega_k$.⁵⁾

The transition matrix element from the initial state characterized by $f(1)$ and $\partial f(1)/\partial \tau_1$ of an "electron" to the final state with $g(2)$ and $\partial g(2)/\partial \tau_2$ of another "electron" is given by the first term of (2.12). Substituting $f(2)$ and $\partial f(2)/\partial \tau_2$ obtained from the first term of (2.13) r is expanded in powers of χ :

$$r = r^{(0)} + r^{(1)} + r^{(2)} + \dots$$

$r^{(1)}$ is given by

$$r^{(1)} = - \int g^\dagger(3) \sqrt{4\pi} g\chi(3) f(3) d^4\omega_3$$

and similarly for $r^{(2)}$:

$$r^{(2)} = - \iint g^\dagger(4) \sqrt{4\pi} g\chi(4) \Xi^{(0)}(4,3) \sqrt{4\pi} g\chi(3) f(3) d^4\omega_4 d^4\omega_3.$$

Each term $r^{(i)}$ permits a simple interpretation which is exactly analogous to the case of a Dirac particle. That is, the scattering amplitude is $\sqrt{4\pi} g\chi$ and the particle travels free between the scattering potentials with the propagator $\Xi^{(0)}$. That there appears the function $\Xi^{(0)}$ in $r^{(i)}$ shows that the propagator in the momentum representation is, in the t -formalism, given by $(k_4^2 - \mathbf{k}^2 - m^2)^{-1}$.

§ 3. Elimination of the field operator in the quantum mechanical case

The Lagrangian form of the scalar particle in a given external field χ will first be discussed in a manner similar to that treated by Feynman⁶⁾. To this end we write (2.9) in the following form:

$$\{\square_\tau^2 - \sqrt{4\pi} g\chi(x, \tau)\} \psi(x, \tau) = m^2 \psi(x, \tau). \quad (3.1)$$

In order to cast this, though indirectly, into the Lagrangian form we consider the space-time coordinates $x_\mu = (x_1 = \tau, \mathbf{x}) = x$ as functions of a parameter σ and introduce a first order differential equation

$$-\partial \phi(x, \sigma) / \partial \sigma = -1/2 \cdot \{\square_\tau^2 - \sqrt{4\pi} g\chi\} \phi(x, \sigma) \quad (3.2)$$

for a new function $\phi(x, \sigma)$. This equation is chosen in such a way that it allows us to use the variational technique at the same time. Equation (3.2) has the form of the Schrödinger equation in which t is replaced by σ and the spatial variables extended to the four-dimensional variables x_μ . It will be shown in Section 6 that if $\phi(x, \sigma)$ is any solution of (3.2) then the solution of (3.1) can be given in terms of it by

$$\psi(x) = \int_{-\infty}^{\infty} \exp(-m^2\sigma/2) \phi(x, \sigma) d\sigma. \quad (3.3)$$

This expression may be made convergent by taking $\sigma' = 0$ in the kernel $K(x, \sigma; x', \sigma')$ of (3.2) and choosing K to vanish for $\sigma < 0$ by definition. However, as far as we confine

ourselves to the study of the kernel K associated with ϕ we can assume any value for σ' and σ . Only in Section 6 where the relation between ϕ and ψ comes into play need we be careful about the finiteness of ϕ .

The kernel of (3.2) is given by⁷⁾

$$K^{(x)}(x, \sigma; x', \sigma') = \int \exp(S^{(x)}) \mathcal{D}x_\mu(\sigma) \quad (3.4)$$

with

$$S^{(x)} = \int_{\sigma'}^{\sigma} \{ -1/2 \cdot (dx_\mu/d\sigma)^2 - 1/2 \cdot \sqrt{4\pi} g \chi \} d\sigma. \quad (3.5)$$

We now proceed to find the kernel corresponding to the transition in which there are no mesons present in the initial and the final states when the nucleon is in quantum mechanical interaction with the meson field. This means that we have to integrate out the meson field operator χ in the coupling term of (3.5). Writing

$$\exp(-1/2 \cdot \sqrt{4\pi} g \int \chi(x_\mu(\sigma)) d\sigma) = \exp(-\int j(1) \chi(1) d^4\omega_1),$$

where

$$j(1) = 1/2 \cdot \sqrt{4\pi} g \int \delta^4(x_\mu(1) - x_\mu(\sigma)) d\sigma,$$

the direct analogy to quantum electrodynamics⁸⁾ gives the desired matrix element as

$$K_{00}(x, \sigma; x', \sigma') = \int \exp[-1/2 \cdot \int \dot{x}_\mu^2(\sigma) d\sigma + g^2/8\pi \cdot \int \{ \{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2} d\sigma' d\sigma'' \}] \mathcal{D}x_\mu(\sigma). \quad (3.6)$$

The evaluation of this kernel (or at least its asymptotic form) represents the central part of our problem. Such a path integral is ordinarily very difficult to evaluate. However, in the next section we will do it by an approximation method whose validity does not require that g^2 be particularly small or particularly large.

§ 4. Evaluation of path integrals

In this section the variational technique will be applied on the path integral obtained in the preceding section. This part of the work keeps a very close parallelism with the three-dimensional case of the polaron problem.¹⁾

In this technique an appropriate trial action S_1 is chosen to imitate the true action S . In case $S - S_1$ is a local function

$$\langle S - S_1 \rangle = \int (S - S_1) \exp(S_1) \mathcal{D}x_\mu(\sigma) / \int \exp(S_1) \mathcal{D}x_\mu(\sigma)$$

is proportional to Σ , a very large interval of σ , and we shall define s by

$$\langle S - S_1 \rangle = s \Sigma.$$

In terms of s and the lowest energy E_1 corresponding to S_1 the best estimate of the lowest energy of the system is found by minimizing

$$E = E_1 - s. \quad (4.1)$$

It is to be noted that the present technique is not restricted to S of the form $\{Ld\tau$ (L : the Lagrangian) but is of much wider generality. For the case of a particle in a potential the new method can be shown to be equivalent to the conventional one (Cf. Appendix A). For more complicated situations, however, the relation between these two methods is not known. Thus, it is not clear for a given trial action S_1 what trial function, if any, in the conventional method leads to the relation given by (4.1).

To save writing we omit the subscripts 00 of (3.6) in the following. Thus

$$K(x, \sigma; x', \sigma') = \int \exp(S) \mathcal{D}x_\mu(\sigma),$$

where

$$S = -1/2 \cdot \int \dot{x}_\mu^2(\sigma) d\sigma + g^2/8\pi \cdot \iint \{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2} d\sigma' d\sigma''. \quad (4.2)$$

As to the trial action we borrow it from the polaron problem with the direct extension to the four dimensions:

$$S_1 = -1/2 \cdot \int \dot{x}_\mu^2(\sigma) d\sigma - C \iint \{x_\mu(\sigma') - x_\mu(\sigma'')\}^2 \exp(-\alpha|\sigma' - \sigma''|) d\sigma' d\sigma'', \quad (4.3)$$

where α and C are parameters. The apparent lack of resemblance of the second terms of S and S_1 may not be objected to seriously. In fact for the polaron problem the approximation of $|\mathbf{x}(\sigma') - \mathbf{x}(\sigma'')|^{-1} \exp(-|\sigma' - \sigma''|)$ by $C\{\mathbf{x}(\sigma') - \mathbf{x}(\sigma'')\}^2 \exp(-\alpha|\sigma' - \sigma''|)$ gave a very satisfactory result. That S_1 be quadratic in x_μ is a very convenient choice because for such a functional S_1 the path integrals can most easily be worked out.⁹⁾ Moreover this form of the second term of S_1 gives a good approximation to the corresponding term of S . For first we note in the second term of S that only small values of $x_\mu(\sigma') - x_\mu(\sigma'')$ are important. When $-\{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2}$ is cut off near the origin it can well be approximated, for small $x_\mu(\sigma') - x_\mu(\sigma'')$, by $C\{x_\mu(\sigma') - x_\mu(\sigma'')\}^2 \exp(-\alpha|\sigma' - \sigma''|)$ provided that the parameters α and C are suitably chosen. The factor $\exp(-\alpha|\sigma' - \sigma''|)$ assures that $x_\mu(\sigma)$'s corresponding to σ 's distant apart to each other do not contribute to the integral. We further observe that a constant added to S_1 does not give any improvement because it will be included in both E_1 and s so that it disappears from E . The range of σ will conveniently be taken from $-\Sigma/2$ to $+\Sigma/2$. Since, however, we eventually consider the limit when Σ tends to infinity the limits of the integrations over σ can be approximated by $-\infty$ and $+\infty$.

We begin with s which is given from (4.2) and (4.3) by

$$\begin{aligned} s &= \langle S - S_1 \rangle / \Sigma \\ &= g^2/8\pi \cdot \int d\sigma'' \langle \{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2} \rangle \\ &\quad + C \int d\sigma'' \langle \{x_\mu(\sigma') - x_\mu(\sigma'')\}^2 \rangle \exp(-\alpha|\sigma' - \sigma''|). \end{aligned} \quad (4.4)$$

We want to evaluate s for the boundary conditions

$$x(-\Sigma/2) = 0, \quad x(+\Sigma/2) = X$$

corresponding to the actual motion of the particle in space-time. To study the first term of (4.4) we write $\{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2}$ as a Fourier integral:

$$\{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2} = (2\pi)^{-2} \{d^4k (k_\mu^2)^{-1} \exp[ik_\mu \{x_\mu(\sigma') - x_\mu(\sigma'')\}]\}.$$

By doing this we see that we only need to evaluate

$$\begin{aligned} & \langle \exp[ik_\mu \{x_\mu(\sigma') - x_\mu(\sigma'')\}] \rangle \\ &= \{ \exp[ik_\mu \{x_\mu(\sigma') - x_\mu(\sigma'')\}] \exp(S_1) \mathcal{D}x_\mu(\sigma) / \exp(S_1) \mathcal{D}x_\mu(\sigma) \}. \end{aligned} \quad (4.5)$$

The result is found to be

$$\exp \{ -F(\sigma) k_\mu^2 + ik_\mu v_\mu \sigma \}. \quad (4.6)$$

Here

$$\begin{aligned} F(\sigma) &= (2\beta^2)^{-1} [\alpha^2 \sigma + (\beta^2 - \alpha^2) / \beta \cdot (1 - \exp(-\beta \sigma))], \\ \beta^2 &= \alpha^2 + 8C/\alpha, \end{aligned} \quad (4.7)$$

and

$$v_\mu = X_\mu / \Sigma.$$

The evaluation of the second term of (4.4) gives

$$(\beta^2 - \alpha^2) \{ \beta^{-1} + v_\mu^2 / 2\alpha^2 \}. \quad (4.8)$$

Finally E_1 is given by

$$E_1 = 2(\beta - \alpha) + (v_\mu^2 / 2) (1 + 8C/\alpha^3). \quad (4.9)$$

From (4.6), (4.8), and (4.9) we obtain

$$E = (\beta - \alpha)^2 / \beta + v_\mu^2 / 2 - (g^2 / 16\pi^3) \int_0^\infty d\sigma \{ d^4k (k_\mu^2)^{-1} \{ -F(\sigma) k_\mu^2 + ik_\mu v_\mu \sigma \} \}. \quad (4.10)$$

§ 5. Removal of divergence by cut-off

The integral involved in (4.10) is logarithmically divergent. We will remove the divergence by the analogue of a procedure applied to quantum electrodynamics¹⁰⁾. In other words we replace $1/k_\mu^2$ in (4.10) by $1/k_\mu^2 - 1/(k_\mu^2 + \Lambda^2)$, where Λ , the cut-off, is taken to be very large as compared to unity and $C(k_\mu^2) = \Lambda^2 / (k_\mu^2 + \Lambda^2)$ may be called the convergence factor.

Using the representation $1/k_\mu^2 = \int_0^\infty \exp(-\gamma k_\mu^2) d\gamma$ the integral is written as

$$\int_0^\infty d\sigma \int_0^\infty d\gamma \int d^4k \exp[-(\gamma + F(\sigma)) k_\mu^2 + ik_\mu v_\mu \sigma].$$

First performing the integration over k and then over γ we have

$$\begin{aligned} & \int_0^\infty d\sigma \int_0^\infty d\gamma (\pi^2 / (\gamma + F(\sigma))^2) \exp \{ -v_\mu^2 \sigma^2 / 4 (\gamma + F(\sigma)) \} \\ &= 4\pi^2 \int_0^\infty d\sigma \{ 1 - \exp(-v_\mu^2 \sigma^2 / 4 F(\sigma)) \} (v_\mu^2 \sigma^2)^{-1}. \end{aligned} \quad (5.1)$$

When $v_\mu^2 \rightarrow 0$ it can be seen that the divergence comes from $\sigma=0$. Since

$$C(k_\mu^2)/k_\mu^2 = \int_0^\infty d\gamma (1 - \exp(-\gamma A^2)) \exp(-\gamma k_\mu^2)$$

the replacement of $1/k_\mu^2$ by $C(k_\mu^2)/k_\mu^2$ in the integral of (4.10) gives

$$\pi^2 \int_0^\infty d\sigma \int_0^\infty d\gamma (1 - \exp(-\gamma A^2)) (\gamma + F(\sigma))^{-2} \exp\{-v_\mu^2 \sigma^2 / 4(\gamma + F(\sigma))\}$$

which we shall call

$$= \pi^2 J.$$

Thus we have ($v^2 = v_\mu^2$, $F = F(\sigma)$)

$$E = (\beta - \alpha)^2 / \beta + v^2 / 2 - g^2 J / 16\pi. \quad (5.2)$$

In order to evaluate E the integral J must now be performed. We will do this in the following for the limiting case where $A^2 \gg 1$. Noticing that when $\sigma \rightarrow 0$ (where the divergence in J comes)

$$F(\sigma) = (2\beta^2)^{-1} [\alpha^2 \sigma + (\beta^2 - \alpha^2) / \beta \cdot (1 - \exp(-\beta\sigma))] \rightarrow \sigma / 2 \quad (5.3)$$

we write J in the following form:

$$\begin{aligned} J = & \iint d\sigma d\gamma [(\gamma + F)^{-2} \exp(-v^2 \sigma^2 / 4(\gamma + F)) - (\gamma + \sigma / 2)^{-2} \exp(-v^2 \sigma^2 / 4(\gamma + \sigma / 2))] \\ & \times (1 - \exp(-\gamma A^2)) \\ & + \iint d\sigma d\gamma (\gamma + \sigma / 2)^{-2} (1 - \exp(-\gamma A^2)) \exp(-v^2 \sigma^2 / 4(\gamma + \sigma / 2)) = J_1 + J_2. \end{aligned} \quad (5.4)$$

This is still exact. The first term J_1 is now convergent even without the factor $(1 - \exp(-\gamma A^2))$ in view of (5.3) so that we can drop this factor, whereas in J_2 we have to still keep it.

After integrating over γ J_1 gives

$$J_1 = \int_0^\infty 4d\sigma [\exp(-v^2 \sigma^2 / 2) - \exp(-v^2 \sigma^2 / 4F)] (v^2 \sigma^2)^{-1}.$$

By changing the variable by

$$v^2 \sigma = \xi$$

and calling

$$\alpha = av^2, \quad \beta = bv^2$$

J_1 results in

$$J_1 = \int_0^\infty 4d\xi [\exp(-\xi^2 / 2) - \exp\{- (\xi^2 b^2 / 2) / [a^2 \xi^2 + (b^2 - a^2) / b \cdot (1 - \exp(-b\xi^2))]\}] / \xi^2. \quad (5.5)$$

It is to be noted that this expression for J_1 is independent of v . The limiting value of J_2 for very large A^2 will be shown (Appendix B) to be just

$$J_2 = 2 \ln(A^2 e / v^2) \quad (5.6)$$

and hence

$$E(a, b, v^2) = (b-a)^2 v^2 / b + v^2 / 2 - (g^2 / 16\pi) \{J_1(a, b) + 2 \ln(\Lambda^2 e / v^2)\}. \quad (5.7)$$

In (5.6) and (5.7) e stands for the base of the natural logarithm. In the next section we will consider the self-energy of the scalar nucleon resulting from this expression.

§ 6. Self-energy of the scalar nucleon

The self-energy of the scalar nucleon can be obtained from the asymptotic form of its kernel \mathcal{K} corresponding to the transition in which there are no mesons present in the initial and the final states. In the following we will show that the kernel \mathcal{K} is so closely related to K that the self-energy of the scalar nucleon can be given in terms of E given by (5.7).

To see the connection between \mathcal{K} and K let us consider the case of a nucleon in an external potential. Throughout this section the range of σ will conveniently be taken from 0 to $+\infty$. The kernel $K^{(x)}$ of (3.2) satisfies then

$$\phi(x, \sigma) = \int K^{(x)}(x, \sigma; x', 0) \phi(x', 0) d^4\omega_{x'}. \quad (6.1)$$

Now consider a function $\psi(x)$ defined by

$$\psi(x) = \int_0^\infty \exp(-m^2\sigma/2) \phi(x, \sigma) d\sigma.$$

From this we obtain

$$\begin{aligned} (\square_x^2 - \sqrt{4\pi} g\chi) \psi(x) &= \int_0^\infty \exp(-m^2\sigma/2) (\square_x^2 - \sqrt{4\pi} g\chi) \phi(x, \sigma) d\sigma \\ &= -2\phi(x, 0) + m^2\psi(x). \end{aligned}$$

Hence ψ obeys the following differential equation:

$$\{\square_x^2 - m^2 - \sqrt{4\pi} g\chi\} \psi(x) = -2\phi(x, 0). \quad (6.2)$$

On the other hand, the kernel $\Xi^{(x)}$ satisfies

$$\{\square_x^2 - m^2 - \sqrt{4\pi} g\chi\} \Xi^{(x)}(x, x') = \delta^4(x - x'). \quad (6.3)$$

From (6.2) and (6.3) it is seen that $\Xi^{(x)}(x, x')$ is given by

$$\Xi^{(x)}(x, x') = \int_0^\infty \exp(-m^2\sigma/2) \phi(x, \sigma) d\sigma \quad (6.4)$$

in which

$$-2\phi(x, 0) = \partial^4(x - x'). \quad (6.5)$$

Substituting (6.5) into (6.1) we have (6.4) as

$$\mathcal{K}^{(x)}(x; x') = \int_0^\infty \exp(-m^2\sigma/2) K^{(x)}(x, \sigma; x', 0) d\sigma, \quad (6.6)$$

where we put

$$-2\Xi(x, x') = \mathcal{K}^{(x)}(x; x').$$

When we calculate the kernel \mathcal{K}_{00} corresponding to the 0-0 transition we again encounter the problem of eliminating the meson field operator as was done in obtaining (3.6) from (3.4). From (3.4) and (3.5), (6.6) is given by

$$\begin{aligned} \mathcal{K}^{(x)}(x; x') = & \int_0^\infty \exp(-m^2\sigma/2) d\sigma \int \exp[-1/2 \cdot \dot{x}_\mu^2(\sigma) d\sigma \\ & - 1/2 \cdot \sqrt{4\pi} g \{ \chi(x_\mu(\sigma)) d\sigma \}] \mathcal{D}x_\mu(\sigma). \end{aligned}$$

Hence we have

$$\begin{aligned} \mathcal{K}_{00}(x; x') = & \int_0^\infty \exp(-m^2\sigma/2) d\sigma \int \exp[-1/2 \cdot \dot{x}_\mu^2(\sigma) d\sigma \\ & + g^2/8\pi \cdot \int \int \{x_\mu(\sigma') - x_\mu(\sigma'')\}^{-2} d\sigma' d\sigma''] \mathcal{D}x_\mu(\sigma). \\ = & \int_0^\infty \exp(-m^2\sigma/2) K_{00}(x, \sigma; x', 0) d\sigma. \end{aligned}$$

Omitting the subscripts 00 following the convention in the foregoing sections and considering the case where $x'(0) = 0$ and $x(\Sigma) = X$ we have

$$\mathcal{K}(X; 0) = \int_0^\infty (-m^2\Sigma/2) K(X, \Sigma; 0, 0) d\Sigma. \quad (6.7)$$

For a free particle $K(X, \Sigma; 0, 0)$ has the asymptotic form $\exp(-E\Sigma)$ where $E = v_\mu^2/2 = X_\mu^2/2\Sigma^2$ so that the main contribution to the integral comes from the vicinity of $\Sigma = X/m$ ($X^2 = X_\mu^2$). It is seen, therefore, that in evaluating the asymptotic form of $\mathcal{K}(X; 0)$, where $X_1 = T$ is sufficiently large, only the contribution from a large value of Σ is important. In dealing with a particle which is not free the trajectory in the five-dimensional world may be approximated by that of a free particle having the same end points in this world. Therefore it will also be expected that the contribution comes mainly from a large value of Σ . This implies that in evaluating the asymptotic form of $\mathcal{K}(X; 0)$ only the asymptotic, and not the exact, form of $K(X, \Sigma; 0, 0)$ is required.

We will now show in general how the best value of the self-energy can be found from (6.7). First we recall that what has been done in the previous sections is to calculate $\exp(-E\Sigma)$ which is to be minimized to give the best estimate of $K \sim \exp(-E_0\Sigma)$. Consequently the best value M of the self-energy \mathcal{E}_0 of $\mathcal{K} \sim \exp(-\mathcal{E}_0 T)$ can be found by minimizing E with respect to the parameters available and then having $\int_0^\infty \exp(-m^2\Sigma/2) \exp(-E\Sigma) d\Sigma$ in the form $\exp(-MT)$. In general E is a function of $v^2 = v_\mu^2$ and, therefore, it has a dependence on Σ through $X_\mu = v_\mu \Sigma$. For simplicity let us assume that only one parameter a is contained in E and consider

$$\int_0^\infty \exp[-(m^2/2 + E(a, v^2))\Sigma] d\Sigma = \int_0^\infty \exp[-(m^2 + W(a, v^2))\Sigma/2] d\Sigma, \quad (6.8)$$

where $W=2E$. We are interested in the exponent of (6.8)

$$[m^2 + W(a, v^2)]\Sigma'/2. \quad (6.9)$$

By varying a we have

$$W_a(a_0, X^2/\Sigma^2) = 0 \quad (6.10)$$

determining the optimum value of a in terms of X^2/Σ^2 :

$$a_0 = a_0(X^2/\Sigma^2).$$

Substitution of this in (6.9) yields

$$[m^2 + W(a_0(X^2/\Sigma^2), X^2/\Sigma^2)]\Sigma/2 = [m^2 + Y(X^2/\Sigma^2)]\Sigma/2, \quad (6.11)$$

where Y is defined by

$$Y(v^2) = W(a_0(v^2), v^2). \quad (6.12)$$

We then calculate (6.8), with its exponent now considered given by (6.11), by the method of steepest descent. Thus by varying Σ' in (6.11) we have

$$m^2 + Y(v_0^2) = 2Y'(v_0^2)v_0^2, \quad (6.13)$$

where Σ_0 and v_0 is related through $\Sigma_0 = X/v_0$. Substituting Σ_0 in (6.11) and by choosing X_μ such that $X=0$ and $X_4=T$ it gives

$$[m^2 + Y(v_0^2)]T/2v_0.$$

Then, since we now have

$$\int_0^\infty \exp[-(m^2 + W(a, v^2))\Sigma/2] d\Sigma \sim \exp[-(m^2 + Y(v_0^2))T/2v_0],$$

we can identify M as

$$M = [m^2 + Y(v_0^2)]/2v_0. \quad (6.14)$$

Noting that (6.12) gives

$$Y'(v^2) = a_0'(v^2)W_a(a_0(v^2), v^2) + W_{,2}(a_0(v^2), v^2),$$

in which the first term vanishes by (6.10), (6.13) and (6.10) may be incorporated to state (6.14) as

$$M = \text{Minimum of } \mathcal{E}(a, v^2)$$

with respect to v and a , where

$$\mathcal{E}(a, v^2) = [m^2 + W(a, v^2)]/2v. \quad (6.15)$$

The procedure for the case of several parameters is obvious.

Applying this general argument to the present problem let us consider (6.8) using the expression for E given by (5.7). Calling

$$m'^2 = m^2 - g^2/4\pi \cdot \ln(A^2 e/v^2) \quad (6.16)$$

and

$$\mu = g^2 / 8\pi m'^2 \quad (6 \cdot 17)$$

we have

$$\int_0^\infty \exp[-m'^2(1-\mu J_1)\Sigma/2] \exp[-\{2(b-a)^2/b+1\}X^2/2\Sigma] d\Sigma. \quad (6 \cdot 18)$$

We notice that the prescription for finding M given in the preceding paragraph is based on the assumption that the integral (6.18) is convergent. In case $m'^2(1-\mu J_1) \leq 0$, however, our integral does not converge. To examine when this is the case we first observe that J_1 is a small positive number for $b \approx a$ whereas it diverges as $2\sqrt{2\pi b}$ for $b \gg a$ (Cf. Appendix C). Hence for very large g^2 we see that $m'^2(1-\mu J_1) < 0$. On the other hand for reasonably small values of g^2 , though $m'^2(1-\mu J_1) < 0$ for $b \gg a$ and $b \gg 1$, we may have $m'^2(1-\mu J_1) > 0$ for $b \approx a$ yielding a finite result for (6.18). That is, (6.18) gives divergent results either for small g^2 when $b \gg a$ (and $b \gg 1$) or for large g^2 but may give a finite answer for small g^2 when $b \approx a$. The former corresponds to the scalar nucleon's having negative self-energy (and in fact it can be made $-\infty$ by letting $b \rightarrow \infty$ keeping $b \gg a$) for any value of g^2 , and the latter gives the self-energy of the metastable state which can exist when the coupling is reasonably weak. We see, therefore, that the scalar theory gives no solution for the self-energy of the scalar nucleon. In the following we will compute the self-energy of the metastable state. It is given as the minimum of

$$\mathcal{E}(a, b, v^2) = [m^2 + v^2 \{2(b-a)^2/b+1\} - g^2 J/8\pi]/2v.$$

Since A^2 is assumed to be very large and v^2 is anticipated to be of reasonable size $2 \ln(A^2 e/v^2)$ may not be very sensitive to v . Then, because J_1 is independent of v , J may be regarded as a constant in the first approximation.

By writing v_0 for the optimum value of v we have

$$(\mathcal{E}(a, b, v_0)/m')^2 = (1-\mu J_1) \{2(b-a)^2/b+1\}.$$

In minimizing \mathcal{E}^2 we consider only the case

$$b = a(1+\epsilon)$$

where ϵ is assumed to be very small as compared to unity. J_1 is then expanded in powers of ϵ :

$$J_1 = \epsilon \mathcal{J}(a) - \epsilon^2 \mathcal{K}(a) + \dots,$$

in which the constant term vanishes as can be seen from (5.5) by taking $b=a$. $\mathcal{J}(a)$, and $\mathcal{K}(a)$, are found to be

$$\mathcal{J}(a) = 4[-1 + (1+1/2a)\ln(1+2a)]$$

and $\mathcal{K}(a) = 2[-1 - (9+19/2a+2/a^2)\ln(1+2a) + (1+2a)(1+4a)/a^2 \cdot \ln(1+4a)]$, respectively. The approximate expression for $(\mathcal{E}/m')^2$ becomes then

$$(\mathcal{E}/m')^2 = \{1 - \mu(\epsilon \mathcal{J} - \epsilon^2 \mathcal{K})\} (2a\epsilon^2 + 1). \quad (6.19)$$

Introducing

$$\eta = \sqrt{2a} \epsilon, \quad A = \mathcal{J}/\sqrt{2a}, \quad \text{and} \quad B = \mathcal{K}/2a \quad (6.20)$$

and by varying a and η (and using (6.20)) we obtain

$$\epsilon = (2a\mathcal{J}' - \mathcal{J})/2(a\mathcal{K}' - \mathcal{K}); \quad \mu^{-1} = (6a\epsilon^2 + 1)\mathcal{J}/4a\epsilon - (4a\epsilon^2 + 1)\mathcal{K}/2a.$$

The substitution of ϵ and μ thus found into (6.19) gives $(M/m')^2$.

In Figure 1 the results for $(M/m')^2$ and some other quantities of interest are given as functions of a . $(M/m')^2$ has a minimum at a maximum of $\mu \approx 0.30$ and no solution exists beyond that value of μ . It is to be noted that in the entire range where a local minimum of $(M/m')^2$ is found the size of ϵ remains small. At least $\epsilon^2 \mathcal{K}/\epsilon \mathcal{J}$ is less than 5% so that we can say that the expansion we have made of J_1 is a very good approximation.

In Figure 2, though it is not drawn to accurate quantitative scale, we plot $(\mathcal{E}/m')^2$ for various values of μ which is a measure of the coupling strength g^2 . The square of the mass M (in units of m'^2) is the minimum of $(\mathcal{E}/m')^2$ as a function of ϵ . From the

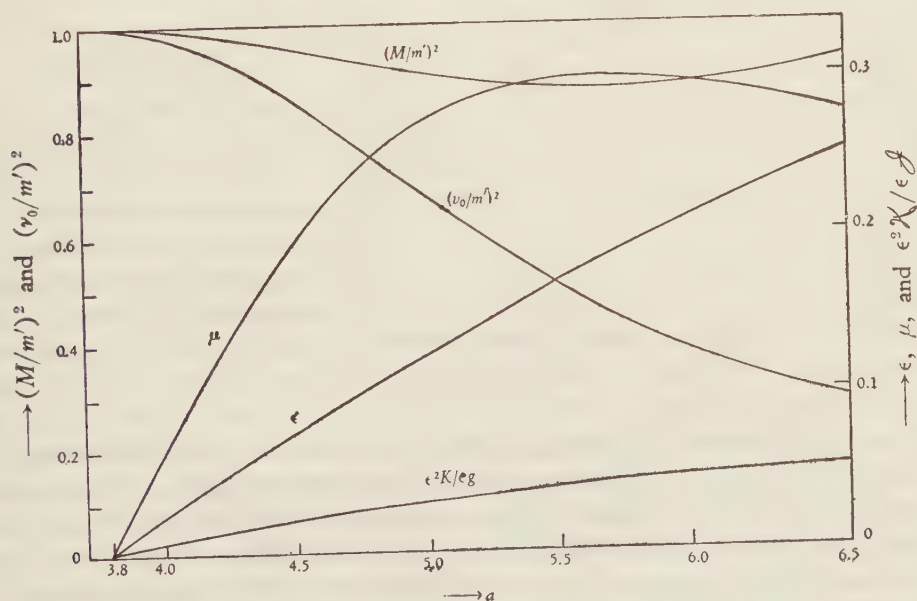


Fig. 1

figure it is seen that no true minimum exists for any value of μ —the scalar meson theory has no actual lowest energy. For small μ (weak coupling), however, a local minimum does exist representing a kind of metastable state of the nucleon. This is the state found by the perturbation theory. Even this state does not exist for $\mu > 0.30$.

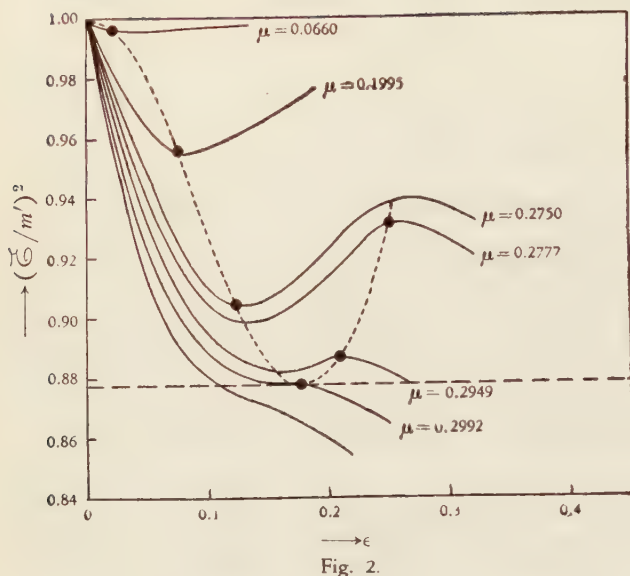


Fig. 2.

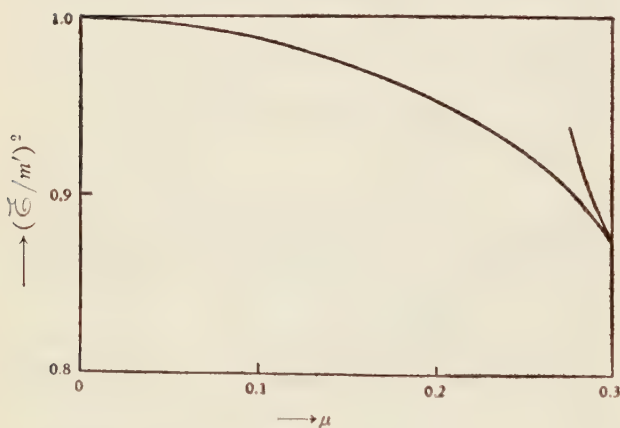


Fig. 3.

The value of the local minimum of $(\xi/m')^2$ (Cf. Figure 2) as a function of μ is given in Figure 3. As was mentioned above no solution exists above $\mu=0.30$. Below it the equation gives a second spurious solution indicated by the upper branch of the curve, corresponding to the maxima in Figure 2.

For more practical point of view it is desirable to have a direct relation among M^2 , m^2 , and g^2 . To do this we define μ' by

$$\mu' = g^2 / 8\pi M^2 \quad (6 \cdot 21)$$

and consider $(M/m')^2$ as a function of μ :

$$(M/m')^2 = G(\mu).$$

Then (6·21) may be written as

$$\mu' = \mu / G(\mu). \quad (6 \cdot 22)$$

In view of (6·22) we can regard $D(\mu) \equiv \mu^{-1} + 2 \ln(M^2 e / v_0^2)$ as a function of μ' and by calling $D(\mu) = F(\mu')$ (6·16) becomes $8\pi m^2 / g^2 = F(\mu') + 2 \ln(\Lambda^2 / M^2)$.

$$(6 \cdot 23)$$

The curve giving $F(\mu')$ is shown in Figure 4. From this the interrelation among M^2 , m^2 , and g^2 can easily be found. Suppose that the experimental mass, which is to be identified as M , and g^2 are given. We first compute μ' from (6·21). Then by reading Figure 4 find $F(\mu')$. Finally the theoretical mass m can be obtained from (6·23) provided that the cut-off Λ is assumed to have a definite value. As an example consider the case $\Lambda = 2m$. From (6·21) and (6·23) we have

$$(m/M)^2 = \mu' \{F(\mu') + 2 \ln 4 + 2 \ln(m^2/M^2)\}$$

and solve this for $(m/M)^2$ for a given value of μ' . The solutions for several values of the coupling constant are given in Table 1. There is no metastable state above $\mu' = 0.34$.

Our conclusion is that the present case of the scalar theory has no solution for the self-energy of the scalar nucleon although metastable states can be found in case the coupling

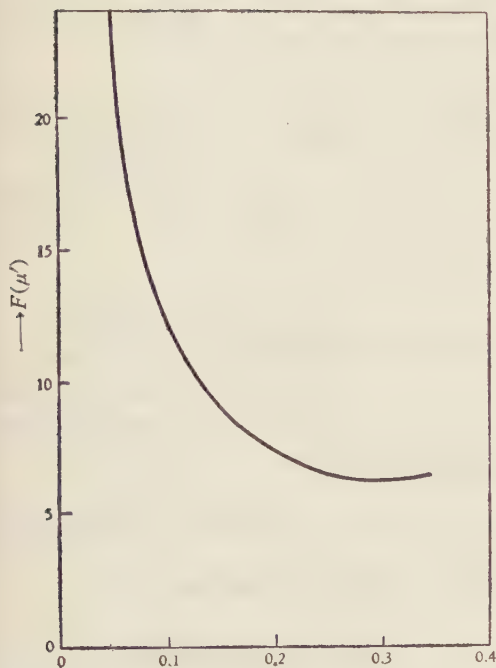


Fig. 4.

the limiting value of J_2 for large Λ^2 may not be obtained in a closed form. And we will probably be forced to resort to the numerical calculation of J .

As to the accuracy of the theory it can not easily be judged because there is no way of comparing our result with experiments.

An attempt to apply the present technique to the realistic problems encounters a great difficulty because of the appearance of the operators γ_μ , σ , and τ . Then the amplitude corresponding to (3.6) will contain these as ordered operators. Therefore the simple path integral treatment will fail for such a problem. However, when a method to work out this kind of problem is found it will be of some importance to note that the Fock's method of parametrizing the Dirac equation¹¹⁾ permits the Lagrangian formulation which is, at least in its form, very similar to the present treatment of the Klein-Gordon equation.

We also note that this method can be extended, though the accuracy of the result may not be very high, to another problem such as the scattering of the meson by the nucleon by using the best estimate of the real action.

The author would like to express his sincere thanks to Professor R. P. Feynman for his kind guidance and encouragement throughout the course of this work. He is also very grateful to Professor G. Araki for his continuous encouragement.

Table 1.

μ'	$(m/M)^2$
0	1
0.066	1.36
0.21	2.48
0.30	3.44
0.34	4.11

is reasonably weak. However, the following should also be noted.

The dependence of the self-energy on the cut-off Λ is completely determined in our case. Therefore it should be easy to renormalize this theory. For other theories, however, the Λ -dependence may not be so simple as the present one.

The effect of non-vanishing meson mass will first show up in (4.10) replacing k_μ^2 by $k_\mu^2 + m^2$. In this case the evaluation of J becomes much more complicated and even

Appendix A

The equivalence of the new variational method to the conventional one for the case of a

potential V can be shown as follows. For this case the action is given by

$$S = \int L d\tau = -1/2 \cdot \int m \dot{x}^2(\tau) d\tau - \int V(x_\tau) d\tau.$$

Let us suppose that the trial action is taken as

$$S_1 = \int L_1 d\tau = -1/2 \cdot \int m \dot{x}^2(\tau) d\tau - \int V_1(x_\tau) d\tau.$$

Then we have

$$sT = \langle S - S_1 \rangle = - \langle \int (V - V_1) d\tau \rangle = -T \langle V - V_1 \rangle.$$

In calculating s consider the numerator of

$$\langle V - V_1 \rangle = \int (V - V_1) \exp(S_1) \mathcal{D}x(\tau) / \int \exp(S_1) \mathcal{D}x(\tau).$$

Corresponding to $T > \tau > 0$ we write

$$\exp(S_1) = \exp\left(\int_\tau^T L_1(\tau'') d\tau''\right) \exp\left(\int_0^\tau L_1(\tau') d\tau'\right).$$

Hence

$$\begin{aligned} \int (V - V_1) \exp(S_1) \mathcal{D}x(\tau) &= \iint \exp\left(\int_\tau^T L_1(\tau'') d\tau''\right) \mathcal{D}x(\tau'') (V(x_\tau) - V_1(x_\tau)) dx_\tau \\ &\quad \times \exp\left(\int_0^\tau L_1 d\tau'\right) \mathcal{D}x(\tau') \\ &= \int K(x_T, T; x_\tau, \tau) (V(x_\tau) - V_1(x_\tau)) dx_\tau K(x_\tau, \tau; x_0, 0). \end{aligned}$$

Since both τ and $T - \tau$ are very large we have

$$K(x_\tau, \tau; x_0, 0) = \sum_n \phi_n(x_\tau) \phi_n^*(x_0) \exp(-E_n \tau) \sim \phi_0(x_\tau) \phi_0^*(x_0) \exp(-E_0 \tau)$$

and

$$K(x_T, T; x_\tau, \tau) \sim \phi_0(x_T) \phi_0^*(x_\tau) \exp(-E_0(T - \tau)),$$

where ϕ_0 is the eigenfunction of the lowest state with energy E_0 (this corresponds to E_1 in (4.1)). Therefore we have

$$\begin{aligned} \int (V - V_1) \exp(S_1) \mathcal{D}x(\tau) &\sim \phi_0(x_T) \phi_0^*(x_0) \exp(-E_0 T) \int \phi_0^*(x_\tau) (V(x_\tau) - V_1(x_\tau)) dx_\tau \phi_0(x_\tau) \\ &= \phi_0(x_T) \phi_0^*(x_0) \exp(-E_0 T) (V - V_1)_{00}. \end{aligned}$$

Similarly for the denominator of $\langle V - V_1 \rangle$ we have

$$\int \exp(S_1) \mathcal{D}x(\tau) \sim \phi_0(x_T) \phi_0^*(x_0) \exp(-E_0 T).$$

Hence

$$\langle V - V_1 \rangle = -s = (V - V_1)_{00}. \quad (\text{A} \cdot 1)$$

From

$$E_1 = (p^2/2m)_{00} + (V_1)_{00}$$

and (A.1) we have

$$E = (p^2/2m)_{00} + (V)_{00}$$

which is expressing the content of the conventional variational method.

It should be noted that in case $C=0$ the energy obtained by our method agrees with that given by the first order perturbation approximation.

Appendix B

The limiting value of J_2 for large Λ^2 given by (5.6) will be worked out here. We first note that for a very rough estimate we may approximate the factor $1 - \exp(-\gamma/\Lambda^2)$ by 0 for $\gamma < \gamma_0$ and by 1 for $\gamma > \gamma_0$ where γ_0 is of order Λ^{-2} . A better estimate can be obtained by choosing a number c such that $c \gg 1$ yet the condition

$$\gamma_m = c/\Lambda^2 \ll 1$$

is still fulfilled. We split J_2 defined by (5.4) as follows:

$$\begin{aligned} J_2 &= \int_0^\infty d\sigma \int_0^{\gamma_m} d\gamma \dots\dots + \int_0^\infty d\sigma \int_{\gamma_m}^\infty d\gamma \dots\dots \\ &= A + B. \end{aligned}$$

Because of our choice of γ_m the factor $1 - \exp(-\gamma/\Lambda^2)$ in B may be replaced by 1.

We consider A first. Let us again choose $b \gg 1$ such that

$$\sigma_m = b\gamma_m \ll 1.$$

Accordingly we split the σ -integration into two:

$$A = \int_0^{b\gamma_m} d\sigma \int_0^{\gamma_m} d\gamma \dots\dots + \int_{b\gamma_m}^\infty d\sigma \int_0^{\gamma_m} d\gamma \dots\dots = A' + A''.$$

In A'' $\sigma \gg \gamma$ so that neglecting γ as compared to σ we have

$$\begin{aligned} A'' &\approx \int_{b\gamma_m}^\infty d\sigma \int_0^{\gamma_m} 4d\gamma \exp(-v^2\sigma/2) \cdot (1 - \exp(-\gamma/\Lambda^2)) / \sigma^2 \\ &= \int_{b\gamma_m}^\infty 4d\sigma \exp(-v^2\sigma/2) \cdot \{\gamma_m - (1 - \exp(-\gamma_m/\Lambda^2)) / \Lambda^2\} / \sigma^2. \end{aligned}$$

Noting that $\exp(-\gamma_m/\Lambda^2) = \exp(-c) \approx 0$ an integration by parts gives

$$\begin{aligned} &\approx [4\exp(-v^2 b\gamma_m/2) / b\gamma_m - 2v^2 \int_{b\gamma_m}^\infty d\sigma \exp(-v^2\sigma/2) / \sigma] (\gamma_m - \Lambda^{-2}) \\ &\approx \{4(b\gamma_m)^{-1} + 2v^2 \ln(1.781v^2 b\gamma_m/2)\} \gamma_m \approx 0 \end{aligned} \quad (\text{A.2})$$

in which use has been made of an approximation

$$\int_a^\infty dx \exp(-\lambda x) / x = -\ln(1.781\lambda a) \quad \text{for } a \ll 1. \quad (\text{A.3})$$

In A' , since $\sigma, \gamma \ll 1$, we put $\exp(-v^2\sigma^2/4(\gamma+\sigma/2)) \approx 1$. Hence

$$\begin{aligned} A' &\approx \int_0^{b\gamma_m} d\sigma \int_0^{\gamma_m} d\gamma (1 - \exp(-\gamma A^2)) / (\gamma + \sigma/2)^2 \\ &= \int_0^{\gamma_m} 2d\gamma \{ \gamma^{-1} - (\gamma + b\gamma_m/2)^{-1} \} (1 - \exp(-\gamma A^2)) \\ &\approx 2 \left[\int_0^{\gamma_m} d\gamma (1 - \exp(-\gamma A^2)) / \gamma - (2/b\gamma_m) (\gamma + \exp(-\gamma A^2)/A^2) \right]_0^{\gamma_m}. \end{aligned} \quad (\text{A} \cdot 4)$$

In the first term the integrand tends to zero when $\gamma \rightarrow 0$, so we replace the lower limit by a small number ϵ . Then

$$\int_0^{\gamma_m} d\gamma (1 - \exp(-\gamma A^2)) / \gamma = \int_\epsilon^{\gamma_m} d\gamma / \gamma - \int_\epsilon^{\gamma_m} d\gamma \exp(-\gamma A^2) / \gamma.$$

Noticing that in the second integral $\exp(-\gamma A^2) = \exp(-c) \approx 0$ when $\gamma = \gamma_m$ we may extend the upper limit to $+\infty$ and by doing so we make use of (A.3). Hence

$$\int_0^{\gamma_m} d\gamma (1 - \exp(-\gamma A^2)) / \gamma \approx \ln(1.781 \gamma_m A^2).$$

The second term of (A.4) becomes

$$(2/b\gamma_m) \{ \gamma_m + (\exp(-\gamma_m A^2) - 1) / A^2 \} \approx 2/b - 2/b\gamma_m A^2 \approx 0.$$

Therefore

$$A' \approx 2 \ln(1.781 \gamma_m A^2). \quad (\text{A} \cdot 5)$$

Now let us proceed to the study of B . Replacing the factor $1 - \exp(-\gamma A^2)$ by 1 we have

$$\begin{aligned} B &= \int_0^\infty d\sigma \int_{\gamma_m}^\infty d\gamma \exp(-v^2\sigma^2/4(\gamma+\sigma/2)) / (\gamma+\sigma/2)^2 \\ &= \int_0^\infty 4d\sigma \{ 1 - \exp(-v^2\sigma^2/4(\gamma_m + \sigma/2)) \} / v^2\sigma^2. \end{aligned}$$

Let $\sigma_m = b\gamma_m$ with $\sigma_m \ll 1$ as before. By splitting the integration into two

$$B = \int_0^{b\gamma_m} d\sigma \dots + \int_{b\gamma_m}^\infty d\sigma \dots = B' + B''$$

we consider B' first. Since $\sigma \ll 1$ in B' we expand the exponential. Then

$$\begin{aligned} B' &= \int_0^{b\gamma_m} 4d\sigma \{ 1 - 1 + v^2\sigma^2/4(\gamma_m + \sigma/2) - \dots \} / v^2\sigma^2 \\ &= 2 \ln(\gamma_m + \sigma/2) \Big|_0^{b\gamma_m} \approx 2 \ln(b/2). \end{aligned} \quad (\text{A} \cdot 6)$$

In the denominator of the exponent of B'' we neglect γ_m as compared to $\sigma/2$. Then we have

$$B'' \approx (4/v^2) (1 - \exp(-v^2 b\gamma_m/2)) / b\gamma_m - 2 \ln(1.781 b\gamma_m/2).$$

Expanding the exponential we obtain

$$B'' \approx 2 - 2 \ln(1.781 v^2 b \gamma_m / 2). \quad (\text{A} \cdot 7)$$

Collecting (A·2), (A·5), (A·6), and (A·7) gives

$$J_2 = 2 \ln(A^2 e / v^2).$$

Appendix C

We will find here the behavior of J_1 for the case $b \gg a$ and $b \gg 1$. We choose $c \gg 1$ such that $c/b \ll 1$. Then we have

$$J_1 \approx \int_0^{c/b} 4d\xi [\exp(-\xi/2) - \exp\{-\xi^2 b/2(1 - \exp(-b\xi))\}] / \xi^2 \\ + \int_{c/b}^{\infty} 4d\xi [\exp(-\xi/2) - \exp(-\xi^2 b/2)] / \xi^2.$$

By changing the variable by

$$\xi = x/b$$

in the first term and by integrating by parts in the second term this becomes

$$= b \int_0^c 4dx [\exp(-x/2b) - \exp\{-x^2/2b(1 - \exp(-x))\}] / x^2 \\ + (4/\xi) [\exp(-\xi/2) - \exp(-\xi^2 b/2)] \Big|_{c/b}^{\infty} - \int_{c/b}^{\infty} 4d\xi [1/2 \cdot \exp(-\xi/2) - \xi b \exp(-\xi^2 b/2)] / \xi^2. \quad (\text{A} \cdot 8)$$

In the first term of this expression we expand the exponential and obtain

$$b \int_0^c 4dx [-x/2b \{1 - x/(1 - \exp(-x))\} + (x^2/8b^2) \{1 - x/(1 - \exp(-x))\}^2 + \dots] / x^2.$$

Neglecting the second term this gives

$$-2 \int_{\epsilon}^c dx/x + 2 \int_{\epsilon}^c dx/(1 - \exp(-x)) = 2c - 2 \ln c. \quad (\text{A} \cdot 9)$$

The third term of (A·8) becomes

$$-2 \int_{c/b}^{\infty} d\xi \exp(-\xi/2) / \xi + 4b \int_0^{\infty} d\xi \exp(-\xi^2 b/2) - 4b \int_0^{c/b} d\xi \exp(-\xi^2 b/2) \\ \approx 2 \ln(1.781 c/2b) + 2b \sqrt{2\pi/b} - 4c. \quad (\text{A} \cdot 10)$$

Then using (A·9) and (A·10) we find

$$J_1 = 2 \sqrt{2\pi b} - 2 \ln(1.781/2b) - 2.$$

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- 9) Appendix C of Reference 5.
- 10) Section 5 of Reference 2.
- 11) V. Fock, *Phys. Z. Sowj.* **12** (1937), 404.

Extensions of Variational Methods, I

— Super-Stationary Variational Method —

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Several methods to improve the degree of approximation for the stationary character in variational methods, are discussed. By selecting a set of trial functions as prescribed in the present paper, it is shown that we can construct quite generally a super-stationary expression for the quantity we wish to find. By "super-stationary" we mean that, the first, second and third variations of that expression vanish for any infinitesimal variations of trial functions. The examples to which this method is applied are: I). To find the eigenvalue in the eigenvalue-problems with discrete spectra only, (for example, to find the potential depth in the deuteron problem). II). To find the discrete eigenvalues for the problems with both discrete and continuous spectra, (for example, to find the energy level of the deuteron). III). To solve problems for continuous spectra by integral equations. IV). To solve problems for continuous spectra by differential equations. Also a method is mentioned for improving approximation of the stationary character in successive manner in each case and it is shown that a simple method for this improvement exists in the cases I) and II).

§ 1. Introduction

The variational methods have been applied extensively to many problems of physics, since W. Ritz applied this method successfully in solving an eigenvalue-problem concerning the vibration of a plate¹⁾. The merits of the variational methods lie, among others, in the following three points. The first is: If we take a rough trial function with errors of $O(\mathcal{A})$, the extremal property guarantees errors of $O(\mathcal{A}^2)$ only for the quantity to be solved. We call it the stationary character of variational methods. In the second place, for the eigenvalue problems, in which the fundamental equation contains an unknown eigenvalue and an unknown function, we must find these two simultaneously. In the usual elementary procedure, we must first choose an approximate value as the eigenvalue and proceed by a trial-and-error method. The variational method, on the contrary, works out this trouble automatically. The third merit of variational method is extensiveness of its range of applicability. We can apply the variational technique to all problems where the fundamental equations can be derived from extremal principle. Indeed some of these problems permit of no other approach other than the variational.

It is the purpose of this series of papers to try to extend the variational methods so as to develop the above mentioned merits to greater extent. The present paper is concerned with the first point, that is to give a general method of constructing the super-stationary expression in stationary-value problems. The subsequent papers will contain extensions of

variational treatments concerning the second point.

The problems which have been considered as the applications in this paper are: I). To obtain any eigenvalue in the eigenvalue-problems having only the discrete spectra. One example is to find the potential depth in the deuteron problem, (§ 2). II). To obtain any discrete eigenvalues in the eigenvalue-problems which contain continuous spectra, for example to find the energy level of the deuteron, (§ 3). III). To solve problems for continuous spectra by integral equation. (§ 5). IV). To solve problems for continuous spectra by differential equation. (§ 6). In § 4 we will discuss several methods for the successive improvement of the degree of approximation.

§2. Eigenvalue problem with discrete spectra only

As already be shown in the preceding paper²⁾, we can construct the super-stationary expression of an eigenvalue, which, by several examples, is shown to be sufficiently accurate to calculate the eigenvalue²⁾. In the present section we will develop a formulation which is suitable for discussions of the subsequent sections.

a) Non-degenerate case

Consider the eigenvalue-problem with real and discrete eigenvalues

$$A\psi_m = \lambda_m B\psi_m, \quad (1)$$

where A and B are hermitian operators for the eigenfunctions ψ_n . We assume that there is no eigenfunction for which $(\psi_m, B\psi_m) = 0$. Let ψ_n be normalized as

$$(\psi_m, B\psi_m) = (-)^{(m)}. \quad (2)$$

Though, without any loss of generality ψ_m can be assumed real, here we will define the scalar product as $(\psi', B\psi') = \int \psi'^* \cdot B\psi' d\tau$. If any eigenvalue is non-degenerate, ψ_n is determined uniquely by (2) except a constant phase. The symbol $(-)^{(m)}$ in the right-hand side of (2) means $+1$ or -1 according to whether the sign of $(\psi_m, B\psi_m)$ is $(+)$ or $(-)$. Let us consider any infinitesimal variations of ψ_m , with the following two restrictions that

$$(\psi_m + \delta\psi_m, B(\psi_m + \delta\psi_m)) = (-)^{(m)}, \quad (3)$$

and $\delta\psi_m$ leaves A and B hermitian. Any variations which satisfy the condition (3) and the hermitian requirement, can generally be expressed in the following form, if ψ_n constitutes a complete set;

$$\delta\psi_m = \sum_{n \neq m} \delta a_{mn} \psi_n, \quad (4)$$

where δa_{mn} can be regarded as any infinitesimal variations of independent variational parameters. Now we will consider infinitesimal variations of the quantity $(\psi'_m, (A - \lambda_n^{(e)} B) \psi_n)$, where $\lambda_n^{(e)} = (\psi_n, A\psi_n) / (\psi_n, B\psi_n)$. Although $\lambda_n^{(e)}$ is actually equal to λ_n , we may regard $\lambda_n^{(e)}$ as a functional of ψ_n , so that $\delta\lambda_n^{(e)} = 0$ as is well-known. Then we get

$$\delta(\psi_m, (A - \lambda_n^{(e)} B) \psi_n) = (\psi_m, (A - \lambda_n^{(e)} B) \delta\psi_n) = (-)^{(m)} (\lambda_m^{(e)} - \lambda_n^{(e)}) \delta a_{nm}. \quad (5)$$

Substituting (5) into (4), we have

$$\delta\psi_m = \sum_{n \neq m} (-)^{(n)} \{ \delta(\psi_n, (A - \lambda_m^{(e)} B) \psi_m) / (\lambda_n^{(e)} - \lambda_m^{(e)}) \} \psi_n. \quad (6)$$

Let trial functions close to ψ_m be ϕ_m ;

$$\phi_m = \psi_m + \delta\psi_m. \quad (7)$$

Using (1), we observe

$$\delta(\psi_n, (A - \lambda_m^{(e)} B) \psi_m) = (\phi_n, (A - \lambda_m^{(e)} B) \phi_m) - (\psi_n, (A - \lambda_m^{(e)} B) \psi_m) = (\phi_n, (A - \lambda_m^{(e)} B) \phi_m). \quad (7a)$$

In the approximation correct to first order of variations, (6) is written as

$$\delta\psi_m = \sum_{n \neq m} (-)^{(n)} \{ (\phi_n, (A - \lambda_m^{(e)} B) \phi_m) / (\lambda_n^{(e)} - \lambda_m^{(e)}) \} \phi_n.$$

So we have from (7)

$$\psi_m = \phi_m - \delta\psi_m = \phi_m - \sum_{n \neq m} (-)^{(n)} \{ (\phi_n, (A - \lambda_m^{(e)} B) \phi_m) / (\lambda_n^{(e)} - \lambda_m^{(e)}) \} \phi_n. \quad (8)$$

This formula has a form similar⁽³⁾ to the ordinary perturbation formula, so that we may call it "Variation-perturbation" formula. Thus we have obtained the following result: Let a set of trial function ϕ_m close to ψ_m be given. We calculate

$$A_{nm} \equiv (\phi_n, A \phi_m), \quad B_{nm} \equiv (\phi_n, B \phi_m), \quad (9 \cdot a)$$

$$a_{mn} \equiv (A_{nm} B_{mm} - A_{mm} B_{nm}) / (A_{nn} B_{mm} - A_{mm} B_{nn}), \quad (9 \cdot b)$$

and construct ϕ_m according to (8)

$$\phi_m \equiv \psi_m - \sum_{n \neq m} a_{mn} \psi_n. \quad (9 \cdot c)$$

For any variations of ϕ_n , it follows that

$$\phi_m = \psi_m + O(\delta^2), \quad \text{namely} \quad \delta\phi_m = \delta\psi_m. \quad (10 \cdot a)$$

Thus the procedure (9) gives a general means to construct the approximate functions with stationary character from a set of trial functions. If we take ϕ_m as a new trial function, the eigenvalue λ_m is calculated as

$$\lambda_m^{(e)} = (\phi_m, A \phi_m) / (\phi_m, B \phi_m), \quad (10 \cdot b)$$

with an error of $O(\delta^4)$. In other words, as a functional of ϕ_n , first, second and third variations of $\lambda_m^{(e)}$ vanish. We call this property the super-stationary character of $\lambda_m^{(e)}$. It should be remarked that the formula (9) are valid even when the normalization of trial functions ϕ_m is not specified. Several examples were given in reference 2 to show the accuracy of the expression of (10·b). But here another example will be given as it is of some interest to compare the example with the ordinary Rayleigh-Ritz method.

(Example) Problem: A neutron and a proton constitute a deuteron in the ground state with zero binding energy by the Yukawa potential. Find the depth of this potential.

In this case A and B are

$$A = -d^2/dr^2, \quad B = e^{-r}/r,$$

with the conditions $\phi_m(0) = 0$; $\phi_m(r) \rightarrow \text{const}$ for $r \rightarrow \infty$. As the set of trial functions we take the correct eigenfunctions for the Hulthén potential; $B = e^{-r}/(1 - e^{-r})$:

$$\phi_m = (-x/m!) d^m/dx^m \cdot [x^{m-1}(1-x)^m], \quad x \equiv 1 - e^{-r}, \quad m \geq 1. \quad (11)$$

The correct eigenvalue corresponding to the ground state is ⁴⁾

$$\lambda_1 = 1.67980.$$

From the usual variational method we obtain

$$\lambda_1^{(t)} = (\phi_1, A\phi_1) / (\phi_1, B\phi_1) = 1.73804, \quad \Delta\lambda_1^{(t)} / \lambda_1 = 0.0347 = (19\%)^2.$$

So we infer that the trial functions (11) may have errors of about 20%. Calculating A_{mn} , B_{mn} and a_{1n} we get

$$A_{mm} = m/2, \quad A_{mn} = 0; \quad m \neq n, \quad B_{11} = \ln(4/3), \quad B_{12} = -\ln(4/3) + 3\ln(9/8), \text{ etc}, \\ a_{12} = -0.15879, \quad a_{13} = 0.01221, \quad \text{etc}. \quad (12)$$

Since we should neglect the quantities of order δ^2 , in this example it is justified to set, comparing the order of a_{13}/a_{12} with a_{12} ,

$$a_{1n} = 0, \quad n \geq 3.$$

Thus

$$\phi_1 = \phi_1 + 0.1588\phi_2.$$

Putting it into (10·b) we obtain

$$\lambda_1^{(t)} = 1.68005, \quad \Delta\lambda_1^{(t)} / \lambda_1 = 0.00015 = (0.012)^2. \quad (13 \cdot a)$$

This result is better than the accuracy estimated by order consideration: $(20\%)^4 = 0.0016$. If we take both a_{12} and a_{13} , we get

$$\lambda_1^{(t)} = 1.68026. \quad (13 \cdot b)$$

Thus even if higher components are included, the result is not always improved, because a_{1n} have errors of $O(\delta^2)$ and in this case the value of a_{1n} has scarcely any meaning. Next we will compare (12), (13) with the calculations of Hulthén and Laurikainen.⁵⁾ They took a linear combination of ϕ_n of (11) as a trial function and obtained the following results for combinations of the first two, three and four terms respectively;

$$\begin{array}{ll} h_1 = 0.5357, & \lambda_1^{(t)} = 1.67993, \\ h_1 = 0.5781, \quad h_2 = -0.0516, & \lambda_1^{(t)} = 1.67985, \\ h_1 = 0.6208, \quad h_2 = -0.1770, \quad h_3 = 0.0982, & \lambda_1^{(t)} = 1.67982, \end{array}$$

where h_n are adjustable parameters in the form

$$\phi(r) = (1 - e^{-r}) \sum_{n=0} h_n e^{-nr}, \quad h_0 = 1. \quad (14)$$

Values of (12) correspond to

$$h_1 = 0.5663; \quad \text{if } a_{12} = -0.15879, \quad a_{1n} = 0; \quad n \geq 3.$$

$$h_1 = 0.6925, \quad h_2 = -0.1473; \quad \text{if } a_{12} = -0.15879, \quad a_{13} = 0.01221, \quad a_{1n} = 0; \quad n \geq 4.$$

From these figures we learn the accuracies and limitations of the present treatment. In the present example simple trial functions (14) give good results, but this may be somewhat accidental. However an important point in our theory is that it makes possible to estimate the error of results and that the accuracy is guaranteed to be $O(\delta^4)$.

b) Degenerate case

If the system has some degenerate eigenvalues, the formula (6) is no longer valid. We will distinguish the eigenfunctions corresponding to the same eigenvalue λ_n by another indices p as ψ_{np} . These ψ_{np} can be chosen to be orthogonal to each other, that is,

$$(\psi_{np}, A\psi_{nq}) = 0, \quad (\psi_{np}, B\psi_{nq}) = 0, \quad p \neq q. \quad (15)$$

In taking the variations of ψ_{np} and ψ_{nq} , we will always preserve the orthogonal relation (15):

$$(\delta\psi_{np}, A\psi_{nq}) = (\psi_{np}, A\delta\psi_{nq}) = (\delta\psi_{np}, B\psi_{nq}) = (\psi_{np}, B\delta\psi_{nq}) = 0, \quad p \neq q. \quad (16)$$

Then the left hand side of (5) will be

$$\delta(\psi_{np}, (A - \lambda_n^{(t)} B) \psi_{nq}) = 0.$$

Let us consider the meaning of (16). $\delta\psi_{np}$ is expressed as

$$\delta\psi_{np} = \sum_m \sum_q \delta a_{np, mq} \psi_{mq}, \quad \delta a_{np, np} = 0,$$

and (16) is put in the form:

$$(\delta\psi_{np}, A\psi_{nq}) = \delta a_{np, nq}^* (\psi_{np}, A\psi_{nq}) = 0, \quad (\delta\psi_{np}, B\psi_{nq}) = \delta a_{np, nq}^* (\psi_{np}, B\psi_{nq}) = 0. \quad (17)$$

Then we have the relation

$$\delta a_{np, nq} = 0, \quad p \neq q. \quad (18)$$

Both $(\lambda_{np}^{(t)} - \lambda_{nq}^{(t)})$ and $\delta a_{np, nq}$ of the right hand side of (5) vanish. Eqs. (4) and (7), (8) are modified by (18) as

$$\begin{aligned} \delta\psi_{mp} &= \sum_{n \neq m} \delta a_{mp, nq} \psi_{nq} = \sum_{n \neq m} \sum_q (-)^{(n)} \{ \delta(\psi_{nq}, (A - \lambda_{mp}^{(t)} B) \psi_{mp}) / (\lambda_{nq}^{(t)} - \lambda_{mp}^{(t)}) \} \psi_{nq} \\ &= \sum_{n \neq m} \sum_q (-)^{(n)} \{ (\phi_{nq}, (A - \lambda_{mp}^{(t)} B) \phi_{mp}) / (\lambda_{nq}^{(t)} - \lambda_{mp}^{(t)}) \} \phi_{nq}. \end{aligned} \quad (19)$$

(19) is also valid even when the degeneracy is approximate $\lambda_{nq}^{(t)} \doteq \lambda_{np}^{(t)}$. In order to impose the condition (16) on the trial function, we must have

$$(\phi_{np}, A\phi_{nq}) = (\phi_{np}, B\phi_{nq}) = 0, \quad p \neq q. \quad (20)$$

(20) is not only necessary but sufficient, because

$$(\phi_{np}, A\phi_{nq}) = \delta a_{np, nq}^* (\phi_{nq}, A\phi_{nq}) + \delta a_{nq, np} (\phi_{np}, A\phi_{np}) = 0,$$

$$(\phi_{np}, B\phi_{nq}) = \delta a_{np, nq}^* (\phi_{nq}, B\phi_{nq}) + \delta a_{nq, np} (\phi_{np}, B\phi_{np}) = 0.$$

Hence (20) implies that

$$\delta a_{np, nq} = 0, \quad (p \neq q).$$

§ 3. Eigenvalue problem including continuous spectra

In the last section we discussed the case where there are point spectra only. Eigenvalue problems which have been widely investigated in physics, however, often contain continuous spectra, so unless the theory is developed to include the continuous spectra, we can not apply it extensively to practical problems. Let indices k, l indicate the continuous spectra and m, n the discrete spectra respectively. First we will specify the normalizations. Taking a parameter which suitably indicates continuous levels, we impose

$$(\psi(k), B\psi(l)) = (-)^{(k)} \delta(k-l), \quad (\psi_m, B\psi_n) = (-)^{(m)} \delta_{mn}, \quad (21a)$$

where $\delta(x)$ is the Dirac delta function. Furthermore we have

$$(\psi(k), B\psi_n) = 0, \quad (\psi_m, B\psi(l)) = 0. \quad (21b)$$

Any infinitesimal variations which satisfy the normalizations of (21) are written as

$$\delta\psi_m = \sum_{n \neq m} \delta a_{mn} \psi_n + \int \delta a_m(l) \psi(l) dl, \quad (22a)$$

$$\delta\psi(k) = \sum_n \delta a(k)_n \psi_n + \int \delta a(k, l) \psi(l) dl, \quad (22b)$$

where we may consider δa as arbitrary independent variations. The orthogonal conditions of (21) may be violated. Next consider variations of the expressions: $(\psi(k), (A - \lambda^{(k)}(l)B)\psi(l))$, $(\psi(k), (A - \lambda_n^{(k)}(l)B)\psi_n)$, etc, where $\lambda^{(k)}(l)$, $\lambda_n^{(k)}$ are defined by

$$\lambda^{(k)}(l) \equiv (\psi(l), A\psi(l)) / (\psi(l), B\psi(l)), \quad \lambda_n^{(k)} \equiv (\psi_n, A\psi_n) / (\psi_n, B\psi_n).$$

The former definition is a form of ∞/∞ , and has no meaning, because its numerator and denominator are normalized by (21). We must take a suitable limiting process, for example, if the scalar product is defined as

$$(\psi(l), A\psi(l)) = \lim_{r \rightarrow \infty} \int_{c(l) \leq |r|}^x \psi(l, r) \cdot A\psi(l, r) dr,$$

$\lambda^{(k)}(l)$ can be specified by the following limiting procedure

$$\lambda^{(k)}(l) = \lim_{r \rightarrow \infty} \left(\int_c^x \psi(l, r) \cdot A\psi(l, r) dr / \int_c^x \psi(l, r) \cdot B\psi(l, r) dr \right).$$

Generally speaking, such limiting process should be taken so that $\lambda^{(k)}(l)$ accords with $\lambda(l)$ actually.

As $\delta\lambda^{(k)}(l) = 0$ and $\delta\lambda_n^{(k)} = 0$ hold, it follows that

$$\delta(\psi(k), (A - \lambda^{(k)}(l)B)\psi(l)) = (\psi(k), (A - \lambda^{(k)}(l)B)\delta\psi(l))$$

$$\begin{aligned}
&= (\psi(k), \int \partial a(l, l') (\lambda^{(t)}(l') - \lambda^{(t)}(l)) B\psi(l') dl') \\
&= (-)^{(k)} \int \partial a(l, l') (\lambda^{(t)}(l') - \lambda^{(t)}(l)) \partial(k-l') dl' \\
&= (-)^{(k)} (\lambda^{(t)}(k) - \lambda^{(t)}(l)) \partial a(l, k). \quad (23 \cdot a)
\end{aligned}$$

Similarly we get

$$\partial(\psi(k), (A - \lambda_n^{(t)} B) \psi_n) = (-)^{(k)} (\lambda^{(t)}(k) - \lambda_n^{(t)}) \partial a_n(k), \quad (23 \cdot b)$$

$$\partial(\psi_m, (A - \lambda^{(t)}(l) B) \psi(l)) = (-)^{(m)} (\lambda_m^{(t)} - \lambda^{(t)}(l)) \partial a(l)_m, \quad (23 \cdot c)$$

$$\partial(\psi_m, (A - \lambda^{(t)} B) \psi_n) = (-)^{(m)} (\lambda_m^{(t)} - \lambda_n^{(t)}) \partial a_{nm}, \quad (23 \cdot d)$$

which are the generalizations of (5). Let the trial functions which are close to ψ_m and $\psi(k)$ be ϕ_m and $\phi(k)$,

$$\phi_m = \psi_m + \delta\psi_m, \quad \phi(k) = \psi(k) + \delta\psi(k). \quad (24)$$

Neglecting higher orders of $\delta\psi_m$ and $\delta\psi(k)$ we have, from (22) and (23),

$$\begin{aligned}
\partial\phi_m &= \sum_{n \neq m} \{ (-)^{(n)} \partial(\psi_n, (A - \lambda_m^{(t)} B) \psi_m) / (\lambda_n^{(t)} - \lambda_m^{(t)}) \} \phi_n \\
&\quad + \int \{ (-)^{(l)} \partial(\psi(l), (A - \lambda_m^{(t)} B) \psi_m) / (\lambda^{(t)}(l) - \lambda_m^{(t)}) \} \phi(l) dl. \quad (25)
\end{aligned}$$

Using (7a) and (26),

$$\partial(\psi(l), (A - \lambda_m^{(t)} B) \psi_m) = (\phi(l), (A - \lambda_m^{(t)} B) \phi_m), \quad (26)$$

(25) is reduced to

$$\begin{aligned}
\partial\phi_m^{(t)} &\equiv \sum_{n \neq m} \{ (-)^{(n)} (A_{nm} - \lambda_m^{(t)} B_{nm}) / (\lambda_n^{(t)} - \lambda_m^{(t)}) \} \phi_n \\
&\quad + \int \{ (-)^{(l)} (A(l)_m - \lambda_m^{(t)} B(l)_m) / (\lambda^{(t)}(l) - \lambda_m^{(t)}) \} \phi(l) dl, \quad (27 \cdot a)
\end{aligned}$$

where $A(l)_m \equiv (\phi(l), A\phi_m)$, etc. Observing (24) and (27·a) we can construct ϕ_m as

$$\phi_m \equiv \psi_m - \partial\phi_m^{(t)} = \psi_n + O(\partial^2). \quad (28 \cdot a)$$

Then we get $\lambda_m^{(t)} \equiv (\phi_m, A\phi_m) / (\phi_m, B\phi_m) = \lambda_m + O(\partial^1)$.

For the solutions of continuous levels it follows similarly that

$$\phi(k) \equiv \psi(k) - \partial\phi^{(t)}(k) = \psi(k) + O(\partial^2), \quad (28 \cdot b)$$

$$\begin{aligned}
\partial\phi^{(t)}(k) &= \sum_n \{ (-)^{(n)} (A_n(k) - \lambda^{(t)}(k) B_n(k)) / (\lambda_n^{(t)} - \lambda^{(t)}(k)) \} \phi_n \\
&\quad + \int \{ (-)^{(l)} (A(l, k) - \lambda^{(t)}(k) B(l, k)) / (\lambda^{(t)}(l) - \lambda^{(t)}(k)) \} \phi(l) dl. \quad (27 \cdot b)
\end{aligned}$$

where $A(l, k) \equiv (\phi(l), A\phi(k))$, $A_n(k) \equiv (\phi_n, A\phi(k))$.

In quantum mechanics the ground state of the system is often of special interest. If

there are many excited discrete levels, the main contribution to $\partial\phi_m^{(t)}$ in (27·a) may come from the first term of (27·a), so the second term containing the effect of continuous levels may not seriously affect to the results, if we take suitable trial functions. In such case we presume that the situation is somewhat similar to one which has been met in the last section. On the contrary in case there is only one bound state, continuous levels have important roles. For illustration we will take the deuteron state as an example.

(Example). Problem: A neutron and a proton constitute a bound state by the exponential potential with given depth. Find the binding energy of the ground state.

In this case A and B are

$$A = -(d^2/dr^2 + Ve^{-r}), \quad B = 1,$$

with the conditions $\psi_0(r=0)=0$, $\psi_0(r \rightarrow \infty) \rightarrow 0$, where ψ_0 is the ground state function. Taking $V=2$ which corresponds to the well-depth parameter¹⁾ $S=1.3833$, this value is near the value obtained from the experiments and hence there is no excited state. We will choose the trial functions as

$$\phi_0(r) = \sqrt{2\gamma(\gamma+1)(2\gamma+1)} (1-e^{-r})e^{-\gamma r}, \quad (29 \cdot a)$$

$$\phi(k, r) = (2/\pi)^{1/2} \sin kr. \quad (29 \cdot b)$$

(29·a) and (29·b) has been normalized according to (21). The correct biniding energy $|E|$ is calculated from reference 4, Table V,

$$-\lambda_0 = |E| = 0.019978.$$

(29·a) involves one variational parameter γ . Adjusting γ according to the Rayleigh-Ritz method we have

$$\gamma = 0.09189.$$

This leads to

$$-\lambda_0^{(t)} \equiv \gamma_t^2 = 0.015114, \quad \gamma_t = 0.12294.$$

We also have $\lambda^{(t)}(k) = k^2$. It can be seen from the example in reference 2, § 2 that $\phi_0(r)$ has errors of about 20%. The trial function (29·b) is roughly approximate because it is the free-state solution obtained by ignoring the effect $V=2$. Calculating (23·b), we have

$$\begin{aligned} \partial a_0(k) = & 4\gamma(\gamma+1)(2\gamma+1)k^2/\pi)^{1/2} [\{ (\gamma^2+k^2)^{-1} - ((\gamma+1)^2+k^2)^{-1} \} \\ & - V \cdot (\gamma_t^2+k^2)^{-1} \{ ((\gamma+1)^2+k^2)^{-1} - ((\gamma+2)^2+k^2)^{-1} \}], \end{aligned}$$

and from (27) and (28),

$$\begin{aligned} \phi_0 = & \sqrt{2\gamma(\gamma+1)(2\gamma+1)} V [(e^{-\gamma r} - e^{-(\gamma+1)r}) / \{ (\gamma+1)^2 - \gamma_t^2 \} \\ & - (e^{-\gamma_t r} - e^{-(\gamma+2)r}) / \{ (\gamma+2)^2 - \gamma_t^2 \}]. \end{aligned}$$

For simplicity if the adjustable γ involved in (29·a) is taken to be γ_t , this gives

$$|E_t| \equiv -\lambda_0^{(t)} = 0.019650.$$

Thus we have obtained a considerably better value for the binding energy. Incidentally it will be remarked that we can not apply the perturbation method assuming V as perturbation, because there is no eigenfunction close to ϕ_0 in the unperturbed system.

§ 4. Higher approximations

In this section two methods for the successive improvement of the stationary character are discussed. Although the higher order approximations involve more laborious calculations in general, it is to some extent worthwhile to point out that we can improve the approximations in several ways.

a) Simple iterative method

First we will consider a general method which allows one to improve the approximations successively. This method is applicable not only to eigenvalue problems such as in § 2-§ 3, but to all other cases for which the present super-stationary method is applicable, for example, to scattering problems which will be discussed in § 5-§ 6.

Let ϕ_m be constructed from the first trial functions ϕ_n which have errors of $O(\delta)$. ϕ_m involves errors of $O(\delta^2)$. Taking ϕ_m as a new set of trial functions, then we can construct the approximate functions ϕ'_m having errors of $O(\delta^3)$ only in the same way. This process can be performed endlessly as $O(\delta^4)$, $O(\delta^5)$, etc. The following is an example of this successive approximation.

(Example). A neutron and a proton constitute a bound state (=deuteron state) with zero binding energy by an exponential potential. Find the depth of this potential. This problem is the same as the example of § 3, but the presentation of the problem is different. A and B are

$$A = -d^2/dr^2, \quad B = e^{-r}.$$

with the conditions $\phi_n(0) = 0$; $\phi'_n(r) \rightarrow \text{const.}$ for $r \rightarrow \infty$. As the set of first trial functions we choose the solution for the Hulthén potential; $B = e^{-r}/(1 - e^{-r})$:

$$\phi_n = (-x/n!) d^n/dx^n [x^{n-1}(1-x)^n], \quad x \equiv 1 - e^{-r}, \quad n \geq 1, \quad (30 \cdot a)$$

$$\phi_1 = x = 1 - e^{-r}, \quad \phi_2 = 2x - 3x^2, \quad \dots$$

The correct eigenvalue corresponding to the ground state is⁽⁶⁾: $\lambda_1 = 1.44580$. A_{mn} and B_{mn} are readily calculated;

$$A_{nn} = n/2, \quad A_{mn} = 0; m \neq n, \quad B_{nn} = n/(4n^2 - 1), \quad B_{n+1,n} = B_{n,n+1} = -1/(8n + 4),$$

$$B_{mn} = 0; |m - n| \geq 2.$$

We have $\lambda_1^{(t)} = A_{11}/B_{11} = 1.5$, $\Delta\lambda_1^{(t)}/\lambda_1 \sim 0.04 = (20\%)^2$. Then it may be inferred that the trial functions (30·a) have errors of about 20%. Constructing the second trial functions as described above, we get

$$\phi_1 = \phi_1 - (5/32)\phi_2, \quad \phi_2 = (5/16)\phi_1 + \phi_2 - (7/16)\phi_3,$$

$$\phi_3 = (21/32)\phi_2 + \phi_5 - (45/64)\phi_4, \quad \phi_4 = (15/16)\phi_3 + \phi_4 - (77/80)\phi_5. \quad (30 \cdot b)$$

From (30·b) we find

$$\lambda_1^{(e)} \equiv (\phi_1, A\phi_1) / (\phi_1, B\phi_1) = 1611/1114 = 1.44614, \quad \Delta\lambda^{(e)} / \lambda_1 \sim 0.0002,$$

which is better than $(20\%)^4 = 0.0016$. Using (30·b), we have the third trial function for ψ_1

$$\phi_1' = \phi_1 + 0.000868\phi_2 + 0.005570\phi_3 + 0.001967\phi_4, \quad (30 \cdot c)$$

and from this ϕ_1'

$$\lambda_1' \equiv (\phi_1', A\phi_1') / (\phi_1', B\phi_1') = 1.44585, \quad \Delta\lambda_1' / \lambda_1 \sim 3 \times 10^{-5} \sim (27\%)^8. \quad (31)$$

The first trial functions (30·a) have larger errors for higher eigenvalues as can be seen from (30·b). Hence the result of (31) is reasonable.

b) Improved approximation in eigenvalue problem

In eigenvalue problems there is a simpler method of approximation. Let us consider the simplest 2×2 matrix case:

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (32)$$

Let the trial functions be

$$\phi_1 = \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix}.$$

From these trial functions we have

$$A_{ii} = \lambda_i + (\lambda_j - \lambda_i) \sin^2 \theta, \quad A_{ij} = (\lambda_1 - \lambda_2) \cos \theta \sin \theta, \quad B_{ij} = \delta_{ij}, \quad (i=1, j=2 \text{ or } i=2, j=1).$$

Using the method explained in § 2, we have

$$\phi_1 = \phi_1 + \{A_{12} / (A_{11} - A_{22})\} \phi_2 = \begin{bmatrix} \cos \theta + \tan(2\theta) \sin \theta / 2 \\ -\sin \theta + \tan(2\theta) \cos \theta / 2 \end{bmatrix} = \begin{bmatrix} 1 + O(\theta^2) \\ O(\theta^3) \end{bmatrix}.$$

Thus ϕ_1 and ϕ_2 have errors of only $O(\theta^3)$ except normalization. Is it possible, then, to construct the accurate approximate function with errors of $O(\theta^3)$ in general cases, too? Expanding trial functions ϕ_n in terms of ψ_n , we consider the expression $(\phi_n, (A - \lambda_n B) \phi_m)$, assuming discrete spectra only,

$$\phi_m = \phi_m + \sum_{n \neq m} \alpha_{mn} \psi_n, \quad (33)$$

$$(\phi_m, (A - \lambda_n B) \phi_n) = \alpha_{nm} (\lambda_m - \lambda_n) (-)^{(m)} + \sum_{\substack{p \neq n \\ p \neq m}} \alpha_{np}^* \alpha_{np} (\lambda_p - \lambda_n) (-)^{(p)}, \quad m \neq n. \quad (34)$$

We have had $a_{mm} = \alpha_{mm} + O(\theta^2)$ in § 2. Inserting a_{mm} to the second term at the right hand side of (34), it follows that

$$a'_{nm} \equiv a_{nm} - \left\{ \sum_{\substack{p \neq n \\ \neq m}} a_{mp}^* a_{np} (\lambda_p^{(e)} - \lambda_n^{(e)}) (-)^{(p)} \right\} / \{ (\lambda_m^{(e)} - \lambda_n^{(e)}) (-)^{(m)} \} = a_{nm} + O(\delta^3).$$

Then inverting (33) and considering the order of the magnitude of various quantities, we have

$$\phi'_n \equiv \phi_n - \sum_{p \neq n} (a'_{np} - \sum_{\substack{q \neq n \\ \neq p}} a_{nq} a_{qp}) \phi_p = \text{const} \times \phi'_n + O(\delta^3),$$

and $\lambda'_n \equiv (\phi'_n, A\phi'_n) / (\phi'_n, B\phi'_n) = \lambda_n + O(\delta^6).$

We can not however proceed to higher degree of approximation so long as λ_n is replaced by $\lambda_n^{(e)}$. As an example the same problem as in a) will be treated by this method. The quantity which are necessary to find the lowest eigenvalue are calculated readily as follows:

$$a_{12} = a'_{12} = 5/32, \quad a_{32} = -21/32, \quad a'_{13} = 5 \cdot (7)^2 / (2)^{12}.$$

Then it follows that

$$\phi'_1 = 22x + 15x^2 + (40/147)(3x - 12x^2 + 10x^3),$$

and $\lambda'_1 = 1.44588, \quad \Delta\lambda'_1 / \lambda_1 \sim 0.00005.$

This shows the correctness of our prediction: $\Delta\lambda'_1 / \lambda_1 \sim (20\%)^6 \sim 0.00006.$

The extensions to degenerate cases or to continuous spectra can be carried out in similar ways.

§ 5. Solution of continuous states by integral equation

In the preceding sections we have discussed the evaluation of discrete eigenvalues. We also constructed a wave function (27·b) (28·b) of continuous spectra with stationary character. The quantity to be solved in continuous spectra is not eigenvalues but, for example, phase shifts or scattering amplitudes. However, the discussions are the same as in § 2 and § 3 in some cases, for example, a case is to determine the depth of a potential from given phase shift at certain energy. In the present and next sections we will discuss the super-stationary method for the phase shift. The variational method for the phase shift based on differential equation was found earlier than that based on integral equation, but we will start from the latter, for this has a nature similar to the eigenvalue problem. For S -scattering by a central force, the equation to be solved is

$$\psi(r) = \sin kr + \lambda \int_0^\infty G(r, r') V(r') \psi(r') dr', \quad (35)$$

$$G(r, r') = k^{-1} \sin(kr_<) \cos(kr_>), \quad r_< \equiv \text{smaller one of } (r, r'), \quad r_> \equiv \text{larger one of } (r, r').$$

For sufficiently large r , $\psi(r)$ tends asymptotically to

$$\psi(r) \rightarrow \sin kr + \tan \delta \cos kr = \sin(kr + \delta) / \cos \delta, \quad (36)$$

where

$$k \tan \delta \equiv \lambda \int_0^\infty \sin kr' V(r') \psi(r') dr'. \quad (37)$$

If $k^{-1} \tan \delta \neq 0$, (35) is written in the form

$$\psi(r) = \lambda \int_0^\infty \{\sin kr \sin kr' / (k \tan \delta) + G(r, r')\} V(r') \psi(r') dr',$$

$$\text{or} \quad \psi(r) - \int_0^\infty G(r, r') \lambda V(r') \psi(r') dr' = k \cot \delta \int_0^\infty (k^{-1} \sin kr_<) (k^{-1} \sin kr_>) \lambda V(r') \psi(r') dr'. \quad (38)$$

(38) is abbreviated in

$$A\psi = k \cot \delta \cdot B\psi, \quad (39)$$

where $A \equiv V(r) - V(r)P$, $N \equiv V(r)Q$, P, Q ; the integral operators defined by (38), (39). Then $k \cot \delta$ turns out to be an eigenvalue in the eigenvalue problem (39). The expression for $k \cot \delta$: $(\psi, A\psi) / (\psi, B\psi)$ is well-known to be stationary for any variations of ψ , for which the scalar product can be defined. This method was applied by Schwinger to nuclear scattering and led to the excellent effective range theory⁷ of nuclear forces. The formulation based on the integral equation (35) has been reduced to the form similar to the ordinary eigenvalue-problem, but there is a difficulty for applying our variation-perturbation method. Because the eigenvalue of Eq. (39) is only one, and the eigenfunction corresponding to it does not constitute a complete set. We can not use the expansion technique as done in the preceding sections.

a) First method

We must start from some different point of view. To do this (38) is changed as

$$\psi(r) = (\lambda/k) \int_0^\infty \sin kr_< (\cos kr_> + \cot \delta \sin kr_>) V(r') \psi(r') dr'. \quad (40)$$

If δ is a given quantity, (40) is an eigenvalue equation, whose eigenvalues are $\lambda_n (n=1, 2, \dots)$ and corresponding eigenfunctions are $\psi_n(r)$. These $\psi_n(r)$ constitute a complete set for weight function $V(r)$. Now in our case δ is unknown, while one of λ_n is a given quantity which we put λ_1 . ($\lambda_1 = \lambda$.) The procedure of applying the super-stationary method is as follows: Let a trial function corresponding to λ_1 be $\phi_1(r)$. Using a stationary expression for δ , we get an approximate value of δ , for which we may take any expression, for example, the Schwinger expression for $k \cot \delta$. Let the value thus obtained be $\delta^{(1)}$.

$$k \cot \delta^{(1)} \equiv (\phi_1, A\phi_1) / (\phi_1, B\phi_1) = \{\lambda_1^{-1} \int_0^\infty V(r) \phi_1^2(r) dr - \int_0^\infty dr V(r) \phi_1(r)\} / \{ \int_0^\infty G(r, r') V(r') \phi_1(r') dr' \} / \{k^{-1} \int_0^\infty V(r) \phi_1(r) \sin kr dr\}^2. \quad (41)$$

Substituting this $\delta^{(1)}$ to (40) instead of δ , there are the eigenvalues λ_n and corresponding eigenfunctions ψ_n ,

$$\psi_n(r) = (\lambda_n/k) \int_0^\infty \sin kr_< (\cos kr_> + \cot \delta^{(1)} \sin kr_>) V(r') \psi_n(r') dr'. \quad (42)$$

Let the order of errors of $\phi_1(r)$ be Δ , then

$$\phi_1(r) - \psi_1(r) = O(\Delta), \quad \cot \delta^{(1)} - \cot \delta = O(\Delta^2),$$

hence
$$A_n - \lambda_n = O(\Delta^2), \quad \Psi_n(r) - \phi_n(r) = O(\Delta^2). \quad (43)$$

We will apply the method in § 2 to (42). Let trial functions close to $\Psi_n(r)$ be $\Phi_n(r)$. Assuming $\phi_n(r) - \Psi_n(r) = O(\Delta)$, approximate values of A_n are calculated, for example, from the following stationary expression of A_n :

$$A_n^{(1)} = \left\{ \int_0^\infty \phi_n^2(r) V(r) dr \right\} / \left\{ \int_0^\infty dr \phi_n(r) V(r) \right. \\ \left. \int_0^\infty k^{-1} \sin kr_< (\cos kr_> + \cot \delta^{(1)} \sin kr_>) V(r') \phi_n(r') dr' \right\},$$

and then $A_n^{(1)} - A_n = O(\Delta^2)$. Constructing $\Phi_n(r)$ as in § 3, we get

$$\Psi_n(r) - \Phi_n(r) = O(\Delta^2), \quad \Phi_n(r) \equiv \phi_n - \sum_{m \neq n} a_{nm} \phi_m, \quad (44)$$

where a_{nm} is the same as (9). To avoid misunderstanding, we write out A_{nm} and B_{mn} appeared in (9) explicitly,

$$A_{nm} = \int_0^\infty \phi_n(r) V(r) \phi_m(r) dr, \\ B_{nm} = \int_0^\infty dr k^{-1} \phi_n(r) V(r) \int_0^\infty \sin kr_< (\cos kr_> + \cot \delta^{(1)} \sin kr_>) V(r') \phi_m(r') dr'.$$

Seeing (43) and (44) we have

$$\Phi_1(r) - \psi_1(r) = O(\Delta^2).$$

Then we finally get the following result.

$$k \cot \delta^{(2)} \equiv (\Phi_1, A\Phi_1) / (\Phi_1, B\Phi_1), \quad k \cot \delta^{(2)} = k \cot \delta + O(\Delta^4). \quad (45)$$

We should notice that the estimations of errors done in this section refer to the weight function $V(r)$, so in outside region of $V(r)$, the error of trial function does not affect the results. In order to extend the degree of approximation further, we can iterate this procedure as in a) of § 4.

(Example) Consider the same example discussed in § 4. The problem is: When depth of exponential potential is $3/2$, find the scattering length $a(=1/\alpha)$, which is the reciprocal value of $(-k \cot \delta)$ when k approaches to zero: $\lambda_1 \equiv \lambda = 3/2$ and $V(r) = e^{-r}$. The asymptotic form of trial function is not important as mentioned above, so we can take the same function in § 4.

$$\Phi_n = (-x/n!) d^n/dx^n \cdot [x^{n-1}(1-x)^n], \quad x \equiv 1 - e^{-r},$$

and also can take $\phi_1 = \Phi_1$. Then from the Schwinger variational method (41) we get

$$\alpha^{(1)} = 1/81 = 0.01235.$$

The correct value is $\alpha = 0.0144$. Computing A_{mn} , B_{mn} , we get $a_{12} = 0.1410$. Then from (44) and (45), $\alpha^{(2)} \equiv (\Phi_1, A\Phi_1) / (\Phi_1, B\Phi_1) = 0.01413$.

b) *Second method*

The fundamental differential equation (46) can be changed into an alternative form of integral equation (47).

$$(-d^2/dr^2 - \lambda V(r))\psi(r) = k^2\psi(r), \quad \psi(0) = 0, \quad (46)$$

$$\psi(r) = k^2 \int_0^l G(r, r') \psi(r') dr',$$

$$(-d^2/dr^2 - \lambda V(r))G(r, r') = \delta(r - r'), \quad G(0, r') = 0, \quad G(l, r') = 0, \quad (47)$$

where $l = (n\pi - \delta)/k$, and n is the smallest integer for which $V(r) > l \neq 0$. This is an eigenvalue equation with eigenvalues k_n^2 and a set of ψ_n which is complete in the region $0 < r < l$. Then we can proceed in a similar way as above by replacing δ as $\delta^{(1)}$. But this procedure is useless for $G(r, r')$ is in general difficult to express in analytical form.

c) *Third method*

We can take $\phi(k)$ of (28·b) as trial function, and use the formula (45). ϕ_m and $\phi(k)$ which are used to construct $\phi(k)$, have errors of $O(\Delta)$, the error of $\delta^{(2)}$ obtainable from (45) is $O(\Delta^4)$.

§ 6. Solution of continuous states by differential equation

In this section let us consider the scattering problem by differential equation. The radial equation for S -wave scattering by a central potential $\lambda V(r)$ is

$$(-d^2/dr^2 - k^2)\psi(r) = \lambda V(r)\psi(r), \quad \psi(0) = 0, \quad (48)$$

Defining operators A and B as

$$A = -d^2/dr^2 - k^2, \quad B = V(r), \quad \psi(0) = 0,$$

and scalar products as

$$(\psi, A\psi) \equiv \int_0^\infty \{\psi^*(r) \cdot A\psi(r)\} dr, \quad \text{etc},$$

and normalizing the correct function ψ and a trial function ϕ as

$$\begin{aligned} \psi(r) &\rightarrow \cos kr + \cot \delta \sin kr, \\ \phi(r) &\rightarrow \cos kr + \cot \delta^{(0)} \sin kr, \end{aligned} \quad (r \rightarrow \infty) \quad (49)$$

we get⁸⁾

$$k \cot \delta = k \cot \delta^{(0)} + (\phi, (A - \lambda B)\phi) - (\Delta\phi, (A - \lambda B)\Delta\phi), \quad (50)$$

where $\delta^{(0)}$ is the phase shift of the trial function ϕ , so in general

$$\Delta\phi \equiv \phi - \psi = O(\Delta), \quad k \cot \delta^{(0)} = k \cot \delta + O(\Delta).$$

Therefore defining $\delta^{(1)}$ as

$$k \cot \delta^{(1)} \equiv k \cot \delta^{(0)} + (\phi, (A - \lambda B)\phi), \quad (51)$$

$\delta^{(1)}$ has stationary character for any variations of ϕ , that is,

$$k \cot \delta^{(1)} = k \cot \delta + O(\mathcal{J}^2). \quad (52)$$

This is the Hulthén variational method.⁽¹⁾ We will extend the variation-perturbation method to this type of variational method. The first method is: We determine l by $l = (n\pi - \delta^{(1)})/k$, where n is the smallest integer for which $V(r > l) = 0$. Instead of (48), we notice the following eigenvalue equation which is defined in the region $0 \leq r \leq l$.

$$(-d^2/dr^2 - \lambda V(r)) \Psi_n = K_n^2 \Psi_n, \quad \Psi_n(0) = \Psi_n(l) = 0. \quad (53)$$

k^2 in (48) is near to one of K_n^2 , which we put K_1^2 . The procedures to be followed hereafter are similar to § 5. A and B are $-d^2/dr^2 - \lambda V(r)$ and unity with conditions $\psi(0) = \psi(l) = 0$. Starting from trial functions Φ_n which are close to Ψ_n , ($\Phi_n - \Psi_n = O(\mathcal{J})$), Φ_1 which satisfy $\Phi_1 - \Psi_1 = O(\mathcal{J}^2)$, can be constructed. As $\Psi_1 - \psi = O(\mathcal{J}^2)$ holds, we get $\Phi_1 - \psi = O(\mathcal{J}^2)$. Though in the variational method of Schwinger's type such prescription as b) in § 5 is not good, because of the complicated Green function $G(r, r')$, in the formulation of Hulthén's type, this method (53) serves for practical use.

The second method corresponds to a) in the Schwinger-type-formulation of § 5. This has been given in reference 2), which we describe here briefly in order to compare with the formulation of Schwinger's type. Consider the following eigenvalue problem:

$$(-d^2/dr^2 - k^2) \Psi_n = \Lambda_n V(r) \Psi_n, \quad \Psi_n(0) = 0, \quad \Psi_n \rightarrow \cos kr + \cot \delta^{(1)} \sin kr, \quad (r \rightarrow \infty) \quad (54)$$

where $\delta^{(1)}$ is defined in (51). Starting from the trial functions Φ_n close to Ψ_n , we can construct Φ_1 for which $\Phi_1 - \Psi_1 = O(\mathcal{J}^2)$ holds, so $\Phi_1 - \psi = O(\mathcal{J}^2)$. In this case A and B are $(-d^2/dr^2 - k^2)$ and $V(r)$ with conditions $\Phi_n \rightarrow \cos kr + \cot \delta^{(1)} \sin kr$, ($r \rightarrow \infty$). Then considering Φ_1 as the trial function, we get

$$k \cot \delta^{(2)} = k \cot \delta^{(1)} + (\Phi_1, (A - \lambda B) \Phi_1) = k \cot \delta + O(\mathcal{J}^4). \quad (55)$$

The calculation in (55) is simpler than that of formulation of Schwinger's type (45), because the Green function does not appear in calculation, but (55) has some trouble that the asymptotic form of trial functions Φ_n must be specified as (54). We will treat the same example as in § 5, in order to compare with Hulthén's and Schwinger's formulations.

$$\Phi_n = (-x/n!) d^n/dx^n \cdot [x^{n-1} (1-x)^n] + (-)^n \alpha r, \quad x \equiv 1 - e^{-r}, \quad n \geq 1. \quad (56)$$

From a Hulthén's variational method: $(\Phi_1, (A - \lambda B) \Phi_1) = 0$, we get

$$\alpha^{(1)} = (9\lambda - 6 - \sqrt{36 + 36\lambda - 15\lambda^2}) / (24\lambda).$$

If we put $\lambda = 3/2$, it follows that $\alpha^{(1)} = 0$, $\alpha_{12} = 0.1563$, then $\alpha^{(2)} = 0.01461$. This $\alpha^{(2)}$ has a comparable accuracy as in Schwinger's formulation, though the accuracy of $\alpha^{(1)}$ of Hulthén's method is inferior to that of Schwinger's method in this example.

The third formulation is quite similar to that of § 5. Taking $\phi(k)$ (28·b) as trial function, we may use the formula (55).

An extension to the scattering problem including tensor forces has been given in reference 2).

We have discussed various applications of our variation-perturbation method. Further extensions to wider problems will be performed in similar way.

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Recoil Effects in the Strong Coupling Theory, II

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In order to take into account the angular correlation between two mesons emitted successively in the calculation of recoil effects of a nucleon, we assume an admixture of S and P states for a trial function of each meson in the center of mass system. An application of our method to the polaron problem is done to estimate the correlation effect numerically, and it is shown that the effect is not so strong in the region of the usual polaron problem. Several approximation methods proposed by Gurari, Lee et al. and Landau concerning the polaron self-energy are discussed in a general point of view.

§ 1. Introduction

Since the conventional perturbation method which has been refined into the covariant formalism was found not applicable directly to the meson theory, many modifications or new approximation methods have been proposed. One of them is Wentzel's strong coupling theory presented before the covariant formalism and the existence of isobar states of a nucleon seems to be favourable to it. As is well known, however, it is very difficult to take account of recoil effects in the strong coupling theory.

Recently Wentzel extended his method to derive the recoil effect in the pseudo-scalar theory.¹⁾ However, it must be noticed that in his treatment the recoil effect was estimated by using the conventional perturbation method and the validity of the method is retained only if the cut-off energy tends to infinity, as pointed by himself in the paper.

On the other hand it has recently become very interesting to treat motion of electrons in the lattice using field-theoretical methods. As Fröhlich pointed out, the coupling constants between an electron and the lattice vibration fields are not weak and the validity of the usual perturbation method seems to break down.²⁾ Formally the case of the interaction between an electron and lattice vibration field corresponds to the so-called neutral scalar coupling theory, so that it seems much simpler than the case of the pion theory. However we can not fix the coordinate of the electron, but its motion plays very essential role in the theory of superconductivity. Therefore it is quite necessary to find the improved treatment where we can take into account the recoil effects in the non-weak coupling case. After Fröhlich several authors proposed some approximation methods.³⁻⁷⁾ Most of them are based on the variational method, in which some kinds of Hartree approximation have been used as trial functions.

The present author developed Tomonaga's intermediate coupling theory to take into

account the relativistic effect in the scalar coupling of the charged scalar meson field, and concluded that the effective coupling constant is the more weakened the stronger coupling due to the recoil effect.⁶⁾ Hereafter the paper will be referred to as I. However, since only the S-wave function is used for trial function in I, the correlation effects between two successive mesons are automatically dropped out. Therefore in this paper we assume an admixture of S and P state wave functions for each meson, and examine the relation between non-correlated terms and correlated terms in the case of the ground state energy of a nucleon. As an analytical expression of the correction by the correlation can not be obtained, some numerical calculations have been performed in the case of the self-energy problem of the polaron. Although our method may be applied to the meson theory, we are content to give some methodological illustration with respect to the behaviour of a particle in the strong coupled field.

The same kind of approach has been done in the case of the neutral scalar field by Lee and Pines, who developed a method assuming a product of S and P state wave functions as a trial function.⁷⁾ However, as will be shown, since the neutral scalar coupling has accidentally very simple property, it seems to us that their method strongly depends on the special property of the neutral scalar coupling.^{3,5,7)}

It is well known that the Hartree approximation is not adequate when the relative motion of each two particles in the system may become important.⁸⁾ Unfortunately, however, we have no other approximation than the variational method when the coupling is strong. Tentatively several authors^{4,5)} and we modify the Hartree approximation introducing some correlation parameters into the trial function. In this paper the relation among such kinds of approximation is discussed and Fröhlich's proposal⁹⁾ is investigated.

§ 2. Mathematical formulation

In order to take into account angular correlation effects between mesons emitted, let us assume a modified wave function for the system in a ground state in the center of mass system as follows:

$$\psi_n = C_n \prod_{\lambda} \left\{ \alpha_0 Y_0 f(k_{\lambda}) + \left(\frac{\alpha_1}{\sqrt{2}} (Y_{1,1} - Y_{1,-1}) + \alpha_2 Y_{1,0} + \frac{i\alpha_3}{\sqrt{2}} (Y_{1,1} + Y_{1,-1}) \right) g(k_{\lambda}) \right\}, \quad (1)$$

where Y_0 and $Y_{1,m}$ are normalized spherical harmonics and $f(k)$ and $g(k)$ are normalized radial wave functions for 'S' and 'P' configuration respectively. Since we normalize parameters α_i as follows

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \quad (2)$$

C_n represents the probability amplitude for finding n mesons around the nucleon. In the evaluation of the total energy it is necessary to calculate the expectation value of the kinetic energy of the nucleon $\mathbf{p}^2/2\mu$ in each Fock space:

$$\frac{1}{2\mu}\langle \mathbf{p}^2 \rangle = \frac{1}{2\mu}\langle \left(\sum_{\lambda}^n \mathbf{k}_{\lambda}\right)^2 \rangle = n \frac{1}{2\mu}\langle k^2 \rangle + \frac{n(n-1)}{2\mu}\langle k_{\lambda}k_{\nu} \cos \omega_{\lambda\nu} \rangle, \quad (3)$$

where $\omega_{\lambda\nu}$ is the angle between two momenta k_{λ} and k_{ν} . It is seen that the correlation effect appears as the second term and has a factor $n(n-1)$. We shall later derive the same formula from the original hamiltonian by means of a canonical transformation assuming the center of mass system at rest.

It is important to notice that the effect of the second term in (3) may play an essential role in the recoil effect and be strong when the coupling constant is very large, because the number of mesons around a nucleon appears in a quadratic form. Lee, Low and Pines estimated the effect of this term using the perturbational technique.⁵⁾ As Fröhlich pointed out, however, such treatment is not valid when the coupling constant is very large, e.g., $\gtrsim 10$ for the polaron problem.

Therefore we shall treat this term in Tomonaga's intermediate coupling formalism, where we take account of all number of mesons even in the ground state. After a simple calculation we obtain

$$\langle k_{\lambda}k_{\nu} \cos \omega_{\lambda\nu} \rangle = \frac{4}{3}\alpha_0^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \left\{ \int k^3 f(k) g(k) dk \right\}^2. \quad (4)$$

Putting $\alpha_0^2 = \alpha$ and

$$\left\{ \int k^3 f_{\pm}(k) g_{\pm}(k) dk \right\}^2 = \langle k_{\pm} \rangle_{fg}^2 = \begin{cases} a' & \text{for } k_{+} \\ b' & \text{for } k_{-}, \end{cases} \quad (5)$$

we have to add the following terms to (7) of I due to this effect :

$$\sum_{n=0}^{\infty} \frac{2\alpha(1-\alpha)}{3\mu} \{ n(n-1)(a'+b')c_{n,n}^2 + ((n+1)na' + n(n-1)b')c_{n+1,n}^2 \}. \quad (6)$$

On the other hand the P wave parts do not contribute to the direct scalar coupling, so that we replace (9) of I to the following expressions :

1) = 4)

$$\sqrt{\alpha} f' \sqrt{n+1} c_{n,n} c_{n+1,n} \left[b - \left\{ \frac{n}{2\mu^2} (\langle k_{+}^2 \rangle + \langle k_{-}^2 \rangle) + \frac{2\alpha(1-\alpha)}{3\mu^2} n(n-1) \right. \right. \\ \left. \left. \times (a' + b') \right\} \int \frac{k^3 f_{+}(k)}{\sqrt{k^2 + \kappa^2}} dk \right], \quad (7)$$

2) = 3)

$$\sqrt{\alpha} f' \sqrt{n+1} c_{n+1,n+1} c_{n+1,n} \left[g - \left\{ \frac{n+1}{2\mu^2} (\langle k_{+}^2 \rangle + \langle k_{-}^2 \rangle) + \frac{2\alpha(1-\alpha)}{3\mu^2} (n+1)n \right. \right. \\ \left. \left. \times (a' + b') \right\} \int \frac{k^3 f_{-}(k)}{\sqrt{k^2 + \kappa^2}} dk \right],$$

where

$$\langle k_+^2 \rangle = \int k^4 (\alpha f_+^2 + (1-\alpha) g_+^2) dk$$

and

$$\langle k_-^2 \rangle = \int k^4 (\alpha f_-^2 + (1-\alpha) g_-^2) dk.$$

If $\alpha=1$ in the above expressions, they correspond to the case of no correlation. Hereafter we can follow the same treatment to solve the algebraic equations which are derived by minimizing the expectation value of the total energy as in I. However, since the charged scalar meson theory has nothing but a methodological meaning, we should like to apply our method to a more realistic problem in order to estimate the correlation effect.

§ 3. The polaron self-energy

As many authors have pointed out, the interaction between an electron in polar crystal and an optical mode of the lattice vibration is not weak but seems to be a good example for intermediate or strong coupling interactions. Fortunately in this case we need only a neutral scalar field and non-relativistic interaction since the energy of phonon, which is almost constant in polar crystals, is negligibly small compared with the electron rest energy. Let us start from the following hamiltonian:

$$H = \mathbf{p}^2/2m + \sum_k \omega a_k^* a_k + l \sum_k (V_k a_k e^{i\mathbf{k} \cdot \mathbf{r}} + V_k^* a_k^* e^{-i\mathbf{k} \cdot \mathbf{r}}), \quad (8)$$

where \mathbf{p} , m and \mathbf{r} are momentum, rest mass and coordinate of the electron respectively, ω is energy of the phonon, and V_k is given as follows,

$$V_k = -\omega i/k (8\pi^2/m\omega)^{1/4}. \quad (9)$$

Here both \hbar and the system volume are taken as units.

Transformation to the center of mass can be done by the well-known Jost transformation,⁹⁾ i.e., the canonical transformation

$$S = \exp \{ -i (\sum_k \mathbf{k} a_k^* a_k) \cdot \mathbf{r} \}. \quad (10)$$

Then we obtain the transformed hamiltonian

$$S^{-1}HS = 1/2m \cdot (\mathbf{P} - \sum_k \mathbf{k} a_k^* a_k)^2 + \sum_k \omega a_k^* a_k + l \sum_k (V_k a_k + V_k^* a_k^*), \quad (11)$$

where \mathbf{P} is the momentum of the total system and we can take it arbitrarily. For simplicity, we assume $\mathbf{P}=0$. Then the recoil effect appears through $1/2m \cdot (\sum_k \mathbf{k} a_k^* a_k)^2$ which can be rewritten as follows:

$$\begin{aligned} (\sum_k \mathbf{k} a_k^* a_k)^2 &= \sum_k \sum_{k'} \mathbf{k} \cdot \mathbf{k}' a_k^* a_k a_{k'}^* a_{k'} \\ &= \sum_k \sum_{k'} \mathbf{k} \cdot \mathbf{k}' a_k^* (\delta(\mathbf{k}, \mathbf{k}') + a_{k'}^* a_{k'}) a_{k'} \\ &= \sum_k k^2 a_k^* a_k + \sum_k \sum_{k'} \mathbf{k} \cdot \mathbf{k}' a_k^* a_{k'}^* a_k a_{k'}. \end{aligned} \quad (12)$$

If we use the Hartree approximation (1) in Fock space, the expectation value of $1/2m \cdot (\sum k a_k^* a_k)^2$ is given, as in (3), by:

$$\frac{1}{2m} \langle (\sum k a_k^* a_k)^2 \rangle = \frac{1}{2m} \sum_{n=0}^{\infty} n \langle k^2 \rangle c_n^2 + \frac{2\alpha(1-\alpha)}{3m} \sum_{n=0}^{\infty} n(n-1) \langle k \rangle_{fg} c_n^2.$$

For the sake of brevity we introduce the following notations:

$$\left. \begin{aligned} \langle \omega \rangle &= a, \quad 1/2m \cdot \langle k^2 \rangle = b, \\ 2\alpha(1-\alpha)/3m \cdot \langle k \rangle_{fg}^2 &= a', \\ \sqrt{\alpha} l \int (V_k + V_k^*) k^2 f(k) dk &= 2b. \end{aligned} \right\} \quad (13)$$

Then the expectation value of the total energy at the state is written as follows:

$$\langle H \rangle = \sum_{n=0}^{\infty} n(a+b+(n-1)a') c_n^2 + \sum_{n=0}^{\infty} 2b \sqrt{n+1} c_n c_{n+1}. \quad (14)$$

Minimizing $\langle H \rangle$ we obtain the simultaneous equations for c_n ,

$$b \sqrt{n} c_{n-1} + \{n(a+b+(n-1)a') - E\} c_n + b \sqrt{n+1} c_{n+1} = 0. \quad (15)$$

Using dimensionless parameters

$$b/a = V, \quad b/a = \beta, \quad a'/a = \gamma \quad \text{and} \quad E/a = W,$$

we rewrite (15) into

$$V \sqrt{n} c_{n-1} + \{n(1+\beta+(n-1)\gamma) - W\} c_n + V \sqrt{n+1} c_{n+1} = 0. \quad (16)$$

If we neglect the angular correlation term, i.e., if $\gamma=0$, the recoil effect simply shifts the phonon energy. It is well known that the neutral scalar field with scalar coupling without recoil term can be solved exactly, because it is an ensemble of simple harmonic oscillators. However, since the term involving $n(n-1)\gamma$ may be interpreted as a fourth order differential operator, it is not easy to get an exact solution. Therefore we solve (16) numerically to estimate the magnitude of the correlation effect.

In our calculation V , α , β , and γ appear as parameters, and last three of which should be determined to minimize the total energy. However, since we are interested in the effect of angular correlation especially, we perform numerical calculation assuming adequate values for the parameters. For simplicity let us assume a normalized gaussian function for $f(k)$ and $g(k)$:

$$f(k) = g(k) = 2\pi^{-1/4} k_0^{-3/2} e^{-k^2/2k_0^2}. \quad (17)$$

Then we obtain

$$\left. \begin{aligned} \beta &= b/a = \langle k^2 \rangle / 2m \langle \omega \rangle = (1/2m\omega) 3k_0^2/2, \\ \gamma &= \frac{a'}{a} = \frac{2\alpha(1-\alpha) \langle k \rangle^2}{3m \langle \omega \rangle} = \left(\frac{1}{2m\omega} \right) \frac{4\alpha(1-\alpha)}{3\pi} k_0^2, \end{aligned} \right\} \quad (18)$$

$$V = \frac{b}{a} = \sqrt{\alpha} l \left(\frac{1}{2m\omega} \right)^{1/4} \sqrt{4\pi} \left\langle \frac{1}{k} \right\rangle = \sqrt{\alpha} l \left(\frac{1}{2m\omega} \right)^{1/4} \pi^{1/4} k_0^{1/2},$$

where k_0 is the cut-off wave number and is the usual Debye limit. For a typical polar crystal, NaCl, $l^2 = 5.2$ and $(2m\omega)^{1/2} \sim 1.55 \times 10^{-8}$ cm. Therefore it may be plausible to give the following values for the above parameters :

$$\beta = 1, \quad \gamma = 0.1, \quad V^2 = 4 \quad \text{and} \quad 6 \quad (19)$$

The result of the calculation is shown in Figs. 1 and 2. The dotted curves and the chain curves show the non-correlation cases and non-recoil cases respectively. It can be seen that the angular correlation effect is not so large even if we take into account all effects from many virtual phonons. Although this agrees with the result by Lee and Pines⁵⁾ in the region of the coupling constant under consideration our method does not depend on the magnitude of the constant, whereas their method can not be extended beyond $l^2 \approx 10$.⁸⁾

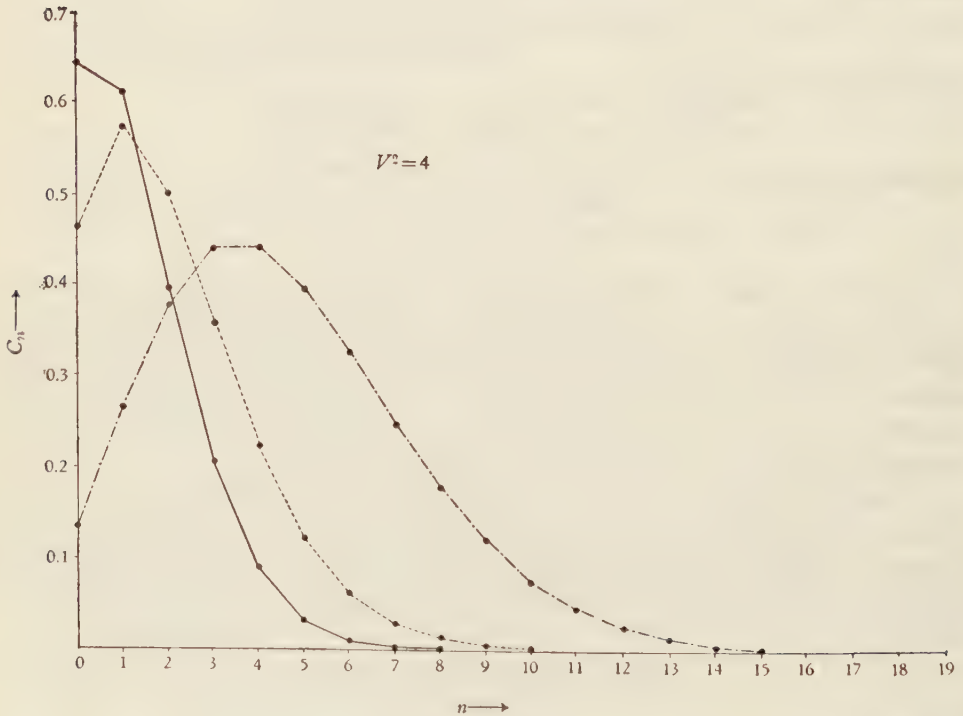
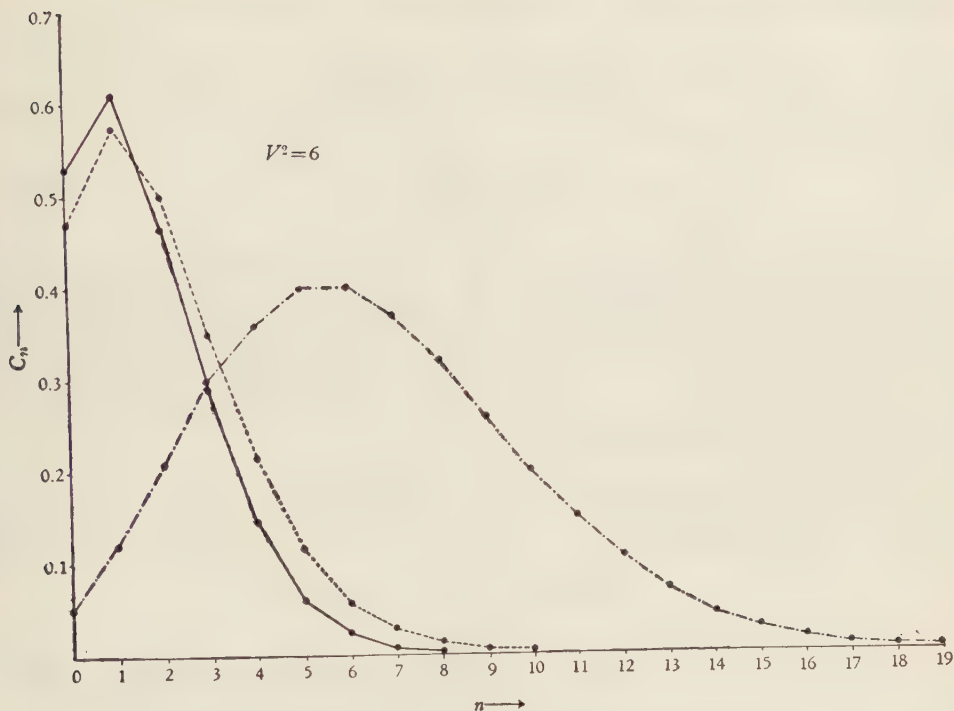


Fig. 1. Probability amplitude for finding n phonons around an electron in the case of $V^2 = 4$:
 ——— recoil effect with correlation
 recoil effect without correlation
 - - - - - no recoil effect


 Fig. 2. The case of $V^2=6$. Notations are the same as in Fig. 1.

§ 4. Discussion

In the non-relativistic approximation the recoil effects appear mainly through the term $1/2m \cdot (\sum k a_k^* a_k)^2$ which consists of non-correlation term $1/2m \cdot \sum k^2 a_k^* a_k$ and correlation term $1/2m \cdot \sum \mathbf{k} \cdot \mathbf{k}' a_k^* a_{k'}^* a_k a_{k'}$ ¹⁰⁾. Lee and Pines tried to explain the 'surprising' result that the exact solution in the strong coupling limit looks like the second order term without the recoil effect in the usual weak coupling perturbation theory.⁷⁾ However it should be noticed that the case of the neutral scalar coupling can be solved exactly if we neglect the correlation term whenever the coupling is weak or strong. Let us rewrite (11) as follows :

$$S^{-1}HS = \underbrace{\sum (\omega + k^2/2m) a_k^* a_k + l \sum (V_k a_k + V_k^* a_k^*)}_{H_0} + \underbrace{1/2m \cdot \sum_k \sum_{k'} \mathbf{k} \cdot \mathbf{k}' a_k^* a_{k'}^* a_k a_{k'}}_{H'}$$

Using the canonical transformation

$$\left. \begin{aligned} \bar{a}_k &= a_k + \frac{l V_k^*}{\omega + k^2/2m}, \\ \bar{a}_k^* &= a_k^* + \frac{l V_k}{\omega + k^2/2m}, \end{aligned} \right\} \quad (20)$$

we can diagonalize H_0 as follows :

$$\bar{H}_0 = \sum \left(\omega + \frac{k^2}{2m} \right) \bar{a}_k^* \bar{a}_k - l^2 \sum \frac{|V_k|^2}{\omega + k^2/2m}.$$

Therefore, as long as we neglect H'

$$E_0 = -l^2 \sum \frac{|V_k|^2}{\omega + k^2/2m} \quad (21)$$

is an exact solution for the ground state, whenever the coupling constant is large or small. Since eq. (21) looks like the usual second order perturbation term which should be valid only in the weak coupling case, Lee and Pines tried to explain the reason in their method. However, it is quite evident in our method. This is due to neglecting of the correlation term H' . In fact, if we transform H' by the same canonical transformation (20), we obtain the following complicated terms :

$$\begin{aligned} \bar{H}' = \frac{1}{2m} \sum \mathbf{k} \cdot \mathbf{k}' & \left[\bar{a}_k^* \bar{a}_{k'}^* \bar{a}_k \bar{a}_{k'} - \bar{a}_k^* \bar{a}_{k'}^* \bar{a}_k \frac{l V_{k'}^*}{\omega + k'^2/2m} - \dots \right. \\ & + \bar{a}_k^* \bar{a}_{k'}^* \frac{l^2 V_k^* V_{k'}^*}{(\omega + k^2/2m)(\omega + k'^2/2m)} + \dots \\ & - \bar{a}_k^* \frac{l^3 V_k |V_{k'}|^2}{(\omega + k^2/2m)(\omega + k'^2/2m)^2} - \dots \\ & \left. + \frac{l^4 |V_k|^2 |V_{k'}|^2}{(\omega + k^2/2m)^2 (\omega + k'^2/2m)^2} \right]. \quad (22) \end{aligned}$$

Clearly these are small in the case of weak coupling. Although it is not easy to diagonalize H' , we find that (22) resembles an expansion of the fourth order perturbation term. On the other hand, in the conventional perturbation theory the angular correlation between two quanta emitted successively in virtual states appears first in the fourth order term in the self-energy problem. Therefore, it is evident that the second order term in a perturbation method⁽¹¹⁾ coincides with the calculation by Gurari¹ and others^{5,7)} who applied Tomonaga's variational method assuming that the angular correlation is small.

The above method is also applicable to the case where the total momentum of the system $\mathbf{P} = \mathbf{p} + \sum \mathbf{k} a_k^* a_k \neq 0$ if we neglect the correlation effect. In the evaluation of the effective mass of the electron Lee, Low and Pines⁵⁾ made an assumption corresponding to

$$\begin{aligned} \sum_k \sum_{k'} \mathbf{k} \cdot \mathbf{k}' a_k^* a_k a_{k'}^* a_{k'} &= \sum_k \left(\sum_{k'} \mathbf{k}' a_{k'}^* a_{k'} \right) \cdot \mathbf{k} a_k^* a_k \\ &= \sum_k \mu \mathbf{P} \cdot \mathbf{k} a_k^* a_k. \quad (23) \end{aligned}$$

The same kind of approximation was used in Gurari's calculation. He modified the Hartree approximation introducing a parameter

$$\tilde{\epsilon} = \left(\sum \mathbf{k}' a_{k'}^* a_{k'} \right) \cdot \mathbf{k} / \left| \sum \mathbf{k}' a_{k'}^* a_{k'} \right| \cdot \mathbf{k}$$

into each phonon wave function. By this modification he takes into account the angular correlation effect.

Finally we should like to touch on the limit of variational principle using a Hartree trial function. The straightforward application of this method to the polaron problem was done by Landau¹²⁾, who reached a different result from those mentioned above. His result depends on the form of the electron density function, and does not contain the recoil effect. Although he treated the phonon field classically and considered the retardation effect due to the electron motion, the recoil by the field was neglected. Therefore, if we want to take the recoil effect correctly into the variational method, we have to modify the Hartree approximation as Gurari or others did. It should be noticed that in our calculation each phonon momentum is measured in the center of mass system, so that we can take into account the recoil effect in our formalism.

Although it seems rather difficult to extend our method to more practical problems, e.g., meson-nucleon scattering, we may expect that the recoil effect can be taken into account by adding some corrections to the nucleon mass or the coupling constant. The author is grateful to Drs. M. Nogami, M. Toda, J. Yamashita for their useful discussions and to Mr. K. Inoue for his assistance in the numerical calculation.

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Errata: for k^4 read k^2 in (8),
for $f'(\sqrt{n} - n\sqrt{n}g')c_{n,n-1} + \dots + f'(\sqrt{n+1}n - n\sqrt{n+1}h')c_{n+1,n}$
read $f'(\sqrt{n}g - n\sqrt{n}g')c_{n,n-1} + \dots + f'(\sqrt{n+1}h - n\sqrt{n+1}h')c_{n+1,n}$
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On the Angular Distribution of Double Pion Production in Pion-Nucleon Collisions at 1.4 Bev

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An angular distribution of doubly produced pions in pion-nucleon collisions in 1.4 Bev is analyzed by the aid of phase space consideration, making some simple assumption on the transition matrix elements of the processes.

§ 1. Introduction and summary

In this paper we intend to perform an analysis of the angular distribution for $\pi-2\pi$ process from the kinematical point of view. In the energy range attained by cosmotron, it will be impossible to apply the methods established in the analysis of low energy phenomena.¹⁾ In spite of such difficult circumstances, it will be noticed that, comparing the results of Eisberg et al.²⁾ with those of Walker et al.³⁾, the common character for angular distribution will be understood by some simple kinematical consideration. A striking feature observed in their experiments is the small deflection of a nucleon in the course of a collision. This leads us to set the following two postulates on the transition matrices of this process:

Postulate I. The matrix element has conspicuously large value, if and only if the nucleon does not change its direction considerably. Then one may put the direction of the final nucleon momentum \mathbf{p} , which appears, for example, in the expression of phase volume, equal to that of the incident momentum \mathbf{p}_0 .

Postulate II. The transition matrix element does not depend strongly upon the direction of the emitted pions.

With these two assumptions, we have examined whether or not the angular distributions of the emitted pions can be explained in terms of phase volume. It is found that the experimental data of Eisberg et al. can be understood fairly well as shown in § 3.

Since the above two postulates are able to explain the important characters of this problem phenomenologically (Postulate I is based directly upon the experimental fact), we take this up as a criterion of existing theories and examine roughly whether or not the postulates are satisfied by the transition matrices calculated from them. The summary of the result is as follows:

- (1) In the case of neutral $P_S(P_S)$ theory, the postulate I is not satisfied by the

matrix element obtained by means of perturbation theory.

(2) Adopting the charge independent $P_S(P_V)$ theory, Kobayashi and Yamanouchi⁴⁾ are examining the $\pi-2\pi$ process by means of perturbation. On the way of their calculation, they have assumed the postulate I and obtained rather flat angular distribution contradicting to the experiments. This tells us the fact that the postulate II is not fulfilled by this model.

(3) If we assume that pion-pion interaction plays a more important role as is done by Kovacs,⁵⁾ both postulates I and II are fulfilled.

§ 2. Experimental evidences

The characteristic features of the double pion production observed by Eisberg et al. and by Walker et al. may be summarized as follows :

(1) In the center of mass system a nucleon does little change its direction through the collision. From the angular distribution of protons for the process $\pi^- + p \rightarrow \pi^- + \pi^0 + p$, the half angle $\vartheta_{1/2}$ of the recoil direction of the protons against the direction of the incident pions is found to be 150° according to Eisberg et al., while approximately 140° according to Walker et al.

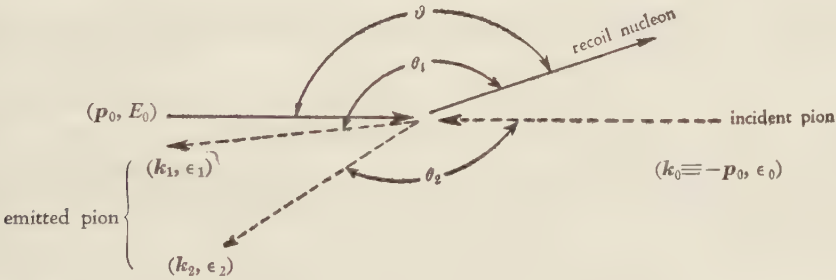


Fig. 1.

(2) Fast pions have a tendency to be emitted in the forward direction, while slow pions in the backward, as has been pointed out by Walker et al., while Eisberg et al. have obtained a similar but less certain evidence for it. Here the words “fast” and “slow” are used to discriminate the pions having larger and smaller momentum than 350 Mev/c respectively.

§ 3. Angular distribution and phase volume

By means of the invariant matrix I introduced by Møller⁶⁾ and of the covariant phase volume dJ' , the differential cross section for the process is expressed as

$$d\sigma = (4\pi^2/v_r) |\langle f|I|i\rangle|^2 dJ', \tag{3.1}$$

$$dJ' = d\mathbf{p}/E \cdot d\mathbf{k}_1/\epsilon_1 \cdot d\mathbf{k}_2/\epsilon_2 \cdot \delta(\mathbf{p} + \mathbf{k}_1 + \mathbf{k}_2) \delta(E + \epsilon_1 + \epsilon_2 - E_0 - \epsilon_0), \tag{3.2}$$

where \mathbf{p} , \mathbf{k}_1 , \mathbf{k}_2 and E , ϵ_1 , ϵ_2 denote respectively the momenta and energies of the nucleon

and the emitted pions in the final state, and E_0 and ϵ_0 indicate the energies of the nucleon and the pion in the initial state respectively. (c. f. Fig. 1)

After the integration over k_2 and k_1 , dJ' is reduced to

$$dJ = \frac{d\mathbf{p}}{E} \frac{k_1 d\Omega_1}{(W_0 - E) + (\epsilon_1/k_1) p \cos \theta_1}, \quad (3.3)$$

where $W_0 \equiv E_0 + \epsilon_0$, $d\Omega_1$ is the element of solid angle for the direction of k_1 and θ_1 is the angle between \mathbf{p} and k_1 .

By the aid of the conservation of energy

$$\sqrt{k_1^2 + \mu^2} = \left(\frac{W_0 - E}{2} \right) \left[1 - \left(\frac{p}{W_0 - E} \right)^2 \right] - \left(\frac{p}{W_0 - E} \right) k_1 \cos \theta_1, \quad (3.4)$$

k_1 is expressed by p and θ_1 . Thus (3.3) is written as

$$dJ = d\mathbf{p}/E \cdot \mathcal{F}(p, \theta_1) d\Omega_1. \quad (3.3)'$$

If the matrix element satisfies both postulates, mentioned in § 1,

$$\mathcal{F}(p, \pi - \theta) \quad (3.5)$$

gives the angular distribution of emitted pions in the collision where the amount of energy transfer is equal to $[W_0 - E(p)]$.

Since Eisberg et al. observed the momenta of the final protons for the process of $\pi^- + p \rightarrow \pi^- + \pi^0 + p$ to be concentrated about $p = 4\mu = 560$ Mev/c (c. f. Fig. 2), let us examine the angular distribution at $p = 3\mu$ and $p = 4\mu$. The angular distribution obtained from the consideration of phase space is found to agree fairly well with that of experiment (c. f. Fig. 3).

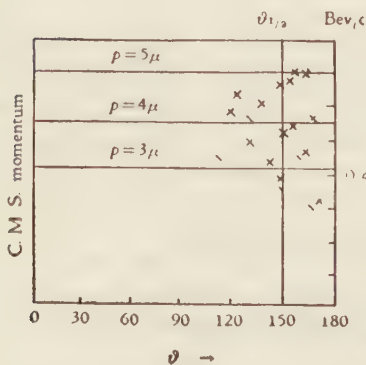


Fig. 2.

Eisberg et al.'s result of angular and momentum distribution of proton from $p + \pi^- \rightarrow p + \pi^- + \pi^0$.

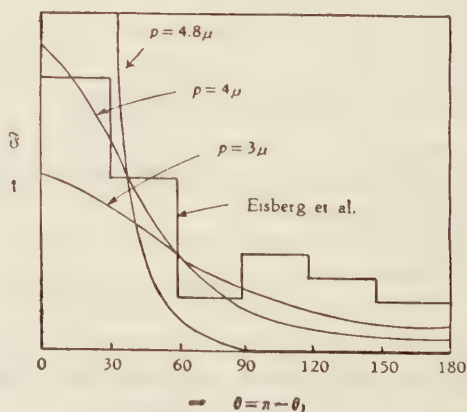


Fig. 3.

Angular distribution of pions taken after the Eisberg et al.'s experiment. Solid curves represent the phase volume predictions.

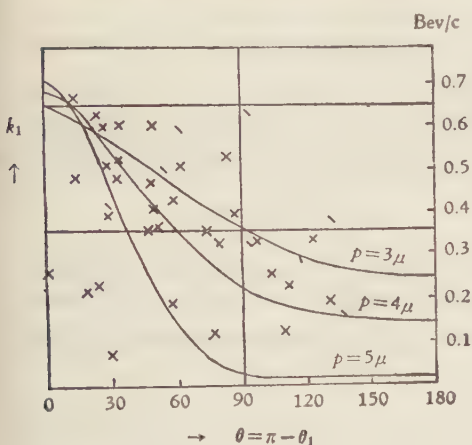


Fig. 4.

Correlation of angular and momentum distribution of π^- and π^0 emitted from the reaction $p + \pi^- \rightarrow p + \pi^- + \pi^0$. Solid curves show the plots of (3.4)' for the values of momentum of the final proton $p = 3\mu$, 4μ and 5μ respectively.

Correlation of the angular distribution with the momentum distribution can be explained to some extent by the simple kinematical investigation.

§ 4. Discussions for transition matrix

We have seen in the previous section that the angular distribution of pions can be interpreted in terms of phase volume provided that the two postulates are satisfied. Since the postulate I is the one deduced from the experimental data, we may think that it plays a part of a criterion for selecting the dynamical model to be adopted. To deal with the dynamical model is not the object of this paper, but as an example we will criticize a few simple models from the above point of view.

(1) Perturbation theory with $Ps(Ps)$ theory

In this case, the main term of the matrix elements has the following form;

$$\left[\frac{(k_0 \cdot k_1)}{(p_0 k_0)(p k_1)} + \frac{(k_0 \cdot k_2)}{(p_0 k_0)(p k_2)} + \frac{(k_1 \cdot k_2)}{(p_0 k_1)(p k_2)} \pm (p_0 \leftrightarrow p) \right] \times (\bar{u}(\mathbf{p}) \gamma_5 u(\mathbf{p}_0)). \quad (4.1)$$

This is in contradiction to our postulates as shown in what follows:

(i) Owing to the factor $|\bar{u}(\mathbf{p}) \gamma_5 u(\mathbf{p}_0)|^2 \sim (\mathbf{p} - \mathbf{p}_0)^2 / M^2$, the matrix element has large value when \mathbf{p} is antiparallel to \mathbf{p}_0 . This contradicts directly with the postulate I. This is because the spread of the pion cloud surrounding a nucleon is of the order $1/M$ for the sake of the γ_5 interaction, and, even when the momentum of the incident pion is in cosmotron region ($\sim M$), only the partial waves of small angular momenta contribute to the process concerned and consequently the nucleon changes its direction considerably in this process.

Here, we have to bear in mind the fact that the angular distribution obtained by Walker et al. shows the presence of too large backward scattering to be explained by the phase volume argument alone. If their result is confirmed, it will not be due to the effect of phase volume, but to the character of the matrix element, such as "final state interaction." In any way, the present experiments are not precise enough to be discussed in such details.

Next problem to be examined is the correlation of the angular distribution with the momentum distribution. As k_1 is a function of p and θ_1 (c. f. (3.4)),

$$k_1 = k_1(p, \theta_1). \quad (3.4)'$$

Fig. 4 shows this relation for the case of $p = 3\mu$, 4μ and 5μ . From these results it may be said that the correlation of the angular

(ii) The factor $\sum_{i,j} (k_i k_j) / \{ (p_i k_i) (p_j k_j) \}$ has a tendency to emit pions in the direction of propagation of the nucleon. This circumstance is due to the fact that the self-field around a nucleon with high velocity suffers Lorentz contraction to its direction of propagation and pions are easily emitted in the direction of the nucleon. This makes it difficult to understand the experimental data.

(2) *Perturbation theory with $P_S(P_V)$ theory*

Adopting this model and assuming the postulate I on the way of calculation, Kobayashi and Yamanouchi⁽¹⁾ have obtained the isotropic angular distribution of the emitted pions. This is attributed to the circumstance that the effect of phase volume is masked by that of matrix element as in the case of (ii) in (1).

(3) *On pion-pion interaction*

Kovacs⁽⁵⁾ has studied this process in terms of pion-pion interaction of $\lambda\phi^4$ -type (c. f. Fig. 5). In this case,

$$|\langle f | I | i \rangle|^2 \sim [(\mathbf{p}_0 - \mathbf{p}) / \mu]^2 / [(\mathbf{p}_0 - \mathbf{p})^2 / \mu^2 + 1]^2. \quad (4.2)$$

It is remarkable that this expression does not contain any pion variables. The angular distribution of recoil nucleon is approximately given as follows:

$$f(\vartheta) d(\cos \vartheta) \approx \sin^2 \frac{\vartheta}{2} d(\cos \vartheta) / [4 \sin^2 \frac{\vartheta}{2} + (\mu^2 / p_0 p)]^2. \quad (4.3)$$

From this, the half angle $\vartheta_{1/2}$ becomes nearly equal to 140° . These results do not contradict with the experimental data.

The authors should like to express their gratitude to Prof. S. Hayakawa and Dr. H. Hasegawa for their valuable discussions.

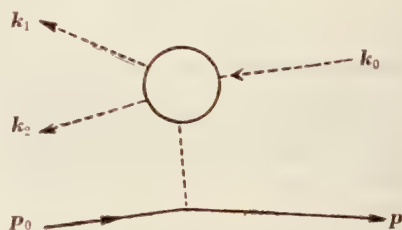


Fig. 5

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Letters to the Editor

Situation of Rarita-Schwinger Particle as a Model of Nucleon Isobar

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September 13, 1955

The nucleon isobar which has played the most important role in some meson reactions at about 200 Mev has been treated in the framework of meson theory by the strong¹⁾, intermediate coupling²⁾ and Tamm-Dancoff approximation.³⁾ On the other hand, it has been treated as if it had phenomenologically the properties of the Rarita-Schwinger particle.⁴⁾

In a sense, the Tamm-Dancoff method is superior to the phenomenological treatment containing the strange model, so called "pseudospinor isobar". But these two models give the similar results in the problem of meson nucleon scattering, and the latter is conveniently used to deal with the problem in relativistically covariant way and to treat the problems of electromagnetic interaction, such as meson production by γ -ray, without any trouble concerning the gauge invariance.

The R.S. method, however, gives results considerably different from those obtained by the T.D. method in the problems of nuclear forces and anomalous magnetic moment of nucleon.⁵⁾⁶⁾⁷⁾ This fact shows that on one hand the R.S. particle describes fairly well the isobar deduced from T.D. approximation, but on the other hand it fails to be its good model.

It is the purpose of this note to examine under what kind of condition R.S. particle can take the place of the isobar deduced from T.D. approximation, and to find out a relation between their coupling constants. For this purpose, let us compare in detail the propagation characters of nucleon isobars in these two methods. (c.f. Fig. 1 and 2.)

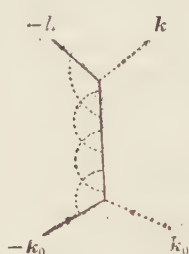


Fig. 1

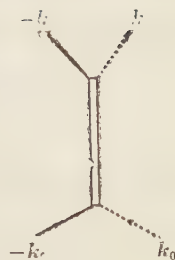


Fig. 2

According to Tamm-Dancoff approximation, the transition matrix $\langle k | t^{T.D.} (3/2, 3/2) | k_0 \rangle$ corresponding to Fig. 1 is given by

$$\langle k | t^{T.D.} (3/2, 3/2) | k_0 \rangle = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_k} \sqrt{2\epsilon_0}} \left(\frac{f}{M} \right)^2 \times \frac{\epsilon_r (4\pi/3) k k_0 E_{3/2} (\phi^* \phi) F_{3/2}(k k_0)}{\epsilon_0 - \epsilon_r},$$

$$\text{where } \frac{1}{\epsilon_r} = \frac{f^2}{4\pi} \frac{4}{3M^2} \int_0^{k_{\max}} \frac{k^4 dk}{\epsilon_k^3 (\epsilon_k - \epsilon_0)}, \quad E_{3/2}, F_{3/2}$$

represent the projection operators on to states of isotopic spin 3/2 and of angular momentum 3/2 respectively, and ϵ_0, ϵ_k denote the energies of mesons in the initial and final states respectively.

The corresponding transition matrix $\langle k | t^{R.S.} | k_0 \rangle$ calculated with the interaction Hamiltonian of R.S. field $\Psi_\mu(x)$, meson $\phi(x)$, and nucleon $\psi(x)$;

$$H(x) = \frac{G}{\mu} \sum_{M=-3/2}^{3/2} \sum_{m,n} \bar{\Psi}_\mu^M(x) \langle 1 \frac{1}{2} \frac{3}{2} M | 1 m \frac{1}{2} n \rangle \times \frac{\partial \phi_m(x)}{\partial x_\mu} \tau_n \psi(x) + \text{c.c.}$$

may be written in the similar form:

$$\langle k | t^{R.S.} | k_0 \rangle = \frac{1}{(2\pi)^3} \frac{1}{4\epsilon_0} \left(\frac{G}{\mu} \right)^2 \frac{\gamma_4 E + M_r}{E^2 - M_r^2} \frac{4\pi}{3} \times \left[k k_0 F_{3/2}(k k_0) - \frac{(E^2 - M_r^2)}{M_r^2} \{ k k_0 F_{1/2}(k k_0) + 3\epsilon_0 F_s \} \right] E_{3/2}(\phi^* \phi),$$

where $F_{3/2}$, $F_{1/2}$ and F_s are projection operators for $p_{3/2}$, $p_{1/2}$ and $s_{1/2}$ states respectively.

For the near resonance energies, that is $\epsilon_0 \approx \epsilon_r$, in T.D. method and $E \equiv M + \epsilon_0 \approx M + \epsilon_r \equiv M_r$ in R.S. method, transition matrices are written as

$$\langle k | t^{\text{T.D.}}(3/2, 3/2) | k_0 \rangle \approx \frac{1}{(2\pi)^3} \frac{1}{4\epsilon_0} \left(\frac{f}{M} \right)^2 \times \frac{(4\pi/3) k k_0 E_{3/2}(\phi^* \phi) F_{3/2}(k k_0)}{\epsilon_0 - \epsilon_r},$$

$$\langle k | t^{\text{R.S.}}(3/2, 3/2) | k_0 \rangle \approx \frac{1}{(2\pi)^3} \frac{1}{4\epsilon_0} \left(\frac{G}{\mu} \right)^2 \times \frac{(4\pi/3) k k_0 E_{3/2}(\phi^* \phi) F_{3/2}(k k_0)}{\epsilon_0 - \epsilon_r}.$$

When we set the following relations between f and G besides M_r and ϵ_r , the results obtained by T.D. method turn out to be equal to those by R.S. method so far as the phenomena in the neighbourhood of resonance is concerned,

$$G^2/4\pi = (f^2/4\pi) (\mu/M)^2,$$

$$M_r = M + \epsilon_r.$$

From this relation, in the case of $f^2/4\pi = 15$ the corresponding $G^2/4\pi$ amounts to 0.335.

It should be noted that, if the energy is far apart from resonance, the propagator of R.S. particle contains $p_{1/2}$ - and s -waves as well as $p_{3/2}$ -wave. Thus in these energy region, R.S. particle can no longer be a substitute for the isobar deduced from T.D. method. We cannot help saying that it will be a matter of course to reach the different results by these two methods in the problems of nuclear forces and of anomalous magnetic moment of nucleons.

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Variation Principle in Relativistic Hydrodynamics

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Recently Halatnikov¹⁾ has given a variation principle to derive the equations of the relativistic hydrodynamics. His procedure is based on the analogy to the relativistic particle dynamics. In this note it will be shown that an alternative variation method, which closely resembles that of field dynamics, can describe the behaviour of an ideal compressible fluid if one restricts oneself to the two limiting cases of extremely high and of extremely low temperatures.

The law of energy-momentum conservation for the compressible fluid is expressed by²⁾

$$\partial T_{\lambda\nu} / \partial x_\lambda = 0 \quad (1)$$

with

$$T_{\lambda\nu} = (\epsilon + p) u_\lambda u_\nu + g_{\lambda\nu} p, \quad (2)$$

where $g_{\lambda\nu}$'s and u_λ 's are the components of the metric tensor and those of the four-velocity of the fluid respectively. ϵ is the internal energy (including rest energy) density per unit volume in the local rest system and p is the pressure. Further we introduce thermodynamical quantities*: the temperature θ , the entropy density s , the chemical potential per particle μ and the number of particles per unit volume ρ . These are connected with the formula

$$\epsilon + p = \theta s + \mu \rho. \quad (3)$$

For a reversible process we have

$$d\epsilon = \theta ds + \mu d\rho, \quad (4)$$

which, when combined with (3), gives

$$dp = s d\theta + \rho d\mu. \quad (5)$$

We can easily derive from these equations

* The international system of transliteration of cyrillic characters has been used throughout this note. Cf. Nuovo Cimento, 10 (1953), Supplemento, 539.

** The densities s and ρ are referred to the rest coordinate of the small part of the fluid considered.

$$\Theta \frac{\partial(su_\lambda)}{\partial x_\lambda} + \mu \frac{\partial(\rho u_\lambda)}{\partial x_\lambda} = 0. \quad (6)$$

The relation (6) indicates that at a sufficiently low temperature, $\Theta \approx 0$ ($\mu \neq 0$), one has the so-called continuity equation (conservation of the particle number)

$$\partial(\rho u_\lambda)/\partial x_\lambda = 0, \quad (7)$$

while at an extremely high temperature, where particles and antiparticles can be created or annihilated almost freely, one may put $\mu \approx 0$ ($\Theta \neq 0$), thus obtaining

$$\partial(su_\lambda)/\partial x_\lambda = 0. \quad (8)$$

These two cases can therefore be treated completely in parallel to each other, as was already pointed out by Halatnikov.¹⁾ In the following I shall write down, for definiteness, the case of high temperature fluid, but all the relations are valid also at very low temperatures, only if the substitution $\Theta \rightarrow \mu$, $s \rightarrow \rho$ is made.

The set of equations (1), (2) and (3), (4), (5) with $\mu=0$ (which implies (8)) can be obtained by a variation principle:

$$\delta \int L d^4x = 0, \quad (9)$$

where we define L , by slightly modifying the non-relativistic case³⁾, as follows:

$$L = s \left\{ \frac{\Theta}{2} u_\nu u^\nu - u_\nu \left(\frac{\partial \phi}{\partial x_\nu} + \alpha \frac{\partial \beta}{\partial x_\nu} \right) - \frac{\Theta}{2} \right\} + p. \quad (10)$$

In (9) u_ν , ϕ , α , β and Θ should be varied with the boundary values being fixed. According to the assumption $\mu=0$, p and s can be regarded as functions of Θ only. Thus one can put for example,

$$p = F(\Theta), \quad s = F'(\Theta), \quad (11)$$

the latter relation is due to (5), $dp = sd\Theta$ ($\mu=0$).

First, the variation of u_ν yields, assuming $s \neq 0$,

$$\Theta u_\nu = \partial \phi / \partial x^\nu + \alpha \cdot \partial \beta / \partial x^\nu. \quad (12)$$

This is a generalization of Clebsch's expression for the velocity field⁴⁾.*

Since u_ν represents the four-velocity, (12) is consistent only when

$$(\partial \phi / \partial x_\nu + \alpha \cdot \partial \beta / \partial x_\nu) (\partial \phi / \partial x^\nu + \alpha \cdot \partial \beta / \partial x^\nu) = -\Theta^2. \quad (13)$$

But this is certainly the case, because the variation of Θ results in

$$\frac{F'(\Theta) - \Theta F''(\Theta)}{\Theta^2} \left\{ \left(\frac{\partial \phi}{\partial x_\nu} + \alpha \frac{\partial \beta}{\partial x_\nu} \right) \left(\frac{\partial \phi}{\partial x^\nu} + \alpha \frac{\partial \beta}{\partial x^\nu} \right) + \Theta^2 \right\} = 0, \quad (14)$$

if (11) and (12) are taken into account. We may impose the condition that $F'(\Theta) = s$ shall not be a linear function**, which implies $F' - \Theta F'' \neq 0$, and then (14) is equivalent to (13).

Next we get, varying ϕ , the entropy conservation law

$$\partial(su_\nu)/\partial x_\nu = 0. \quad (15)$$

Further, variations of α and β give respectively

$$d\beta/d\tau = 0 \quad (16)$$

and

$$d\alpha/d\tau = 0, \quad (17)$$

where

$$d/d\tau \equiv u_\nu \partial / \partial x_\nu \quad (18)$$

means the so-called substantial derivative, or derivative with respect to "proper time" $\tau = (-x_\nu x^\nu)^{1/2}$. In deriving (16) and (17) we have made use of (15) and also assumed $s \neq 0$. These equations indicate that $\alpha = \text{const.}$ and $\beta = \text{const.}$ represent a generalized vortex line.

The equations (12), (13), (15), (16) and (17) just represent the relativistic motion of fluid with $\mu=0$. Indeed the behaviour of the fluid in this case is described by the entropy conservation law (15) and the following equation¹⁾:

$$\frac{d}{d\tau} \Theta u_\nu + \frac{\partial \Theta}{\partial x^\nu} = 0. \quad (19)$$

But we have, after some calculation using (12) and (13),

* The case with $\alpha=0$ was called quasi-potential motion by Halatnikov.¹⁾ It is well-known that, for a specified vector field Θu_ν , ϕ , α and β cannot be determined uniquely.⁵⁾ For example, a simple "gauge transformation"

$$\phi \rightarrow \phi' = \phi - \beta k,$$

$$\alpha \rightarrow \alpha' = \alpha + k,$$

$$\beta \rightarrow \beta' = \beta,$$

$$k = \text{const.}$$

leaves Θu_ν unchanged. But such an invariance does not lead to any other physical consequence than those obtained by variation.

** This is satisfied by ordinary fluids.

$$\frac{d}{d\tau} \theta u_\nu + \frac{\partial \theta}{\partial x^\nu} = \frac{d\alpha}{d\tau} \frac{\partial \beta}{\partial x^\nu} - \frac{d\beta}{d\tau} \frac{\partial \alpha}{\partial x^\nu}$$

and this expression vanishes owing to (16) and (17).

The Lagrange density function (10) can also be expressed as

$$L = -\frac{F'(\theta)}{2\theta} \left(\frac{\partial \phi}{\partial x_\nu} + \alpha \frac{\partial \beta}{\partial x_\nu} \right) \left(\frac{\partial \phi}{\partial x^\nu} + \alpha \frac{\partial \beta}{\partial x^\nu} \right) - \frac{\theta F'(\theta)}{2} + F(\theta). \quad (20)$$

Here one has to take variation with respect to ϕ , α , β and θ and assume that the velocity field is given by (12).

In both forms, the Lagrange density function takes the value

$$L = F(\theta) = p, \quad (21)$$

when the equations of motion hold. This relation should have been expected since it is known to be valid in the non-relativistic case and moreover both L and p are Lorentz-invariant quantities.

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On the Binding Energies of Very Light Nuclei

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September 21, 1955

So far, little work has been done on H^3 and He^4 basing on the independent particle model or other nuclear models usually discussed in the nuclear theory. It is because these nuclear models are thought to be applicable to nuclei with relatively

large number of nucleons but not to very light nuclei without serious modification.

Even if we assume that only two body forces, whatever meson-theoretical or phenomenological, act between nucleons in a nucleus, we can not analytically solve the Schrödinger equation for it in general. So if we had tried to solve the equation, we had to use the variational method for three- or four-particle problem. On the other hand, according to Post,¹⁾ we can get an analytical solution for the Schrödinger equation of any nucleus, if we take an oscillator potential, $V_{ij} = (\hbar^2/2M) \nu^2 r_{ij}^2 - V_0$, as the interaction between two nucleons, under the condition that the wave function of the system does not contain the centre of mass coordinate. The solution for the ground state of a nucleus is easily written down as follows:

$$\Psi_E = e^{\frac{\nu}{2\sqrt{A}} (\sum_{i,j} r_{ij}^2)} \sum_{(n_1 \dots n_A)} c_{n_1 n_2 \dots n_A} \phi_{n_1}(A^{1/4} \mathbf{r}_1) \dots \phi_{n_A}(A^{1/4} \mathbf{r}_A) \quad (A: \text{mass number}),$$

where ϕ_n is a single particle wave function for the harmonic oscillator potential and the summation is taken over all the combinations of n_1, n_2, \dots, n_A giving the total energy E . We can easily express this wave function in terms of the relative coordinates of the nucleons. Formally we can apply this model to all nuclei of any mass numbers. For the three or four-nucleon system, this ground state wave function is $\Psi = N^{1/2} e^{-\frac{\nu}{2\sqrt{A}} \sum_{i,j} r_{ij}^2}$.

This procedure may be too fantastic to apply to any real nucleus. Especially it may not be suitable for heavy nuclei to approximate the interaction between nucleons in a nucleus by an oscillator potential. For very light nuclei, however, the approximation by this potential may be allowed, because, for very light nuclei, the probability of the distance between two nucleons becoming larger than the range of true nuclear forces is so small that the effect of the infinite wall of the oscillator potential would not be decisive. Of course, it is the oscillator potential as a whole, not the bottom part of it, that approximates the exact nuclear forces. The fact that the extent of the nuclear wave function assuming this two body oscillator potential is proportional to $A^{1/4}$, which is not so different from the usual $A^{1/3}$ dependence of the nuclear radius for very light nuclei, seems to be a support to our treatment.

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According to these considerations, we apply this picture to very light nuclei such as H^3 , He^3 and He^4 and study their binding energies, Coulomb energies and nuclear radii. Also we investigate the properties of their wave functions indicated by the photodisintegration experiments.²⁾

For this purpose, we determine the oscillator potential between two nucleons as follows: We assume a relation $V_{ij}^s = qV_{ij}^t$ between the triplet even state potential V_{ij}^t and the singlet even state potential V_{ij}^s , and put $q=0.6$. Moreover we assume for simplicity that the odd state potentials vanish. Then we can express $V_{ij} = [(3+q)/4 + (1-q)/4 \cdot (\sigma_i \cdot \sigma_j)] V_{ij}^t$. V_{ij}^t has two parameters, ν and V_0 . We determine them according to the following conditions: (I) The energy of the ground state of V_{ij}^t is equal to the binding energy of the deuteron, B. From this condition, V_0 is expressed as $V_0 = B + (3/\sqrt{2}) (\hbar^2/2M) \nu$. (II) The wave function for V_{ij}^t resembles to that of the deuteron. For this purpose we take the following prescription. The wave function $u(r)^{in} = A \exp [(-\nu/2\sqrt{2}) r^2]$ for V_{ij}^t continues smoothly to the outside wave function of the deuteron $u(r)^{out} = A' \cdot \exp(-\gamma_t r)$ at $r=r_0$. From this condition, we get a relation $\nu = \sqrt{2} (\gamma_t r_0 + 1)/r_0^2$. It may be reasonable to take $r_0 = 2 \sim 3 \times 10^{-13} \text{cm}$ because r_0 can be identified as the usual force range. We use $B = 2.227 \text{ Mev}$ and $\gamma_t = 0.232 \times 10^{13} \text{cm}^{-1}$.

The values of ν and V_0 for V_{ij}^t are tabulated in Table I.

Table I. The nuclear parameters for oscillator potential

$r_0 (\times 10^{-13} \text{cm})$	$(\hbar^2/2M)$	$V_0 (\text{Mev})$
2	10.69	24.92
3	5.52	13.93

The potential V_{ij} determined by this prescription is spin-dependent, so that we can not apply the Post method directly to this potential. Therefore we average V_{ij} over the spin states of a nucleus, namely we calculate $\langle \psi | (\sigma_i \cdot \sigma_j) | \psi \rangle_{\text{spin}}$ and average it over the number of the nucleon pairs.

Neglecting the Coulomb energy, the binding energies of H^3 and He^4 are given by

$$E(H^3) = 3\sqrt{3} \sqrt{0.8} (\hbar^2/2M) \nu - 2.4V_0,$$

$$E(He^4) = 18\sqrt{0.8} (\hbar^2/2M) \nu - 4.8V_0,$$

respectively. We calculate the Coulomb energy as a perturbation. The results of the calculations are summarized in Table II, III and IV. The results due to the variational method by Irving³⁾ using the exponential- and Yukawa-potential are tabulated in comparison. According to these tables our results agree fairly well with the experimental data. Especially, contrary to the results of variational calculations, the binding energies of H^3 and He^4 are consistently given for nearly equal values of r_0 , and H^3 - He^3 mass difference also agrees with the experimental value and He^4 has the larger radius than H^3 .

Table II. The binding energies of H^3 and He^4 (Mev)

Potential	$r_0 (\times 10^{-13} \text{cm})$	H^3 B.E. = 8.492 Mev			He^4 B.E. = 28.2 Mev		
		K.E.	P.E.	B.E.	K.E.	P.E.	B.E.
Oscillator	2	5.236	15.354	10.118	9.084	-42.634	33.550
	3	2.703	10.492	7.789	4.686	-27.134	22.448
Exponential		29.1	-36.7	7.6	70.0	-102.9	31.9
	$V(r) = -A \exp(-\kappa r)$, $1/\kappa = 0.856 \times 10^{-13} \text{cm}$, $A = 123.3 \text{ Mev}$, $q = 0.6$						
Yukawa		56.5	-65.0	8.5	200.4	-255.4	55.0
	$V(r) = -A \exp(-\kappa r)/\kappa r$, $1/\kappa = 1.17 \times 10^{-13} \text{cm}$, $A = 67.3 \text{ Mev}$, $q = 0.69$						

The authors wish to express their cordial thanks to Professors M. Kobayasi, S. Nakamura and S. Takagi for their valuable discussion about this problem and other members in their laboratory for their

continual encouragement. They also wish to express their thanks to the Yukawa Yomiuri Fellowship for the financial aid.

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Table III Coulomb energies of He^3 and He^4

Potential		He^3	He^4
Osc.	$r_0 = 2 \times 10^{-13} \text{cm}$	1.09	0.826
	$r_0 = 3 \times 10^{-13} \text{cm}$	0.783	0.594
Exponential		1.02	1.00
Yukawa		1.13	1.73
Experiment		0.764	

Table IV. The nuclear radii

Potential		H^3	He^4
Osc.	$r_0 = 2 \times 10^{-13} \text{cm}$	2.59	2.78
	$r_0 = 3 \times 10^{-13} \text{cm}$	3.61	3.88
Exponential		1.46	1.087
Yukawa		1.32	0.735
Experiment	Inela. Scatt. of neutron		~ 2.9
	Photodisintegration	(1)	~ 2.5
		(2)	~ 5
	Coulomb energy	2.25	

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A Device to Avoid Difficult Integrals in Calculations on Molecular Structure

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September 28, 1955

An attempt is made to avoid the difficulties in molecular integrals. The idea is simple. Just as in an atom we describe all the electrons with reference to one single center appropriately chosen in a molecule.

By this device we can side-step the difficulties of integrations concerning the electron-electron interaction. Many centers and different values of attractive charges offer no troubles.

As a simplest example we shall calculate the ground state energy of H_2^+ . A calculation along the same line was already published by Matsen¹⁾. We take the following form of the wave function:

$$\Psi(r, \vartheta, \varphi) = c_1 \varphi_1(r, \vartheta, \varphi) + c_2 \varphi_2(r, \vartheta, \varphi),$$

$$\varphi_1 = N(1, \zeta) e^{-\zeta r} Y_{00} + a N(n, \zeta') r^{n-1} e^{-\zeta'/r} Y_{00},$$

$$\varphi_2 = N(n, \zeta') r^{n-1} e^{-\zeta'/r} Y_{20}(\vartheta, \varphi),$$

$$Y_{00} = [1/4\pi]^{1/2}, \quad Y_{20}(\vartheta, \varphi) = [5/4\pi]^{1/2} P_2(\cos \vartheta),$$

where n, ζ, ζ' and a are variable parameters and c_1, c_2 are determined by the Ritz method. $N(n, \zeta)$ is a normalization constant for a radial function. The origin of coordinates is at the mid-point between two nuclei. Calculations are performed for the nuclear distance $R=2$ a.u. (equilibrium distance of H_2^+) and $R=1.4$ a.u. (approximate equilibrium distance of H_2). The results are as follows:

$$(I) \quad R=2.0: \quad n=5, \quad \zeta=1, \quad \zeta'=4, \quad a=0.401,$$

$$c_1/c_2 = 4.237114,$$

$$\text{Electronic energy: } -1.0742 \text{ a.u.}$$

$$(II) \quad R=1.4: \quad n=4, \quad \zeta=1.1, \quad \zeta'=4.3, \quad a=0.333,$$

$$c_1/c_2 = 6.877142,$$

$$\text{Electronic energy: } -1.2642 \text{ a.u.}$$

Corresponding exact values of energy²⁾ are -1.1026 ($R=2$) and -1.2843 ($R=1.4$) respectively. The results obtained above are not so excellent but may not forbid us to apply this method to other problems. Next we shall treat the H_2 molecule. Two types of wave function were adopted in the calculation of the ground state energy at $R=1.4$. The first of these is

$$\Psi(1, 2) \sim \phi_{\text{in}}(1) \phi_{\text{out}}(2) + \phi_{\text{out}}(1) \phi_{\text{in}}(2),$$

$$\phi_{\text{in}} = N(1, 1.1) e^{-1.1r} Y_{00} + 0.5 N(4, 4.3) r^3 e^{-4.3r} Y_{00}$$

$$+ 0.24406 N(4, 4.3) r^3 e^{-4.3r} Y_{20},$$

$$\phi_{\text{out}} = N(1, 0.8) e^{-0.8r} Y_{00},$$

where ϕ_{in} is almost the same form as the wave function Ψ of H_2^+ at $R=1.4$ (see above). ϕ_{out} is simply an s -type function. The above is so called "in-out" type function though in very primitive fashion. The mathematics of the calculation is quite elementary. It gives $D_e = 2.5766$ e.v. as the dissociation energy. This result tells us that the single term approximation for an outer electron is too crude for the ground state. However it is probable

that it works well for the higher excited states. As a next step we try to include in-out effect more reasonably by using the following function;

$$\Psi(1, 2) = c_1\varphi_1(1, 2) + c_2\varphi_2(1, 2) + c_3\varphi_3(1, 2),$$

$$\varphi_1(1, 2) = [s(1) + as'(1)] [s(2) + as'(2)],$$

$$\varphi_2(1, 2) = [s(1) + as'(1)] d(2) + d(1) [s(2) + as'(2)],$$

$$\varphi_3(1, 2) = d(1)d(2),$$

$$s = N(1, \zeta) e^{-\zeta r} Y_{00}, \quad s' = N(4, \zeta') r^3 e^{-\zeta' r} Y_{00},$$

$$d = N(4, \zeta') r^3 e^{-\zeta' r} Y_{20}.$$

This yields the following result with the parameter $\zeta=1$, $\zeta'=4.3$, $a=0.25$.

$$D_e = 3.80 \text{ e.v. } (R=1.4),$$

$$c_1 = 0.66109, \quad c_2 = 0.07881, \quad c_3 = 0.06788.$$

This result is seemingly not so bad in view of the simplicity of calculation.

Mathematically, the present method bears some interesting connections with the works of Barnett and Coulson³⁾ and Lundquist and Löwdin⁴⁾. We are contemplating a series of applications which includes, for example, a calculation on a linear H_3 system.

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Heat Conductivity and Viscosity of Liquid ^3He

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Theoretical calculations of the viscosity of liquid ^3He , based essentially on the ideal Fermi-Dirac gas model, have been carried out by Singwi and Kothari¹⁾ and by Pomeranchuk²⁾. They have shown that for $T \ll T_0$, the heat conductivity K according to this model must vary inversely as T and the viscosity

must vary inversely as T^2 . Although the observed variation of viscosity with temperature is not inconsistent with the requirement of an ideal Fermi-Dirac model, it certainly does not support this model because the condition $T \ll T_0$ is not fulfilled by the experiments. The validity of this assumption may also be questioned for the calculations of heat conductivity. Therefore, it is worthwhile to enquire if the predictions of Singwi and Kothari and of Pomeranchuk are also supported by the solid like model of liquid ^3He as against the Fermi-Dirac gas model, especially when Mikura³⁾ and also other workers have found that the treatment of ^3He as an ideal Fermi-Dirac gas of liquid density is far from the reality.

In the solid like model of liquid ^3He it is possible to introduce mean free path λ defined as⁴⁾

$$\lambda = A_0 / (\alpha \gamma T), \quad (1)$$

where A_0 is the lattice constant (in our case the minimum distance of approach of two atoms), α is the coefficient of thermal expansion and γ is the Grüneisen parameter. The heat conductivity K is given by the usual formula

$$K = C_v \lambda v / 3, \quad (2)$$

where C_v is the specific heat of liquid ^3He and v is the velocity of sound in the medium. Combining (1) and (2) we have

$$K = \frac{1}{3} \frac{C_v A_0 v}{\alpha \gamma T}. \quad (3)$$

Since by Grüneisen's law α can be taken to be proportional to specific heat C_v^* , eq. (3) may be written as

$$K = A / T, \quad (4)$$

where A is a constant to be determined experimentally for a particular width of the slit. In the absence of any data on the velocity of sound v in liquid ^3He , it may be taken to be constant over a wide range of temperature as calculated by Mikura⁵⁾. So far no experimental data exists on the variation of heat conductivity of liquid ^3He with temperature. Hence, it is sufficient to show that the form of eq. (4) is consistent with the existing theoretical calculations of Singwi and Kothari and of Pomeranchuk. Since it has not been assumed that $T \ll T_0$, eq. (4) is true for a wider range of temperature.

Thermal conductivity K and viscosity are related,

* For longitudinal Debye waves in liquid ^4He the validity of this law has been reported by K. R. Atkins and M. H. Edwards, Phys. Rev. **93** (1954), 1416.

to a first approximation, as

$$K = f\eta C_v, \quad (5)$$

f being a numerical factor. From (4) and (5) we obtain a relation for the viscosity of liquid ^3He in the form

$$\eta = A/(fC_v T). \quad (6)$$

As the data on the specific heat of liquid ^3He is found to fit best with an empirical relation of the type $C = aT + bT^3$, eq. (6) may be written as

$$\eta = A/f(aT + bT^3)T. \quad (7)$$

For temperatures less than 1°K the term bT^3 in the denominator becomes negligible. Then eq. (7) reduces to

$$\eta = B/T^2, \quad (8)$$

which gives the variation of η at low temperatures in agreement with the findings of Singwi and Kothari, of Daunt⁶⁾, and of Pomeranchuk.

Thus starting with entirely different considerations it has been possible to deduce the variations of heat conductivity and of viscosity with temperature and the results so obtained are in agreement with the earlier works.

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The Anomalous Magnetic Moment of the μ -Meson

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In order to decide the upper limit of the magnitude of the anomalous magnetic moment (a.m.m.) of

the μ -meson, we have analyzed the burst produced by cosmic ray μ mesons. The burst produced by the electromagnetic interaction of the μ -meson with matter is considered, where the spin of μ -meson is assumed to be of one-half. The energy spectrum of μ -mesons at sea level is given by the same procedure as that of Hayakawa and Tomonaga¹⁾. They have taken into account the energy losses due to ionization, bremsstrahlung and pair production by μ -mesons, but we further include, in this note, the μ - π creation and the Cerenkov radiation. The frequencies of the burst produced by the bremsstrahlung due to μ -mesons with and without a.m.m. can be calculated along the same line as that of Christy and Kusaka²⁾.

The results of the calculation and the corresponding experimental values³⁾ are shown in Fig. 1. The standard error in our calculation is 56 percent, the main part of which (50 percent) comes from the shower function we have used.

From Fig. 1 we can see the followings: the larger the magnitude of a.m.m. becomes the slower is the decrease of the rate of burst frequencies. Thus we may conclude that, the magnitude of a.m.m. of μ -mesons does not exceed 80 percent of the normal Dirac moment.

Recently, Peaslee⁴⁾ has also given the qualitative information on the a.m.m. through the analysis of the intensity of μ -mesons underground, but our result is more decisive.

In another way, the study on the X-ray coming from the μ -mesonic atoms⁵⁾ may give an appropriate method to decide the magnitude of a.m.m., but there are many difficulties in doing so, because, in addition to the a.m.m., we must consider the charge distribution of nucleus. On the other hand, Fialho⁶⁾ pointed out that the magnitude of a.m.m. of μ -mesons might be determined by the radiative π - μ decays. This would be the best method available at present.

For our present purpose, it is necessary to take into account the shower particles from the pair production and the ionization by μ -mesons. This will be done in the forthcoming paper to be published elsewhere, in which we shall also consider the limit of applicability of the theories of electromagnetic interactions of μ -mesons at high energies.

The authors wish to express their sincere thanks to Prof. S. Hayakawa for his valuable discussions and to Prof. S. Sakata for his interest in their work.

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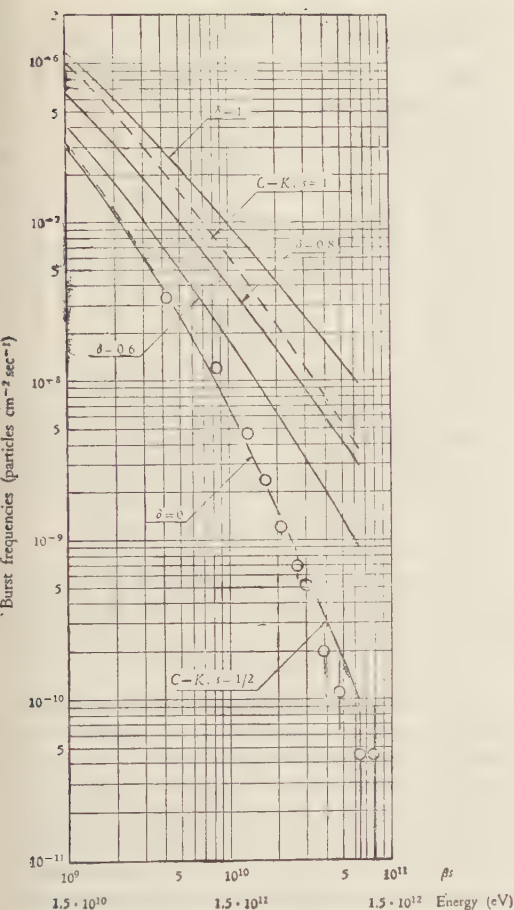


Fig. 1. The burst frequency as functions of the number of particles, S , and of the energy. Theoretical values indicate the frequencies of burst produced by the bremsstrahlung due to μ -mesons with and without a.m.m. β is the critical energy of matter. C-K represents the result of Christy and Kusaka for spin 1/2 and spin 0 respectively. The experimental values of Schein and Gill are represented by circles. δ is the magnitude of a.m.m.

- 2) R. F. Christy and S. Kusaka, Phys. Rev. 59 (1941), 414.
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In their experiment, the burst with more than 1.9×10^7 ion pairs is measured. Their result satisfy our requirements. On this connection, see H. Carmichael, Phys. Rev. 74 (1948), 1667.

We are indebted to Mr. H. Hasegawa for calling our attention to the work of Carmichael.

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We are indebted to Prof. S. Sakata for calling our attention to this work.

Deuteron-Deuteron Reaction at High-Energy

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Recently Godfrey¹⁾ measured the differential cross section for the process $d+d \rightarrow H^3+p$, using the deuterons of 190 Mev from the Berkeley synchrocyclotron. This is one of the simplest processes in the high-energy stripping or pickup reactions. He showed that it was inadequate to explain the angular distribution merely in terms of the pure stripping (or pickup). So far two mechanisms have been proposed about the pickup process in the high-energy nucleon bombardment on nuclei. One of them leads to the direct process, as proposed by Chew and Goldberger²⁾, which contributes strongly to the small angle part of the angular distribution of the deuterons; and the other to the indirect process, so called by Bransden³⁾, contributing chiefly to its large angle part. As the nuclei concerned here are of simple constitution it will be expected that the above process can be treated from the unified standpoint. The main purpose of this note is to study the behavior of the differential cross sections for the above process from this respect.

Using the Born approximation and the charge-independent Serber force between two nucleons, we calculated the differential cross section in the center-of-mass system. We obtained the following expression:

$$\begin{aligned} d\sigma(\theta) = & 4K/3k(3M/8\pi\hbar^2)^2 \{ 2/3 \cdot [I(\theta) - I(\pi-\theta)]^2 \\ & + 1/3 \cdot [I(\theta) + I(\pi-\theta)]^2 + 1/12 \cdot (1+3q)^2 [J(\theta) \\ & + J(\pi-\theta)]^2 \} \end{aligned}$$

$$+1/3 \cdot (1+3q) [I(\theta) + I(\pi-\theta)] [J(\theta) + J(\pi-\theta)] \}.$$

Here k and K are the wave numbers of the incoming deuteron and the outgoing triton, respectively, in the center-of-mass system; and θ is the angle between k and K . q is the ratio of the singlet to the triplet potential depth in the neutron-proton interaction. It will be more convenient to consider the inverse process $H^3 + p \rightarrow d + d$ to see the physical meanings of the integrals I and J , because the matrix element remains the same as the original.

As the result of the interaction between the incoming proton and one nucleon in the triton, there come out two cases where these two interacting nucleons end up in the same deuteron or in different deuteron. The former corresponds to the direct process mentioned above, while the latter is similar to the indirect process. The integral I shows the interaction matrix element of the former case and J that of the latter one. Therefore, we may call I and J the direct integral, respectively. Using the Yukawa potential, we calculated both integrals under the assumptions of the Hulthén wave function for the deuteron and the Irving wave function ($n=0$) for the triton. Both integrals had to be solved with numerical integrations. The cross sections consist of three contributions from direct, indirect and interference term.

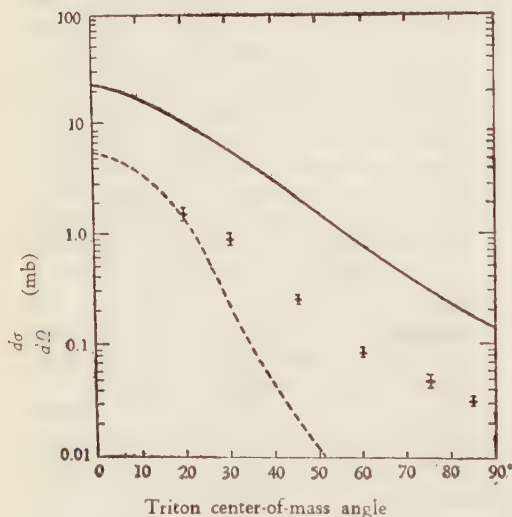


Fig. 1. The angular distribution of the triton in the center-of-mass system. — Yukawa potential ($r_0 = 1.18 \times 10^{-13}$ cm, $V_0 T = 67.8$ Mev, $q = 0.686$); Gaussian potential ($r_0 = 1.94 \times 10^{-13}$ cm, $V_0 T = 45.0$ Mev, $q = 0.578$).

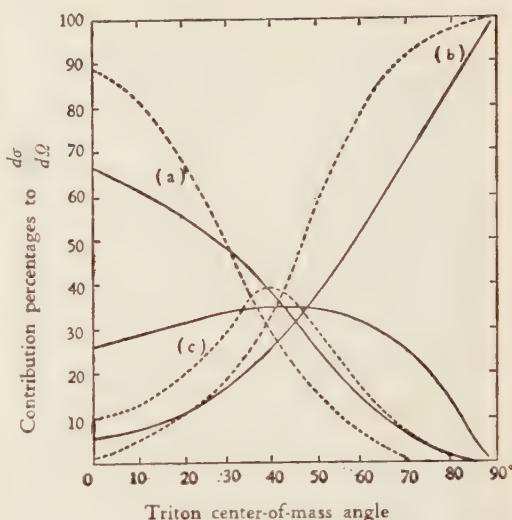


Fig. 2. The angular dependences of the contributions to the cross sections from (a) direct, (b) indirect and (c) interference term. — Yukawa potential, Gaussian potential.

The direct term is dominant at the small angle part of the angular distribution, and negligible at the large angle part. Then the slope of its curve is greater than that of the experimental curve. On the contrary, the indirect term shows the appreciable contribution at these large angles, and thereby the slope of the resultant curve becomes much closer to that of the experimental curve. The interference term shows the considerable effect at the intermediate angles.

The details are shown in Figs. 1 and 2. The theoretical value of the total cross section (18.2 mb) is much larger than the experimental value (4.1 mb) in our approximations. By the way, the calculations with the Gaussian potential were also carried out, because both integrals could be analytically solved. The results are also shown in the same figures for comparison. The theoretical value of the total cross section (3.5 mb) is close to the experimental value, but the slope of the angular distribution is much greater than that of the experimental curve.

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Positive Temperature Effect of Cosmic Rays

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The temperature effect for the intensity of μ -mesons at sea-level is investigated by taking π - μ decays into account. The positive temperature coefficient is calculated by paying special attention to the difference in the attenuation mean free path of the N -component and the absorption mean free path of π -mesons and to the atmospheric depth which gives main contributions to the coefficient. The magnitude of the coefficient is obtained as $0.05\%/^{\circ}\text{C}$ and the depth of main contributions is found to be 200 mb. Influence of heavy mesons is qualitatively discussed

§ 1. Introduction

The atmospheric effect of cosmic rays has been interpreted in terms of the absorption of cosmic ray particles and of the spontaneous decay of μ -mesons. The former is correlated with the barometric effect, while the latter with the average height of the production of μ -mesons. These effects were thoroughly analysed by Olbert¹⁾ and there remains almost nothing added to his analysis, as far as the role of μ -mesons is concerned. However, a possible interpretation of the positive temperature effect suggested by him seems to be an open question. Although Olbert tried to attribute the positive temperature effect to the difference in variations of two kinds of average heights introduced in his theory, these two are found to correlate with one another, so that Olbert's suggestion seems to be of little validity, as shown by Wada and Kudō²⁾.

The positive temperature effect was first noticed by Forró³⁾ for cosmic rays underground and was interpreted by Miyazima⁴⁾ in terms of π - μ decay. A detailed analysis of this effect showed that the upper limit of the positive temperature coefficient is about $0.5\%/^{\circ}\text{C}$ and is attained in the limit of high energies⁵⁾. The calculation of the coefficient was extended to lower energies by Barrett et al⁶⁾ and their result predicted the magnitude of the coefficient appreciably smaller than $0.1\%/^{\circ}\text{C}$ for the bulk of the hard component observed at sea level. This had already been anticipated with a qualitative estimate by Duperier⁷⁾ who observed a positive temperature coefficient of $0.1\%/^{\circ}\text{C}$ or larger for the hard component at sea level. This was found to be too high to reconcile with the theoretical prediction. On the other hand, several authors⁸⁻¹⁰⁾ obtained lower values in rough agreement with the theoretical estimate.

Such divergent experimental results are thought to be due partly to the varieties of ways of taking correlation with atmospheric temperatures. One can hardly say, at what

height the temperature for correlation calculation should be taken for the comparison with the theoretical temperature coefficient, unless a more careful analysis is made. For this purpose Maeda and Wada¹¹⁾ attempted to extend the theoretical calculation of the atmospheric effect by taking π - μ decays into account. Unfortunately, however, their calculation seems to be subjected to the following drawbacks. Firstly, they disregarded the difference between the attenuation mean free path of the N -component and the absorption mean free path of π -mesons. Secondly, an inadequate energy spectrum of π -mesons at production was assumed. Thirdly, a great part of their calculation was carried out numerically, so that one could hardly know the height at which the temperature should be taken and that any modification in basic assumptions would require almost a complete recalculation.

In this paper our aim is mainly directed to improve the above points and also to complete the work of Olbert by taking π - μ decays into account. The first point was already noticed by Duperier⁷⁾, but the scarcity in the knowledge of the absorption mean free path forbids us to get a definite answer. The ambiguity arising from this will be avoided by separating a part depending on the difference between the two mean free paths.

The production spectra of π -mesons and μ -mesons are now given by Olbert¹²⁾ on the basis of observations and properties of mesons. These are adopted in our work, supplementing their high energy parts in reference to underground data.

Our calculation is carried out in close analogy to Olbert's¹¹⁾. A number of the average atmospheric depths are introduced. These are considered to such depths that give main contributions to respective terms and are shown in Fig. 1 as functions of momentum of μ -mesons. These values may serve to analyse the atmospheric effect from observations.

It must be noticed that the average depth responsible for π - μ decays is larger than that for μ - e decays. The former, denoted as $x_1^{*)}$, is the first order moment with respect to the generating intensity multiplied by the survival probability, while the latter, denoted as $x_0^{*)}$, is the average with weight favouring small x . x_1 is found to be about 200 mb and decreases with increasing momentum of mesons. This value of depth is larger than that currently adopted, say 100 mb⁷⁾, for the correlation calculation of the temperature effect. Since the temperature variation at 100 mb could be smaller than that at 200 mb, the large temperature coefficient obtained through observations might be due to the inadequate choice of the depth at which the temperature variation was taken.

We added the discussion on the influence of heavy mesons in § 4. The κ - μ decay has been considered by Barrett et al¹³⁾. Here we are mainly concerned with K - π decays that may increase the positive temperature coefficient. From existent observations we can only deduce the production rate of such K -mesons not higher than that of π -mesons.

§ 2. Diffusion of π -mesons

Let the differential intensity of π -mesons with momenta between p and $p+dp$ at atmospheric depth x be $\pi(p, x)dp$. As is well known, $\pi(p, x)$ follows the diffusion equation

*) x_0 and x_1 in the present paper correspond to x_1 and x_2 in Olbert's paper¹¹⁾, respectively.

$$\frac{\partial \pi(p, x)}{\partial x} = -\left(\frac{1}{\lambda_\pi} + \frac{B_\pi}{px}\right) \pi(p, x) + Q(p, x). \quad (2.1)$$

λ_π is the mean free path of π -mesons for the nuclear absorption and B_π the characteristic momentum for the decay of π -mesons, which is given by

$$B_\pi = (m_\pi c / \tau_\pi) (x / \rho(x)) = 120 \text{ GeV}/c, \quad (2.2)$$

where m_π and τ_π are the mass and the lifetime of π -mesons respectively. We fix the numerical values of these quantities as

$$m_\pi = 273 m_e, \quad \tau_\pi = 2.5 \times 10^{-8} \text{ sec},$$

in reference to latest experiments. $\rho(x)$ represents the density of air at atmospheric depth x . As this is nearly proportional to x and the proportionality considerably simplifies the following analytical calculation, we take

$$x / \rho(x) = 6.38 \times 10^5 \text{ cm},$$

which is a good choice at such depths that π -mesons play a dominant role in the atmospheric effect.

$Q(p, x)$ is the source of π -mesons that are supplied by N -component. There is experimental indication that the momentum spectrum of π -mesons at production is nearly independent of atmospheric depth, so that the production spectrum $S(p)$ can be separated in $Q(p, x)$. This assumption seems to fail at low momenta, but the contribution from low momentum π -mesons to the meson intensity at low altitudes is unappreciable. The x -dependence of $Q(p, x)$ follows the attenuation of the N -component, expressed as $\exp(-x/\lambda_n)$. With the above considerations we may assume

$$Q(p, x) = \lambda_n^{-1} S(p) \exp(-x/\lambda_n). \quad (2.3)$$

It is important to note that the temperature coefficient caused by the decay of π -mesons depends upon the difference between λ_n and λ_π . Introducing a quantity λ' by*)

$$\lambda'^{-1} = \lambda_\pi^{-1} - \lambda_n^{-1}, \quad (2.4)$$

the intensity change of π -mesons at low momenta ($\ll B_\pi$) can be estimated as follows. The intensity of π -mesons of momentum p observed at depth x is contributed mainly from their production in a layer between x and $x(1 - p/B_\pi)$. In this layer of thickness px/B_π the fraction of π -mesons supplied is $1/\lambda_n$ and that lost is $1/\lambda_\pi$. Consequently the fractional change of the intensity caused by fractional temperature change, $\delta T/T = \partial B_\pi/B_\pi$, is given by

$$\frac{\delta B_\pi}{B_\pi} \cdot \frac{xp}{B_\pi} \cdot \left(\frac{1}{\lambda_\pi} - \frac{1}{\lambda_n}\right) = -\frac{\delta T}{T} \cdot \frac{xp}{B_\pi} \cdot \frac{1}{\lambda'}. \quad (2.5)$$

The intensity change is proportional to $-1/\lambda'$ and positive or negative according to $\lambda_\pi > \lambda_n$ or $\lambda_\pi < \lambda_n$. The intensity variation caused by this partially compensates the variation of the

*) The sign of λ' here is opposite to that adopted by Barrett et al⁶⁾. The N -cascade at the energies of interest makes us infer the positive value of λ' .

rate of π - μ decays due to the temperature change at a decaying point. This fact has already been noticed by Duperier⁷⁾ in his original work, but overlooked by Mreda and Wada¹¹⁾. Since the magnitude of λ' is not accurately known, we shall separate the temperature coefficient into two parts, one being independent of λ' and the other dependent on λ' .

Quantitative calculation is carried out in a similar way to Barrett et al.⁶⁾, but with slight differences. For instance, a part dependent on λ' is explicitly separated out. The fractional temperature change at depth x ,

$$\eta(x) = \delta T(x)/T(x), \quad (2.6)$$

induces the variation of the characteristic momentum,

$$B_\pi \rightarrow B_\pi(1 + \eta(x)). \quad (2.7)$$

This in turn results in the intensity variation, $\pi_1(p, x)$, that is proportional to $\eta(x)$. The average intensity, $\pi_0(p, x)$, given by

$$\pi_0(p, x) = \frac{1}{\lambda_n} S(p) e^{-x/\lambda_n} \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda'} dx' \quad (2.8)$$

turns out to be the source of π_1 , as shown by

$$\pi_1(p, x) = -\frac{B_\pi}{p} e^{-x/\lambda_n} \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda_n} \eta(x') \pi_0(p, x') \frac{dx'}{x'}. \quad (2.9)$$

Substituting (2.8) into (2.9) and changing the order of the integrals, we have

$$\pi_1(p, x) = -\frac{1}{\lambda_n} \frac{B_\pi}{p} S(p) e^{-x/\lambda_n} \int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \int_{x'}^x \frac{dx''}{x''} \eta(x''). \quad (2.10)$$

If there is no abrupt change of temperature near x , $\int_{x'}^x dx'' \cdot \eta(x'')/x'' / \log(x/x')$ is regarded as slowly varying. This part is, therefore, taken out of the integral with the aid of the mean value theorem as

$$\pi_1(p, x) = -\frac{B_\pi}{p} \frac{\int_{x_\pi}^x \frac{dx''}{x''} \eta(x'')}{\log(x/x_\pi)} \cdot \frac{\int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \log(x/x')}{\int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'}} \cdot \pi_0(p, x), \quad (2.10')$$

where x_π is given by

$$\begin{aligned} x_\pi &= \int_0^x dx' \cdot x' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \log(x/x') \bigg/ \int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \log(x/x') \\ &\cong [(B_\pi + p)/(B_\pi + 2p)]^2 x + \frac{p^2 (B_\pi + p)^2 (2B_\pi^2 + 8B_\pi p + 7p^2)}{(B_\pi + 2p)^4 (B_\pi + 3p)^2} \frac{x^2}{\lambda'}. \end{aligned} \quad (2.11)$$

Further we have

$$\int_{x_\pi}^x \frac{dx''}{x''} \eta(x'') / \log(x/x_\pi) \cong \eta((x + x_\pi)/2) \equiv \eta(H(p) \cdot x),$$

$$\int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \log(x/x') \Big/ \int_0^x dx' (x'/x)^{B_\pi/p} e^{x'/\lambda'} \cong L(p) - M(p) (x/\lambda'),$$

where

$$H(p) \cong (1/2) \cdot (1 + [(B_\pi + p)/(B_\pi + 2p)]^2), \quad L(p) = p/(B_\pi + p) \\ \text{and} \quad M(p) = p^2/(B_\pi + 2p)^2. \quad (2.12)$$

In the expression of $H(p)$ we keep only the first term of (2.11). This is valid provided that λ' has not a too small value. Thus (2.10') is reduced to

$$\pi_1(p, x) = \frac{B_\pi}{p} \cdot \eta(H(p)x) \left[-L(p) + M(p) \frac{x}{\lambda'} \right] \pi_0(p, x). \quad (2.13)$$

Now the term depending upon λ' is separated explicitly.

§ 3. Diffusion of μ -mesons

A μ -meson of momentum p is the decay product of a π -meson whose momentum lies between p and $(m_\pi/m_\mu)^2 p$ for relativistic μ -mesons. However, one may be allowed to take a single value, $(m_\pi/m_\mu)p$, with a sufficient approximation in place of the continuous value of momentum. Taking the mass of μ -mesons as $m_\mu = 207 m_e$, there enters a parameter

$$r = m_\mu/m_\pi = 0.76. \quad (3.1)$$

With this approximation the source of μ -mesons in the average atmosphere is obtained as

$$s_0(p, x) = (B_\pi/px) \pi_0(p/r, x). \quad (3.2)$$

The part proportional to η consists of the one due to the temperature variation at the depth of disintegrations and the other due to the intensity variation of π -mesons, π_1 . Hence the source proportional to η is

$$s_1(p, x) = \frac{B_\pi}{px} \eta(x) \pi_0(p/r, x) + \frac{B_\pi}{px} \pi_1(p/r, x). \quad (3.3)$$

Such decomposition into two parts has not been made explicitly in reference 6. The two terms in the right hand side of (3.3) are of opposite signs, so that the temperature effect due to π - μ decays is rather small. At high momenta, $p \gg B_\pi$, the second term is smaller than the first one by factor B_π/p and only the latter is taken into account, as in some previous works.

Introducing (2.13) with (2.12) into (3.3), an approximate estimate of s_1 can be made as

$$s_1(p, x) \cong \left[\eta(x) \left(1 - \frac{rB_\pi}{p} L \right) + \eta(Hx) \frac{rB_\pi}{p} \frac{x}{\lambda'} M \right] s_0(p, x), \quad (3.4)$$

where H , L and M stand for $H(p/r)$, $L(p/r)$ and $M(p/r)$, respectively. The first term contributes to the positive temperature coefficient, whereas the second term to the positive

or negative one, according as λ' is positive or negative.

Analogous to the procedure in § 2, the intensity of μ -mesons is decomposed into two parts. The average intensity is given by

$$\mu_0(p, x) = \int_0^x w(p, x, x') s_0(p', x') dx', \quad (3.5)$$

where $w(p, x, x')$ is the survival probability that a μ -meson of momentum p' at depth x' survives at x . Its momentum at x is denoted as p which is related to p' through the momentum-range relation

$$p = p(R), \quad p' = p(R + x - x'),$$

where R represents the residual range of the meson at x . Introducing the characteristic momentum for μ -mesons,

$$B_\mu(x) = (m_\mu c / \tau_\mu) (x / \rho(x)) \\ = 1.05 \text{ GeV}/c \quad \text{for} \quad x / \rho(x) \cong 6.38 \times 10^5 \text{ cm}, \quad (3.6)$$

where $\tau_\mu = 2.15 \times 10^{-6}$ sec is the lifetime of μ -mesons, the survival probability is expressed as

$$w(p, x, x') = \exp \left[- \int_{x'}^x \frac{B_\mu(x'') dx''}{x'' p(R + x - x'')} \right]. \quad (3.7)$$

The numerical values of this function are given by Olbert¹²⁾.

The intensity variation of μ -mesons consists of two parts

$$\mu_1(p, x) = \mu_1^{(-)}(p, x) + \mu_1^{(+)}(p, x). \quad (3.8)$$

$\mu_1^{(-)}$ comes from the decay of μ -mesons and contributes to the negative temperature effect :

$$\mu_1^{(-)}(p, x) = - \int_0^x \frac{B_\mu(x') \eta(x')}{x'} w(p, x, x') \mu_0(p', x') dx'. \quad (3.9a)$$

$\mu_1^{(+)}$ represents the influence of π -mesons and contributes possibly to the positive temperature effect :

$$\mu_1^{(+)}(p, x) = \int_0^x w(p, x, x') s_1(p', x') dx'. \quad (3.9b)$$

$\mu_1^{(-)}$ has thoroughly been calculated by Olbert¹⁾ and is expressed as

$$\mu_1^{(-)}(p, x) = \{a_H(p, x) \eta_H(x_0) + a_\pi(p, x) \eta_\pi(x_1)\} \cdot \int_0^x \Phi(p, x, x') dx', \quad (3.10)$$

where

$$\Phi(p, x, x') \equiv w(p, x, x') s_0(p', x'). \quad (3.11)$$

η_H represents the fractional variation of the mean production height of μ -mesons and η_π the fractional temperature variation averaged between x and x_1 . They are given by

$$\eta_H(x_0) = \int_{x_0}^x \frac{\eta(x')}{x'} dx', \quad \eta_\pi(x_1) = \frac{1}{x - x_1} \int_{x_1}^x \eta(x') dx', \quad (3.12)$$

with

$$\begin{aligned}x_0(p) &= \int_0^x x' \log(x/x') \Phi(p, x, x') dx' / \int_0^x \log(x/x') \Phi(p, x, x') dx', \\x_1(p) &= \int_0^x x' \Phi(p, x, x') dx' / \int_0^x \Phi(p, x, x') dx'.\end{aligned}\quad (3.13)$$

$a_H(p, x)$ and $a_K(p, x)$ are differential temperature coefficients corresponding to the appropriate averages of temperature as in (3.12). They are expressed as

$$\begin{aligned}a_H(p, x) &= -\frac{B_\mu}{m_\mu c^2} \left(\frac{a}{x_e} - k_0 \right) \cdot \int_0^x \log(x/x') \Phi(p, x, x') dx' / \log(x/x_0) \\&\quad \times \int_0^x \Phi(p, x, x') dx',\end{aligned}\quad (3.14)$$

$$a_K(p, x) = -\frac{B_\mu}{m_\mu c^2} \frac{a}{x_e} \log \left(\frac{x_e - x_1}{x_e - x} \right),$$

where $a = 53.5 \text{ g cm}^{-2}$, $k_0 = 2.07 \times 10^{-3}$ and $x_e = R + x + 56 \text{ g cm}^{-2}$, R being the residual range corresponding to momentum p .

$\mu_1^{(+)}$ can be obtained by substituting (3.4) into (3.9b) as

$$\mu_1^{(+)}(p, x) = \int_0^x \left[\eta(x') \left(1 - \frac{rB_\pi}{p'} L' \right) + \eta(H'x') \frac{rB_\pi}{p'} \frac{x'}{\lambda'} M' \right] \Phi(p, x, x') dx', \quad (3.15)$$

where $H' = H(p'/r)$, $L' = L(p'/r)$ and $M' = M(p'/r)$. Taking out slowly varying parts of the integral we obtain

$$\mu_1^{(+)}(p, x) = \left[\eta(x_1) \left(1 - \frac{rB_\pi}{p_1} L_1 \right) + \eta(H_2 x_2) \frac{rB_\pi}{p_2} \frac{x_1}{\lambda'} M_2 \right] \int_0^x \Phi(p, x, x') dx', \quad (3.16)$$

where $L_1 = L(p_1/r)$, $M_2 = M(p_2/r)$ and $H_2 = H(p_2/r)$, and $p_i = p(R + x - x_i)$, $i = 1, 2$. x_1 and x_2 are the average depths defined respectively by (3.13) and

$$x_2(p) = \int_0^x x'^2 \Phi(p, x, x') dx' / \int_0^x x' \Phi(p, x, x') dx'. \quad (3.17)$$

Introducing the temperature coefficients

$$\begin{aligned}b_1(p, x) &= 1 - (rB_\pi/p_1) L_1, \\b_2(p, x) &= (rB_\pi/p_2) (x_1/\lambda') \cdot M_2,\end{aligned}\quad (3.18)$$

(3.16) is expressed as

$$\mu_1^{(+)}(p, x) = [b_1(p, x) \eta(x_1) + b_2(p, x) \eta(H_2 x_2)] \mu_0(p, x). \quad (3.19)$$

In comparison with observations, we have to obtain the temperature coefficients for the integral intensity. The integral intensity of μ -mesons with momenta larger than p_0 is given, in the average atmosphere, by

$$I_0(p_0, x) = \int_{p_0}^{\infty} \mu_0(p, x) dp. \quad (3.20)$$

Correspondingly, μ_1 has to be integrated over p , giving $I_1(p_0, x)$. For $I_1^{(-)}$, a_H and a_K are replaced, as given by Olbert¹⁾, by the averages over the momentum spectrum of μ -mesons, A_H and A_K . For $I_1^{(+)}$, however, one has to notice that x_1 and x_2 are functions of p . Since these are slowly varying with respect to p , they are evaluated as

$$\overline{x_1}(p_0) = \int_{p_0}^{\infty} x_1(p) b_1(p, x) \mu_0(p, x) dp / \int_{p_0}^{\infty} b_1(p, x) \mu_0(p, x) dp, \quad (3.21)$$

$$H_2 \overline{x_2}(p_0) = \int_{p_0}^{\infty} H_2 x_2(p) b_2(p, x) \mu_0(p, x) dp / \int_{p_0}^{\infty} b_2(p, x) \mu_0(p, x) dp. \quad (3.22)$$

Thus we have

$$I_1^{(+)}(p_0, x) \equiv \int_{p_0}^{\infty} \mu_1^{(+)}(p, x) dp \\ = [B_1(p_0, x) \eta(\bar{x}_1) + B_2(p_0, x) \eta(\overline{H_2 x_2})] I_0(p_0, x). \quad (3.23)$$

The temperature coefficients $B_i (i=1, 2)$ are given by

$$B_i(p_0, x) = \int_{p_0}^{\infty} b_i(p, x) \mu_0(p, x) dp / I_0(p_0, x). \quad (3.24)$$

§ 4. Influence of heavy mesons.

It is now known that π - and μ -mesons are generated also from hyperons and heavy mesons. These new sources could have influences on the atmospheric effect. In this connection Barrett et al.⁵⁾ estimated the influence of κ - μ decays. Increasing knowledge about hyperons and heavy mesons allows us to discuss such influences in more detail.

The following two reasons lead us to infer that hyperons play a less important role than heavy mesons. Firstly, the mass ratios between primary particles and secondary mesons are larger for hyperons than for heavy mesons. Consequently, the average energy of the secondary mesons is, in comparison with the energy of the primary particles, reduced by a larger factor for hyperons than for heavy mesons. Thus the contribution to the meson intensity from hyperons is smaller due to the energy reduction. Secondly, Λ -particles, which are the commonest hyperons, are produced mainly with small energies. If, therefore, the number of Λ -particles produced is as many as that of K -mesons, the intensity of the former at a given energy is much smaller than that of the latter. These two reasons may allow us to neglect the influence of hyperons and to concentrate ourselves only to heavy mesons.

For our purpose heavy mesons are divided into two groups, according to their decay products. One consists of those which give π -mesons and possibly contributes to the positive temperature effect. They are such as θ^0 and $K_{\pi 2}$, and denoted collectively as K for brevity. The other consists of those which give μ -mesons and decreases the positive temperature effect compared with such a case if entire μ -mesons are due to π - μ decays.

They are such as $K_{\mu,2}$ and $K_{\mu,3}$, and denoted collectively as k . All of them have masses about $1000 m_e$ and their lifetimes are around 10^{-8} sec but θ^0 . No definite evidence for the frequency of heavy mesons has yet been obtained, but some experiments indicate the frequency of about ten percent of that of π -mesons. However, it seems doubtful that the frequency remains so high also at high energy nucleon-nucleon collisions of our interest.

Such ambiguities in the properties of heavy mesons force us to start from the following tentative assumptions which may be tested by comparing with observations. Firstly, the velocity spectra of K and k are the same as that of π -mesons in the center of mass system. Namely, the production spectra are assumed as

$$\lambda_n^{-1} f_K(m_\pi/m_K) S((m_\pi/m_K)p) dp e^{-x/\lambda_n}, \quad (4.1)$$

where m_K is the mass of heavy mesons. f_K represents the frequency of production of heavy mesons in comparison with that of π -mesons. For k mesons suffix K is replaced by k . Secondly the mean free path of heavy mesons for the nuclear absorption is assumed to be the same as that of π -mesons, λ_n . With these assumptions the formulae presented in § 2 are employed as they are, with only exception in the characteristic momenta for decays which are denoted as B_K and B_k . In our order of magnitude calculation, we take

$$B_K \simeq B_k \simeq 10^3 \text{ Gev}/c, \quad (4.2)$$

corresponding to the lifetime of about 10^{-8} sec. The decay modes of heavy mesons suggest us to take the average energy of the secondary mesons as r_K times the primary energy, with

$$r_K \simeq r_k \simeq 1/2. \quad (4.3)$$

For k the temperature coefficient can be obtained in exactly the same way as in § 2 and § 3. In place of (3.19) we have

$$R_k f_k [b_{1k}(p, x) \eta(x_1) + b_{2k}(p, x) \eta(H_2 x_2)], \quad (4.4)$$

with

$$\left. \begin{aligned} b_{1k} &= 1 - \frac{r_k B_i}{r_k B_k + p_1} \simeq \frac{p_1}{r_k B_k}, \\ b_{2k} &= \frac{r_i B_k p_2}{(r_k B_k + 2p_2)^2} \frac{x_1}{\lambda'} \simeq \frac{p_2}{r_k B_k} \frac{x_1}{\lambda'}, \end{aligned} \right\} \quad (4.5)$$

and

$$R_k = \frac{r_i B_k}{r_k B_k + p} \frac{r B_\pi + p}{r B_\pi}. \quad (4.6)$$

The last expressions in (4.5) hold in the limit of $p_i \ll B_i$, $i=1, 2$. As B_k is one order larger than B_π , the contribution to the temperature effect, in comparison with π - μ decays, is rather small and is of the order of $f_k p_1/B_k$ at low momenta. At high momenta, $p \gg B$, the influence of k - μ decays may be detectable, as seen from the calculation of Barrett et al.

The influence on the temperature effect may be more important for K -particles, because

they contribute to the time delay in addition to π - μ decays, namely to the increase of the positive temperature effect.

The evaluation of effects of K - π decays can be made by the same way as in § 2, and § 3 but by replacing π by K . Thus we find the source of π -mesons due to K , in the average atmosphere, as

$$S_0^K(p, x) = \lambda_n^{-1} R_K f_K S(p) \exp(-x/\lambda_n), \quad (4.7)$$

with

$$R_K = r_K B_K / (r_K B_K + p). \quad (4.8)$$

Comparing this with (2.3), one can see the contribution to the intensity of π -mesons from K to be $f_K R_K$ times that from the direct production of π -mesons. The temperature variation, $\gamma(x)$, causes the variation of the source by

$$S_1^K(p, x) = \left[\gamma(x) \left(1 - \frac{r_K B_K}{p} L_K \right) + \gamma(H_K x) \frac{r_K B_K}{p} \frac{x}{\lambda'} M_K \right] S^K(p, x), \quad (4.9)$$

where

$$H_K = (1/2) (1 + [(r_K B_K + p) / (r_K B_K + 2p)]^2), \quad L_K = p / (r_K B_K + p) \\ \text{and } M_K = p^2 / (r_K B_K + 2p)^2. \quad (4.10)$$

The variation of the source, $S_1^K(p, x)$, results in the variation of the intensity of π -mesons as

$$\pi_1^K(p, x) = -\frac{B_\pi}{p} e^{-x/\lambda_\pi} \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda'} \gamma(x') \pi_0^K(p, x') \frac{dx'}{x'} \\ + \int_0^x (x'/x)^{B_\pi/p} e^{(x'-x)/\lambda_\pi} S_1^K(p, x') dx' \\ \cong -\frac{B_\pi}{p} e^{-x/\lambda_\pi} \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda'} \gamma(x') \pi_0^K(p, x') \frac{dx'}{x'} \\ + \left[\gamma(x) \left(1 - \frac{r_K B_K}{p} L_K \right) + \gamma(H_K x) \frac{r_K B_K}{p} \frac{x_K}{\lambda'} M_K \right] \pi_0^K(p, x), \quad (4.11)$$

where x_K is defined like x_π in (2.11) as

$$x_K = \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda'} dx' / \int_0^x (x'/x)^{B_\pi/p} e^{x'/\lambda'} dx' \cong ((B_\pi + p) / (B_\pi + 2p)) x. \quad (4.12)$$

The last expression of (4.11) is obtained by taking an appropriate average of x/λ' in S_1^K to be x_K/λ' . π_0^K is the average intensity of π -mesons supplied by K through the source S_0^K .

Adding (4.11) to (2.9), we have the intensity variation of π -mesons to the first order of γ . The first term in the last expression of (4.11) is nothing to do with the modification of the temperature coefficient, because the contributions from K are added with the same weight, $f_K R_K$, to π_0 as well as π_1 . Only the second term gives rise to a net

modification that increases the positive temperature effect, unless λ' takes a small, negative value. The temperature coefficient due to this part can be calculated in the same way as in § 3 and is expressed as

$$\mu_1^K(p, x) = [b_{1K}(p, x)\eta(x_1) + b_{2K}(p, x)\eta(H_{K2}x_2)]\mu_0^K(p, x), \quad (4.13)$$

with

$$b_{1K} = 1 - \frac{rr_K B_K}{rr_K B_K + p_1}, \quad (4.14)$$

$$b_{2K} = \frac{rr_K B_K p_2}{(rr_K B_K + 2p_2)^2} \frac{rB_K + p_2}{rB_K + 2p_2} \frac{x_1}{\lambda'}.$$

H_{K2} is obtained from H_K in (4.10) by substituting p_2 into p , p_1 and p_2 being defined below (3.16). In most cases, $p_1, p_2 \ll B_K$, so that

$$b_{1K} \cong p_1 / rr_K B_K$$

$$b_{2K} \cong (p_2 / rr_K B_K) [(rB_K + p_2) / (rB_K + 2p_2)] (x_1 / \lambda'). \quad (4.14')$$

For $p_1, p_2 \gg B_K$ only b_{1K} remains.

Adding the influences of k and K , we obtain the "positive" temperature coefficient to be added to as

$$(1 + R_k f_k + R_K f_K)^{-1} [(1 + R_K f_K)(b_1 + b_2) + R_k f_k(b_{1k} + b_{2k}) + R_K f_K(b_{1K} + b_{2K})]$$

$$\cong (b_1 + b_2) - R_k f_k(b_1 + b_2 - b_{1k} - b_{2k}) + R_K f_K(b_{1K} + b_{2K}). \quad (4.15)$$

For the integral intensity a similar expression holds. From this one can see that the presence of k decreases the temperature coefficient, while the presence of K increases it, as far as $b_{1K} + b_{2K}$ is positive.

§ 5. Numerical results and discussion

For numerical calculations we must fix the values of parameters which are not so precisely determined as masses and lifetimes. The attenuation mean free path of the N -component, which by itself is not much sensitive to our results, may be fixed as

$$\lambda_n = 125 \text{ g cm}^{-2} \quad (5.1)$$

from cosmic ray experiments. The absorption mean free path of π -mesons, λ_π , is supposed to be smaller than λ_n , because nucleons produced by the collisions of π -mesons with nuclei are disregarded in the absorption of π -mesons, whereas all energetic N -particles are taken into account in the attenuation of N -particles. Hence λ_π may be assumed as close to the collision mean free path of π -mesons, that is,

$$\lambda_\pi = 60 \text{ g cm}^{-2}. \quad (5.2)$$

(5.1) and (5.2) lead us to the positive value of λ' , which we take as

$$\lambda' = 120 \text{ g cm}^{-2}. \quad (5.3)$$

With this choice of λ' there certainly results the positive temperature effect due to π - μ decays.

The momentum spectrum of π -mesons at production has been determined by Olbert¹²⁾ for momenta between 0.5 and 15 Gev/c. At momenta higher than this range the spectrum can be inferred from the absorption of cosmic rays underground. Thus we take

$$S(p) = \text{const.} (p + 3.5)^{-2.75}, \quad (5.4)$$

where momentum p is measured in Gev/c. What is directly to do with the temperature coefficient is not $S(p)$, but the production spectrum of μ -mesons, which has a shape similar to (5.4). The latter is used in computations.

Finally we need the survival probability. For $p < 15$ Gev/c this is given by Olbert¹²⁾ and for $p > 15$ Gev/c we calculated it by assuming the energy loss independent of momenta.

Thus we are able to calculate the values of x_1 , x_2 and $H_2 x_2$, also \bar{x}_1 and $\bar{H}_2 x_2$ as functions of momenta. These are plotted in Fig. 1.

For the temperature coefficient we are interested only in that part which contributes to the positive temperature effect due to π - μ decays. The coefficient is divided into b_1 and b_2 for the differential intensity and into B_1 and B_2 for the integral intensity. These are plotted in Fig. 2 against momenta. The part that depends on λ' is given in such a way that one gets correct values if multiplied by $(120/\lambda')$. This way of representation allows one to adjust the temperature coefficient, if a better value of λ' is known in future. At present we can merely draw b_2 , B_2 and $B_1 + B_2$ with λ' given in (5.3), as shown in Fig. 2.

The temperature coefficients due to heavy mesons are also plotted in Fig. 3, dropping factors $R_K f_K / (1 + R_K f_K + R_K f_K)$ etc. On account of the smallness of f_K the influence of heavy mesons would be appreciable only for $p \gtrsim B_K$.

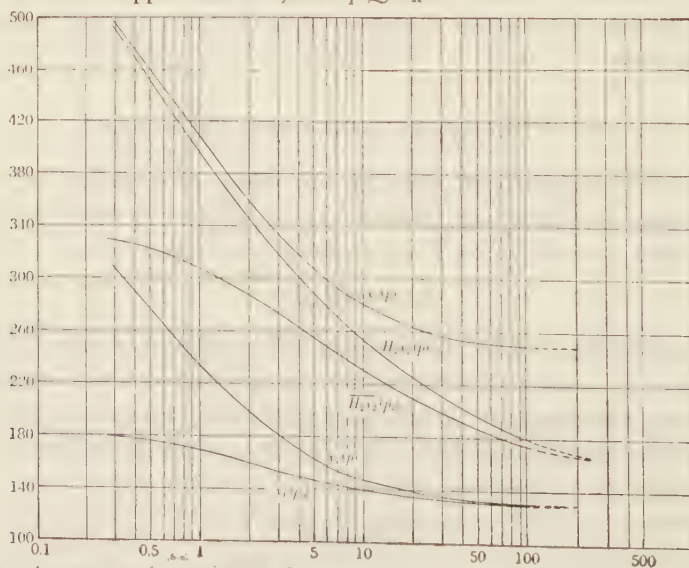


Fig. 1. Average depths responsible for the positive temperature effect. Definitions are given in the text for $x_1(p)$ by (3.13), $\bar{x}_1(p_0)$ by (3.21), $x_2(p)$ by (3.17), $H_2 x_2(p)$ by (3.16) and (3.17), $\bar{H}_2 x_2(p_0)$ by (3.22).

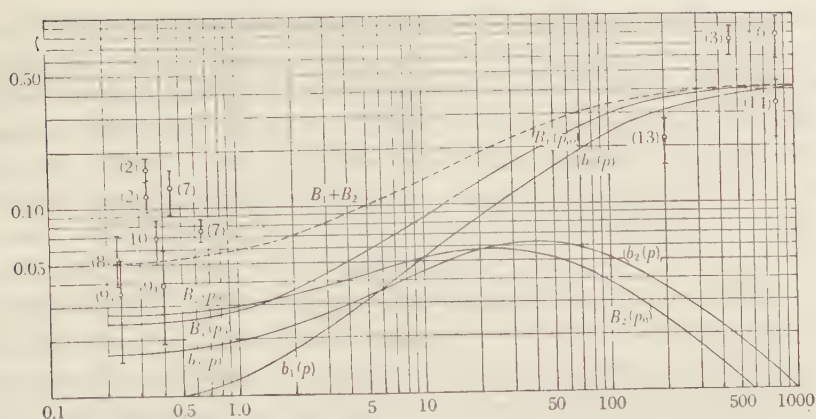


Fig. 2. Positive temperature coefficients. b_1 and b_2 are the coefficients for the differential intensity of μ -mesons at sea level. They are independent of and dependent on λ' respectively. B_1 and B_2 are the corresponding ones for the integral intensity. These coefficients are expressed in unit of percent per degree, dividing the corresponding quantities in the text by the average temperature 225°K. Figures attached on the experimental values indicate the reference numbers.

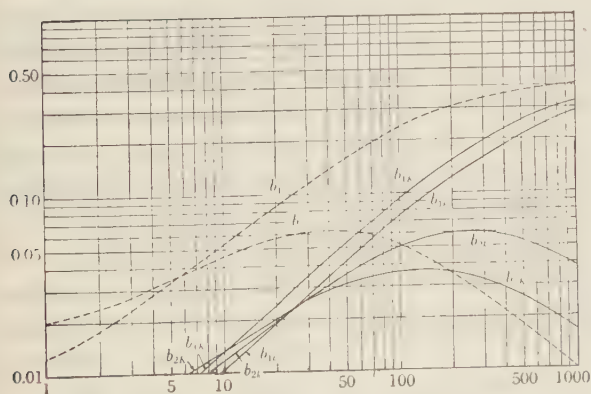


Fig. 3. Temperature coefficients due to heavy mesons. $b_{K\lambda}$ and b_k represent the coefficients for the differential intensity of μ -mesons generated through $K-\pi-\mu$ and $k-\mu$ decays respectively. Suffices 1 and 2 indicate λ independent and dependent parts. These coefficients are expressed in unit of percent per degree, dividing the corresponding quantities in the text by the average temperature 225°K. Weight factors in these coefficients, $(1+R_{k1}f_K+R_{k2}f_k)^{-1}R_{k1}f_K$ for b_{K1} and $(1+R_{k1}f_K+R_{k2}f_k)^{-1}R_{k2}f_k$ for b_{K2} , are omitted. f_k and f_K represent the frequencies of heavy mesons in comparison with that of π -mesons. R_k and R_K are given by (3.6) and (4.8) respectively.

A comparison with observed data has been made by Wada and Kudō²⁾ in Fig. 7 of their paper. The theoretical temperature coefficient calculated by us is definitely higher than that calculated by Maeda and Wada¹¹⁾, which are shown by curve *W* in their figure. This is due chiefly to their incorrect assumption on the production spectrum of mesons. Wada and Kudō corrected for the spectrum and also approximately for λ' , taking $\lambda' = 120 \text{ g cm}^{-2}$, the same value as (5.3), and obtained curves *R* and *S*. These are nearly in accordance with our result. The positive value of λ' increases the coefficient to certain extent and the resultant positive temperature coefficient is found to agree with some observations, for example, by Dawton and Elliot⁸⁾, by Chasson¹⁰⁾ and by Trumphy and Trefall⁹⁾. However, some high coefficients obtained by Duperier⁷⁾ and by Wada and Kudō²⁾, higher than

0.07%/°C, can not be made agreeable with our theoretical prediction. The discrepancy, if real, can not be explained by taking account of oblique rays. If these are included, the temperature coefficient should decrease in comparison with that based only on vertical rays.

It must be noticed, however, that the atmospheric depth at which the temperature is adopted for correlation calculations plays a very important role. Usually one adopts the temperature at 100 mb level⁷⁾ or the distance between 100 mb and 200 mb levels instead²⁾. The former choice is certainly too high on account of $\bar{x}_1 \cong 180$ mb. The latter choice may be good enough, as far as the part dependent on λ' is negligible. But this part is not always negligible, since \bar{x}_1/λ' is presumably larger than unity. For this part the temperature should be taken at \bar{H}_{x_2} that is greater than 200 mb in practical cases. Consequently, one should pay attention to the temperature at about 200 mb level. As this level lies in the troposphere, the temperature variation may be larger than that at 100 mb level. If one takes the correlation with temperatures at 100 mb levels, one may thus obtain too large a temperature coefficient*).

About the influence of heavy mesons very little can be drawn from our calculation. We can only set on an upper limit for the production rate of heavy mesons. Experimental uncertainties in the temperature coefficient at high momenta may not exceed 50%¹³⁾¹⁴⁾. Within this uncertainty the temperature coefficient is well accounted for in terms of π - μ decays alone. This leads us to an upper limit of $R_K f_K$ to be 0.5, unless there arises fortuitous cancellation between the contributions from K and k (taking account of $R_K \sim 1/2$, therefore, we obtain $f_K < 1$). This seems to be consistent with other observational facts, such as the behaviour of cosmic rays underground and the mass spectrum of secondary particles from high energy showers.

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*) Mr. K. Maeda kindly called our attention to the observed fact that the temperature variations at 100 mb and at 200 mb were not much different on the average but could sometimes behave in opposite ways in day-to-day variations.

On Quantum Electrodynamics without Subsidiary Conditions

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The formalism of quantum electrodynamics without subsidiary conditions was obtained in a relativistically covariant way, starting from Heisenberg's representation. Transforming the formalism of quantum electrodynamics in Heisenberg's representation to the one in the interaction representation, the equivalence of this formalism to the ordinary quantum electrodynamics introduced by Fermi was proved.

§ 1. Introduction

The divergence difficulties of the norm of the state vector accompanying Lorentz's condition in quantum electrodynamics are well-known facts from the old time.¹⁾ The procedure avoiding these divergence difficulties was proposed by Gupta²⁾ and Bleuler,³⁾ and a rather radical interpretation of this procedure was tried by using the indefinite metric in Hilbert's space introduced by Dirac.⁴⁾ Furthermore, the formulation of quantum electrodynamics without subsidiary conditions was proposed by Valatin,⁵⁾ whose method starts out from the interaction representation, assuming that the interaction Hamiltonian is composed of the transverse part and the longitudinal photon part leading to the Coulomb energy.

In their above methods there are some defects: Gupta and Bleuler have used the concept of the indefinite metric, which seems to me to be inadequate and Valatin has adopted the method mentioned above, which seems to lack the necessity with respect to the derivation of the formulation.

The method proposed here starts from a formulation of quantum electrodynamics expressed rigorously in Heisenberg's representation, and after dividing the electromagnetic field potential into the transverse and the other parts, the integrals of these wave equations were exactly obtained by using Green's functions. And furthermore, by employing the procedure derived by Glauber⁶⁾ and Umezawa⁷⁾, the formulation in Heisenberg's representation is transformed into the gauge invariant one in the interaction representation and the equivalence of this formalism to the ordinary quantum electrodynamics is proved. Here the gauge difference in a wide sense produced from the other parts of the electromagnetic field potential A_μ gives the ambiguities of Coulomb potential energy in the interaction Hamiltonian, which can be vanished. This circumstance is due to the reason that only three components of A_μ can be determined, while the remaining one component of A_μ is undetermined.

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§ 2. Quantum electrodynamics in Heisenberg's representation

The quantized wave equations for the system of electrons and electromagnetic field interacting with each other without using Lorentz's condition are expressed as follows :

$$\{\gamma_\mu (\partial_\mu - ieA_\mu(x)) + \kappa\} \psi(x) = 0, \quad (1)$$

$$(\partial_{\mu\nu} \square - \partial_\mu \partial_\nu) A_\nu(x) = -j_\mu(x), \quad (2)$$

$$j_\mu(x) = ie\psi(x) \gamma_\mu \psi(x), \quad (3)$$

where the dynamical variables in Heisenberg's representations are denoted by the bold letters and the ones in the interaction representation by the usual letters. $A_\mu(x)$ and $\psi(x)$ are the quantized electromagnetic potential and the electron wave functions respectively.

Here we divide $A_\mu(x)$ into the following two parts :

$$A_\mu(x) = \mathfrak{A}_\mu(x) + \mathfrak{B}_\mu(x), \quad (4)$$

where $\mathfrak{A}_\mu(x)$ indicates the transverse part of the electromagnetic potential and satisfies two conditions :

$$\partial_\mu \mathfrak{A}_\mu(x) = 0, \quad n_\mu \mathfrak{A}_\mu(x) = 0, \quad (5)$$

where n_μ is a time-like unit vector $n_\mu^2 = -1$. Now we introduce the projection operator $T_{\mu\nu}$, which produces the transverse part of A_μ :

$$\mathfrak{A}_\mu = T_{\mu\nu} A_\nu. \quad (6)$$

Here $T_{\mu\nu}$ operator satisfies the following conditions :

$$\partial_\mu T_{\mu\nu} = 0, \quad n_\mu T_{\mu\nu} = 0, \quad T_{\lambda\mu} T_{\mu\nu} = T_{\lambda\nu}. \quad (7)$$

Such a projection operator satisfying the above condition (7) is expressed as follows :

$$\begin{aligned} T_{\mu\nu} = & \delta_{\mu\nu} + (n_\mu \square - \partial_\mu \partial_\nu) \{ (\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2) \} n_\nu \\ & - (n_\mu \partial + \partial_\mu) \{ (\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2) \} \partial_\nu, \end{aligned} \quad (8)$$

where α is any numerical constant.

Then the quantized wave equations of $\mathfrak{A}_\mu(x)$, which satisfy the conditions (5), are given in the following by operating $T_{\lambda\mu}$ to the equation (2) :

$$\square \mathfrak{A}_\mu(x) = -j_\mu(x) - (n_\mu \square - \partial \partial_\mu) \{ (\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2) \} n_\lambda j_\lambda(x), \quad (9)$$

where $\partial = n_\mu \partial_\mu$, and ∂^{-1} , $(\square + \partial^2)^{-1}$ expresses the symbolical operator, which means the multiplication by $(in_\lambda k_\lambda)^{-1}$, $- \{k_\lambda^2 + (n_\lambda k_\lambda)^2\}^{-1}$ in Fourier's transformation of functions considered now respectively. $(\square + \partial^2)^{-1} n_\lambda j_\lambda(x)$ is the expression related to the Coulomb potential energy, i.e.,

$$(\square + \partial^2)^{-1} n_\lambda j_\lambda(x) = \int_\sigma n_\lambda j_\lambda(x') \partial \mathfrak{D}(x-x') d\sigma', \quad \text{if } n_\mu(x_\mu - x'_\mu) = 0, \quad (10)$$

where σ is the space-like surface and $\mathfrak{D}(x-x')$ is defined by

$$\partial^2 \mathcal{D}(x-x') = D(x-x'), \quad (11)$$

and $\partial \mathcal{D}(x)$ is the following function:⁽⁸⁾

$$\left. \begin{aligned} \partial \mathcal{D}(x) &= (1/4\pi) 1/\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}, & \text{for } x_\mu^2 > 0, \\ &= 0, & \text{for } x_\mu^2 < 0, \\ &= (1/8\pi) (1/n_\mu x_\mu), & \text{for } x_\mu^2 = 0. \end{aligned} \right\} \quad (12)$$

From equations (2) and (9), we get the quantized wave equation of $\mathfrak{B}_\mu(x)$:

$$\square \mathfrak{B}_\mu(x) - \partial_\mu \partial_\lambda \mathfrak{B}_\lambda(x) = (\square n_\mu - \partial_\mu \partial) \{ (\square + \partial^2)^{-1} + \alpha \delta(\square + \partial^2) \} n_\lambda \mathbf{j}_\lambda(x). \quad (13)$$

These expressions (9) and (13) are gauge invariant, where the former in a narrow sense ($\square A = 0$) and the latter in a wide sense. Before solving these equations, we prove that the original A_μ has only three determined solutions and the remaining one component of A_μ can not be determined. If we substitute the Fourier's transformation of A_μ :

$$A_\mu(x) = \int a_\mu(k) e^{ikx} (dk) \quad (14)$$

into (2), we get from (2):

$$k_\lambda^2 a_\mu(k) - k_\mu k_\lambda a_\lambda(k) = F_\mu(k),$$

where $F_\mu(k)$ expresses the Fourier's transformation of the source function and satisfies the following condition

$$k_\mu F_\mu(k) = 0.$$

The rank of the following matrix to give the coefficients of the simultaneous algebraic equations to determine $a_\mu(k)$ is 3:

$$\begin{pmatrix} k_1^2 - k_\lambda^2 & k_1 k_2 & k_1 k_3 & k_1 k_4 \\ k_2 k_1 & k_2^2 - k_\lambda^2 & k_2 k_3 & k_2 k_4 \\ k_3 k_1 & k_3 k_2 & k_3^2 - k_\lambda^2 & k_3 k_4 \\ k_4 k_1 & k_4 k_2 & k_4 k_3 & k_4^2 - k_\lambda^2 \end{pmatrix}.$$

Therefore only three components of $a_\mu(k)$ can be determined, while the remaining one component of $a_\mu(k)$ is undetermined.

The solutions of quantized wave equations (1), (9) and (13) are expressed in the following integral forms:

$$\phi(x) = \phi(x, \sigma) - ie \int S^\sigma(x, x') \gamma_\mu A_\mu(x') \phi(x') dx', \quad (15)$$

$$\begin{aligned} \mathfrak{A}_\mu(x) &= \mathfrak{A}_\mu(x, \sigma) + \int \mathbf{j}_\nu(x'') d_{\nu\mu}^1(x) D^\sigma(x, x'') dx'' \\ &+ \int (n_\nu \square - \partial_\nu \partial) d_{\nu\mu}^1(x) D^\sigma(x, x'') [(\square'' + \partial''^2)^{-1} + \alpha \delta(\square'' + \partial''^2)] n_\lambda \mathbf{j}_\lambda(x'') dx'', \end{aligned} \quad (16)$$

$$\mathfrak{B}_\mu(x) = \mathfrak{B}_\mu(x, \sigma) + n_\mu \{(\square + \partial^2)^{-1} + \alpha \delta(\square + \partial^2)\} n_\lambda j_\lambda(x), \quad (17)$$

where

$$S^\sigma(x, x'') = (\gamma_\mu \partial_\mu - \kappa) D^\sigma(x, x''), \quad (18)$$

$$\Delta^\sigma(x, x'') = 1/2 \cdot \{\epsilon(\sigma, x'') - \epsilon(x - x'')\} D(x - x''), \quad (19)$$

$$D^\sigma(x, x'') = 1/2 \cdot \{\epsilon(\sigma, x'') - \epsilon(x - x'')\} D(x - x''), \quad (20)$$

$$d_{\nu\mu}^1 = \delta_{\mu\nu} - (\partial_\mu \partial_\nu \partial^{-2} + n_\mu \partial_\nu \partial^{-1} + n_\nu \partial_\mu \partial^{-1}), \quad (21)$$

$$\epsilon(x - x'') = \begin{cases} +1, & \text{for } x_0 > x_0'', \\ -1, & \text{for } x_0 < x_0'', \end{cases}$$

$$\epsilon(\sigma, x'') = \begin{cases} +1, & \text{if } x'' \text{ is earlier than } \sigma, \\ -1, & \text{if } x'' \text{ is later than } \sigma. \end{cases}$$

Here x does not necessarily lie on σ . $\mathfrak{I}_\mu(x, \sigma)$, $\mathfrak{B}_\mu(x, \sigma)$ and $\psi(x, \sigma)$ satisfy the following free equations and the conditions:

$$(\gamma_\mu \partial_\mu + \kappa) \psi(x, \sigma) = 0, \quad \square \mathfrak{I}_\mu(x, \sigma) = 0, \quad (\square \delta_{\mu\nu} - \partial_\mu \partial_\nu) \mathfrak{B}_\nu(x, \sigma) = 0, \quad (22)$$

$$\partial_\mu \mathfrak{I}_\mu(x, \sigma) = 0, \quad n_\mu \mathfrak{I}_\mu(x, \sigma) = 0. \quad (23)$$

(16) is reduced to the following form:

$$\begin{aligned} \mathfrak{I}_\mu(x) = & \mathfrak{I}_\mu(x, \sigma) + \int j_\nu(x'') d_{\nu\mu}^1 D^\sigma(x, x'') dx'' \\ & - (n_\mu \partial + \partial_\mu) \partial^{-1} [(\square + \partial^2)^{-1} + \alpha \delta(\square + \partial^2)] n_\lambda j_\lambda(x), \end{aligned} \quad (24)$$

by making use of the properties:

$$n_\mu d_{\mu\nu}^1 = 0, \quad \partial_\mu d_{\nu\mu}^1 = -(n_\nu \partial + \partial_\nu) \partial^{-2} \square. \quad (25)$$

It is easily proved that the integral (24) satisfies the two conditions (5);

$$\begin{aligned} \partial_\mu \mathfrak{I}_\mu(x) &= \partial_\mu \mathfrak{I}_\mu(x, \sigma) - \int j_\nu(x'') (n_\nu \partial + \partial_\nu) \partial^{-2} \square D^\sigma(x, x'') dx'' - \partial^{-1} (n_\lambda j_\lambda(x)) \\ &= \int j_\nu(x'') (n_\nu \partial + \partial_\nu) \partial^{-2} \delta^4(x - x'') dx'' - \partial^{-1} (n_\lambda j_\lambda(x)) \\ &= (n_\nu \partial + \partial_\nu) \partial^{-2} j_\nu(x) - \partial^{-1} (n_\lambda j_\lambda(x)) = 0. \end{aligned}$$

Since the fact that the solution (15) satisfies the wave equation (1) is well known, the proof is omitted. As the peculiar Green's function: $d_{\nu\mu}^1 D^\sigma(x, x'')$ is used in (16) or (24), we prove that (24) satisfies the equation (9).

$$\begin{aligned} \square \mathfrak{I}_\mu(x) = & - \int j_\nu(x'') d_{\nu\mu}^1 \partial^4(x - x'') dx'' - (n_\mu \square + \partial^{-1} \partial_\mu \square) [(\square + \partial^2)^{-1} + \alpha \delta(\square + \partial^2)] \\ & \times n_\lambda j_\lambda(x) \end{aligned}$$

$$\begin{aligned}
&= -\mathbf{j}_\mu(x) + \partial_\mu \partial^{-1} (n_\nu \mathbf{j}_\nu(x)) - (\square n_\mu - \partial \partial_\mu) [(\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2)] n_\lambda \mathbf{j}_\lambda(x) \\
&\quad - \partial_\mu \partial^{-1} (\square + \partial^2) [(\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2)] n_\lambda \mathbf{j}_\lambda(x) \\
&= -\mathbf{j}_\mu(x) - (\square n_\mu - \partial \partial_\mu) [(\square + \partial^2)^{-1} + \alpha \partial (\square + \partial^2)] n_\lambda \mathbf{j}_\lambda(x).
\end{aligned}$$

As $\mathfrak{A}_\mu(x)$ has two components, $\mathfrak{B}_\mu(x)$ must have one determined component. Then the remaining undetermined component $\mathfrak{A}_\mu(x, \sigma)$ is related to the difference of gauge $\partial_\mu A(x)$ in a wide sense, where $A(x)$ is an arbitrary function, because the wave equation which $\partial_\mu A$ satisfies becomes an identity relation:

$$(\square \partial_{\mu\nu} - \partial_\mu \partial_\nu) \partial_\nu A(x) \equiv 0.$$

It is proved in the next section that this arbitrary term $A(x)$ and the term related to α do not contribute to the final result. Thus the final solutions in Heisenberg's representation are given by (15), (24) and (17).

§ 3. Quantum electrodynamics in the interaction representation

Here we transform the formulation obtained above into the one in the interaction representation by using Glauber's procedure⁹⁾ (quite similar to Umezawa's method). If x lies on σ , we get from (15), (24) and (17):

$$\begin{aligned}
\phi(x)|_{x\sigma\sigma} &= \phi(x/\sigma), \quad \mathfrak{A}_\mu(x)|_{x\sigma\sigma} = \mathfrak{A}_\mu(x/\sigma) - \alpha(n_\mu \partial + \partial_\mu) \partial^{-1} \delta(\square + \partial^2) n_\lambda j_\lambda(x/\sigma), \\
\mathfrak{B}_\mu(x)|_{x\sigma\sigma} &= \partial_\mu A(x/\sigma) + n_\mu (\square + \partial^2)^{-1} n_\lambda j_\lambda(x/\sigma) + \alpha n_\mu \partial (\square + \partial^2) n_\lambda j_\lambda(x/\sigma), \quad (26)
\end{aligned}$$

by using the properties:

$$S^\sigma(x, x')|_{x\sigma\sigma} = 0, \quad D^\sigma(x, x')|_{x\sigma\sigma} = 0, \quad (27)$$

where $x\sigma$ and x/σ mean that x lies on σ . The interaction Hamiltonian $H(x', n)$ can be determined by the following relations and commutation relations:

$$[\phi(x/\sigma), H(x', n)] = i \cdot \partial \phi(x/\sigma) / \partial \Omega(x'), \quad (28)$$

$$[\mathfrak{A}_\mu(x/\sigma), H(x', n)] = i \partial \mathfrak{A}_\mu(x/\sigma) / \partial \Omega(x'), \quad (29)$$

$$\left. \begin{aligned}
\{\bar{\phi}_\alpha(x), \phi_\beta(x')\} &= i \delta_{\alpha\beta}(x-x'), \\
[\mathfrak{A}_\mu(x), \mathfrak{A}_\nu(x')] &= i d_{\nu\mu}^1 D(x-x'), \\
\text{other commutators} &= 0.
\end{aligned} \right\} \quad (30)$$

By differentiating functionally on the equations (15) and (24), we get:

$$\begin{aligned}
i \cdot \partial \phi(x/\sigma) / \partial \Omega(x') &= -e S(x-x') \gamma_\mu \phi(x'/\sigma) \{ \mathfrak{A}_\mu(x'/\sigma) + 1/2 \cdot n_\mu (\square' + \partial'^2)^{-1} n_\lambda j_\lambda(x'/\sigma) \\
&\quad - \alpha / 2 \cdot \partial'_\mu \partial'^{-1} \delta(\square' + \partial'^2) n_\lambda j_\lambda(x'/\sigma) + \partial'_\mu A(x'/\sigma) \} \\
&\quad - e / 2 \cdot \{ n_\mu (\square' + \partial'^2)^{-1} n_\lambda j_\lambda(x'/\sigma) - \alpha \partial'_\mu \partial'^{-1} \delta(\square' + \partial'^2) n_\lambda j_\lambda(x'/\sigma) \} \\
&\quad \times S(x-x') \gamma_\mu \phi(x'/\sigma), \quad (31)
\end{aligned}$$

$$i \cdot \partial \mathfrak{A}_\mu(x/\sigma) / \partial \Omega(x') = -i j_\nu(x'/\sigma) d_{\nu\mu}^1 D(x-x'), \quad (32)$$

through the use of

$$\begin{aligned}\partial \in (\sigma, x'') / \partial \Omega(x') &= 2\delta^1(x' - x''), \quad \partial S^\sigma(x, x'') / \partial \Omega(x') = \delta^1(x' - x'') S(x - x'), \\ \partial D^\sigma(x, x'') / \partial \Omega(x') &= \delta^1(x' - x'') D(x - x').\end{aligned}$$

where we used the following integral instead of (12) :

$$\phi(x) = \phi(x, \sigma) - \frac{ie}{2} \int S^\sigma(x, x'') \gamma_\mu \{A_\mu(x'') \phi(x'') + \phi(x'') A_\mu(x'')\} dx''. \quad (33)$$

One can show that (31), (32) are satisfied by the following Hamiltonian by making use of (30) :

$$\begin{aligned}H(x) &= -j_\mu(x) \mathfrak{H}_\mu(x) - \frac{1}{2} n_\mu j_\mu(x) \int_\sigma n_\lambda j_\lambda(x') \partial' \mathfrak{D}(x - x') d\sigma' \\ &\quad + \alpha/2 \{ [\partial_\mu \partial^{-1} \partial (\square + \partial^2) n_\lambda j_\lambda(x)] j_\mu(x) + j_\mu(x) [\partial_\mu \partial^{-1} \partial (\square + \partial^2) n_\lambda j_\lambda(x)] \} \\ &\quad - j_\mu(x) \partial_\mu A(x),\end{aligned} \quad (34)$$

where we put

$$\phi(x, -\infty) = \phi(x), \quad \mathfrak{H}_\mu(x, -\infty) = \mathfrak{H}_\mu(x), \quad n_\mu(x_\mu - x'_\mu) = 0 \quad (35)$$

and $\phi(x)$ and $\mathfrak{H}(x)$ satisfy the following relations :

$$(\gamma_\mu \partial_\mu + \kappa) \phi(x) = 0, \quad \square \mathfrak{H}_\mu(x) = 0, \quad n_\mu \mathfrak{H}_\mu(x) = 0, \quad \partial_\mu \mathfrak{H}_\mu(x) = 0. \quad (36)$$

The equation of motion for state vector $\Psi[\sigma]$ is

$$i \cdot \partial \Psi[\sigma] / \partial \Omega(x) = H(x) \Psi[\sigma], \quad (37)$$

where the third and the 4th terms vanish by making use of the following unitary transformation of $\Psi[\sigma]$:

$$\Psi[\sigma] = \exp(iA) \phi[\sigma], \quad A = \frac{\alpha}{2} \int_\sigma \{ j_\mu(x') [\partial'^{-1} \partial (\square' + \partial'^2) n_\lambda j_\lambda(x')] \} \quad (38)$$

$$+ [\partial'^{-1} \partial (\square' + \partial'^2) n_\lambda j_\lambda(x')] j_\mu(x') \} d\sigma'_\mu - \int_\sigma j_\mu(x') A(x') d\sigma,$$

$$ie^{-iA} \frac{\partial \Psi}{\partial \Omega(x)} = - \frac{\partial A}{\partial \Omega(x)} \phi + i \frac{\partial \phi}{\partial \Omega(x)}$$

$$= - \frac{\alpha}{2} \{ j_\mu(x) [\partial_\mu \partial^{-1} \partial (\square + \partial^2) n_\lambda j_\lambda(x)] + [\partial_\mu \partial^{-1} \partial (\square + \partial^2) n_\lambda j_\lambda(x)] j_\mu(x) \} \phi$$

$$+ j_\mu(x) \partial_\mu A(x) \phi + i \frac{\partial \phi}{\partial \Omega(x)},$$

$$\exp(-iA) H(x) \exp(iA) = H(x) - i[A, H(x)] + \dots$$

$$\begin{aligned}
[A, H(x)] = & \int_{\sigma} [j_{\mu}(x) \mathfrak{I}_{\mu}(x) + \frac{1}{2} n_{\mu} j_{\mu}(x) \int_{\sigma} n_{\lambda} j_{\lambda}(x') \partial' \mathcal{D}(x-x') d\sigma' - j_{\mu}(x) \partial_{\mu} A(x) \\
& + \frac{\alpha}{2} \{j_{\mu}(x) (\partial_{\mu} \partial^{-1} \partial (\square + \partial^2) n_{\lambda} j_{\lambda}(x)) + (\partial_{\mu} \partial^{-1} \partial (\square + \partial^2) n_{\lambda} j_{\lambda}(x)) j_{\mu}(x)\}, \\
& \frac{\alpha}{2} \{j_{\mu}(x') (\partial'^{-1} \partial (\square' + \partial'^2) n_{\lambda} j_{\lambda}(x')) + (\partial'^{-1} \partial (\square' + \partial'^2) n_{\lambda} j_{\lambda}(x')) j_{\mu}(x')\} \\
& - j_{\nu}(x') A(x') d\sigma_{\nu}' \}.
\end{aligned}$$

Here the ambiguous terms relating to α become zero and if $A(x)$ is a c -number, all terms of the right hand side containing $A(x)$ vanish, while if $A(x)$ is any q -number independent of $\psi(x)$, quite ambiguous results are produced. However, as the equation of $A(x)$ does not exist and furthermore $A(x)$ has not any definite commutation relation, $A(x)$ can not describe the real physical field. Then we must consider $A(x)$ as the c -number not corresponding to any physical field. Therefore the final equation of motion for state vector $\Psi[\sigma]$ becomes as follows:

$$i \cdot \delta \Psi[\sigma] / \delta \mathcal{U}(x) = \{-j_{\mu}(x) \mathfrak{I}_{\mu}(x) - \frac{1}{2} \int_{\sigma} n_{\mu} j_{\mu}(x) n_{\lambda} j_{\lambda}(x') \partial \mathcal{D}(x-x') d\sigma'\} \Psi[\sigma]. \quad (39)$$

This formula is completely equivalent to the form of the ordinary quantum electrodynamics. If we adopt only 2 times of the second term, (31) becomes as the following:

$$[\psi(x), H(x')] = -e \mathcal{S}(x-x') \gamma_{\mu} \psi(x') \{ \mathfrak{I}_{\mu}(x') + n_{\mu} \int_{\sigma} n_{\lambda} j_{\lambda}(x') \partial'' \mathcal{D}(x'-x') d\sigma'' \}. \quad (40)$$

If we calculate the left hand side of (40) by using (34), we get the formula (31). By making use of the following relation,

$$[\psi(x), j_{\mu}(x')] = e \mathcal{S}(x-x') \gamma_{\mu} \psi(x'),$$

(31) becomes as follows:

$$\begin{aligned}
[\psi(x), H(x')] = & -e \mathcal{S}(x-x') \gamma_{\mu} \psi(x') \{ \mathfrak{I}_{\mu}(x') + n_{\mu} \int_{\sigma} n_{\lambda} j_{\lambda}(x') \partial'' \mathcal{D}(x'-x') d\sigma'' \} \\
& - \frac{e}{2} \mathcal{S}(x-x') \gamma_{\lambda} n_{\lambda} \psi(x') \int_{\sigma} \partial'' \mathcal{D}(x'-x'') \mathcal{S}(x''-x') \gamma_{\mu} n_{\mu} d\sigma''.
\end{aligned}$$

The second term is related to the Coulomb static self energy of electrons. Then in order to eliminate such a term, we must adopt the symmetrical expression $1/2 \{A_{\mu}(x'') \psi(x'') + \psi(x'') A_{\mu}(x'')\}$ instead of $A_{\mu}(x'') \psi(x'')$ in (12).

§ 4. Conclusions

We obtained the relativistically covariant formulation of quantum electrodynamics without subsidiary conditions starting from Heisenberg's representation, by dividing the electromagnetic field potential into the transverse and other parts. The latter part has the connection to the Coulomb static potential, which has the ambiguities relating to the

difference of gauge in a wide sense. But these ambiguous parts can be vanished by the unitary transformation of the state vector in the interaction representation. By making use of the suitable Green's functions and putting the differential operator of the source function outside the integrals, the equations of motion in Heisenberg's representation are solved. And furthermore, the equivalence of this formalism to the one of the ordinary quantum electrodynamics introduced by Fermi was proved by transforming the formalism in Heisenberg's representation into the one in the interaction representation. But in order to avoid the term related to the Coulomb static self energy in the final interaction Hamiltonian, we must take the symmetrized interaction term in the expression of the interaction term between electrons and electromagnetic field potential.

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Mathematical appendix

Calculation of $\mathcal{D}(x)$ and $\partial\mathcal{D}(x)$ functions are performed as follows. As the integral of $D(x)$ function is expressed as follows:¹⁰⁾

$$D(x) = -\frac{i}{(2\pi)^4} \int (dk) \int_{-\infty}^{\infty} da \exp[iak_{\mu}^2 + ik_{\mu}x_{\mu}] \epsilon(k), \quad (\text{A} \cdot 1)$$

$\partial\mathcal{D}(x)$ is written in the following integral form :

$$\partial\mathcal{D}(x) = \frac{1}{(2\pi)^4} \int \frac{(dk)}{n_{\mu}k_{\mu}} \int_{-\infty}^{\infty} da \exp[iak_{\mu}^2 + ik_{\mu}x_{\mu}] \epsilon(k). \quad (\text{A} \cdot 2)$$

The integral representations

$$\epsilon(k) = -\frac{\epsilon_{\mu}k_{\mu}}{|\epsilon_{\mu}k_{\mu}|} = \frac{i}{\pi} \int_{-\infty}^{\infty} \exp[i\epsilon_{\mu}k_{\mu}\tau] \frac{d\tau}{\tau}, \quad (\text{A} \cdot 3)$$

$$P \frac{1}{n_{\mu}k_{\mu}} = -\frac{i}{2} \int_{-\infty}^{\infty} \exp(in_{\mu}k_{\mu}a) \frac{a}{|a|} da \quad (\text{A} \cdot 4)$$

enable the integration over k space to be effected :

$$\begin{aligned} \partial\mathcal{D}(x) &= \frac{-i}{\pi(2\pi)^6} \int (dk) \int_{-\infty}^{\infty} \exp(ibn_{\mu}k_{\mu}) db \int_{-\infty}^{\infty} \frac{dc}{c} \exp(ibc) \\ &\quad \times \int_{-\infty}^{\infty} da \exp[iak_{\mu}^2 + ik_{\mu}x_{\mu}] \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp(i\epsilon_{\mu}k_{\mu}\tau) \\ &= \frac{-2i}{(2\pi)^6} \int (dk) \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{d}{c} \int_{-\infty}^{\infty} \frac{da}{a} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp[i(ak_{\mu}^2 + k_{\mu}x_{\mu} + bn_{\mu}k_{\mu} + \epsilon_{\mu}k_{\mu}\tau)] \exp(ibc). \end{aligned}$$

If we put

$$k'_{\mu} = k_{\mu} + \frac{1}{2a}(x_{\mu} + bn_{\mu} + \epsilon_{\mu}\tau)$$

and use the following relation :

$$\int (dk') \exp(ik'_{\mu}{}'^2) = \frac{i\pi^2}{a|a|},$$

we get

$$\begin{aligned} \partial \mathcal{D}(x) &= \frac{1}{2(2\pi)^4} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp \left[-\frac{i}{4a} (x_{\mu} + bn_{\mu} + \epsilon_{\mu}\tau)^2 \right] \exp(ibc) \\ &= \frac{2}{(2\pi)^4} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp(ibc) \int_0^{\infty} d\alpha \{ \exp[-i\alpha(x_{\mu} + bn_{\mu} + \epsilon_{\mu}\tau)^2] \\ &\quad - \exp[i\alpha(x_{\mu} + bn_{\mu} + \epsilon_{\mu}\tau)^2] \}. \end{aligned}$$

If we perform the b' integration by using the relation

$$\int_{-\infty}^{\infty} db' \exp(\pm iab'^2) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\pm i\frac{\pi}{4}\right),$$

and putting

$$b' = b - \left(n_{\mu}x_{\mu} + \tau n_{\mu}\epsilon_{\mu} - \frac{c}{2\alpha} \right),$$

we get

$$\begin{aligned} \partial \mathcal{D}(x) &= \frac{2\sqrt{\pi}}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp[ic(n_{\mu}x_{\mu} - \tau)] \times \\ &\quad \times \int_0^{\infty} \frac{d\alpha}{\sqrt{\alpha}} \left\{ \exp \left[-i\alpha \left\{ x_{\mu}^2 + (n_{\mu}x_{\mu})^2 + \frac{c^2}{4\alpha^2} \right\} + i\frac{\pi}{4} \right] \right. \\ &\quad \left. - \exp \left[i\alpha \left\{ x_{\mu}^2 + (n_{\mu}x_{\mu})^2 + \frac{c^2}{4\alpha^2} \right\} - i\frac{\pi}{4} \right] \right\}, \end{aligned}$$

where we put $\epsilon_{\mu} = n_{\mu}$, because ϵ_{μ} is any time-like vector. If we put $\alpha = a^2$ and use the next relation

$$\int_0^{\infty} da \exp \pm i \left[x^2 a^2 + \frac{c}{4a^2} - \frac{\pi}{4} \right] = \frac{\sqrt{\pi}}{2x} \exp(\pm icx),$$

the following result is obtained :

$$\begin{aligned} \partial \mathcal{D}(x) &= \frac{4\sqrt{\pi}}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp[ic(n_{\mu}x_{\mu} - \tau)] \times \\ &\quad \times \int_0^{\infty} da \left\{ \exp \left[-i \left\{ (x_{\mu}^2 + (n_{\mu}x_{\mu})^2) a^2 + \frac{c^2}{4a^2} \right\} \right] \cdot \exp\left(i\frac{\pi}{4}\right) \right. \\ &\quad \left. - \exp \left[i \left\{ (x_{\mu}^2 + (n_{\mu}x_{\mu})^2) a^2 + \frac{c^2}{4a^2} \right\} \right] \exp\left(-i\frac{\pi}{4}\right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{8i\pi}{2(2\pi)^4} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \int_0^{\infty} \frac{dc}{c} \sin \{ (n_{\mu} x_{\mu} - \tau) c \} \frac{1}{\sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}} \{ \exp [-ic \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}] \\
&\quad - \exp [ic \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}] \} \\
&= \frac{4\pi}{(2\pi)^3} \frac{1}{\sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}} \int_0^{\infty} \frac{dc}{c} \cos (n_{\mu} x_{\mu} c) \sin (c \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}),
\end{aligned}$$

where the next relation was used :

$$\int_{-\infty}^{\infty} \frac{d\tau}{\tau} \cos \tau c = 0, \quad \int_0^{\infty} \frac{d\tau}{\tau} \sin \tau c = \pi \quad (c > 0).$$

By making use of the following formula :

$$\left. \begin{aligned} \int_0^{\infty} \sin qx \cos px \frac{dx}{x} &= \frac{\pi}{2} & \text{for } q > p, \\ &= 0 & \text{for } q < p, \\ &= \frac{\pi}{4} & \text{for } q = p, \end{aligned} \right\}$$

we get the final result :

$$\left. \begin{aligned} \partial \mathcal{D}(x) &= \frac{1}{4\pi} \frac{1}{\sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}}, & \text{for } \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} > n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 > 0, \\ &= 0, & \text{for } \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} < n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 < 0, \\ &= \frac{1}{8\pi} \frac{1}{n_{\mu} x_{\mu}}, & \text{for } \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} = n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 = 0. \end{aligned} \right\} \quad (\text{A} \cdot 5)$$

Here in the case of $n_{\mu} x_{\mu} = 0$, $\partial \mathcal{D}(x) = \frac{1}{4\pi} \frac{1}{\sqrt{x_{\mu}^2}}$ expresses the covariant formulation of Coulomb potential.

Next we perform the explicit integration of $\mathcal{D}(x)$ function similarly as before.

$$\begin{aligned}
\mathcal{D}(x) &= \frac{i}{(2\pi)^4} \int \frac{(dk)}{(n_{\lambda} k_{\lambda})^2} \int_{-\infty}^{\infty} da \epsilon(k) \exp [i a k_{\mu}^2 + i k_{\mu} x_{\mu}] \\
&= \frac{-1}{\pi (2\pi)^6} \int (dk) \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{dh}{h} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} da \times \\
&\quad \times \exp [i (n_{\mu} k_{\mu}) (b + c + \tau)] \exp i (bg + ch) \exp (i a k_{\mu}^2 + i k_{\mu} x_{\mu}),
\end{aligned}$$

where we put $\epsilon_{\mu} = n_{\mu}$. Integrating over k_{μ} , we get

$$\begin{aligned}
\mathcal{D}(x) &= - \frac{i}{2(2\pi)^6} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} \frac{dc}{c} \int_{-\infty}^{\infty} \frac{dh}{h} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp i (bg + ch) \\
&\quad \times \int_{-\infty}^{\infty} \frac{da}{a|a|} \exp \left[- \frac{i}{4a} \{ x_{\mu} + (b + c + \tau) n_{\mu} \}^2 \right]
\end{aligned}$$

$$= -\frac{2i}{(2\pi)^6} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} \frac{dh}{h} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \exp i(bg+ch) \\ \times \int_0^{\infty} d\alpha \{ \exp[-i\alpha \{x_\mu + (b+c+\tau)n_\mu\}^2] - \exp[i\alpha \{x_\mu + (b+c+\tau)n_\mu\}^2] \}$$

and integrating over b' after transforming b , we obtain the following result:

$$\mathcal{D}(x) = \frac{2\sqrt{\pi}i}{(2\pi)^6} \int_{-\infty}^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} \frac{dh}{h} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \int_0^{\infty} \frac{d\alpha}{\sqrt{\alpha}} \exp i \{ gn_\mu x_\mu + ch - (c+\tau)g \} \times \\ \times \left\{ \exp -i \left[\alpha (x_\mu^2 + (n_\mu x_\mu)^2) + \frac{g^2}{4\alpha} - \frac{\pi}{4} \right] - \exp i \left[\alpha (x_\mu^2 + (n_\mu x_\mu)^2) + \frac{g^2}{4\alpha} - \frac{\pi}{4} \right] \right\} \\ = \frac{-4i}{(2\pi)^4} \frac{1}{\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}} \int_0^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} \frac{dh}{h} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \sin(g\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}) \sin\{g(n_\mu x_\mu - c - \tau)\} \\ \times \exp(ich) \\ = \frac{4\pi i}{(2\pi)^4} \frac{1}{\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}} \int_0^{\infty} \frac{dg}{g} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} \frac{dh}{h} \exp(ich) \cos g(n_\mu x_\mu - c) \sin g\sqrt{x_\mu^2 + (n_\mu x_\mu)^2},$$

where we used the next formula:

$$\int_{-\infty}^{\infty} \frac{d\tau}{\tau} \sin g(n_\mu x_\mu - c - \tau) = -\pi \cos g(n_\mu x_\mu - c).$$

Making use of the following relation:

$$\int_{-\infty}^{\infty} dc \exp(ich) \cos g(n_\mu x_\mu - c) = \pi \{ \delta(h-g) \exp(ig n_\mu x_\mu) + \delta(h+g) \exp(-ig n_\mu x_\mu) \},$$

we get

$$\mathcal{D}(x) = \frac{-1}{(2\pi)^2} \frac{1}{\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}} \int_0^{\infty} \frac{dg}{g^2} \sin g\sqrt{x_\mu^2 + (n_\mu x_\mu)^2} \sin gn_\mu x_\mu \\ = -\frac{1}{4\pi} \frac{n_\mu x_\mu}{\sqrt{x_\mu^2 + (n_\mu x_\mu)^2}}, \quad \text{for } \sqrt{x_\mu^2 + (n_\mu x_\mu)^2} \geq n_\mu x_\mu, \quad x_\mu^2 > 0, \\ = -\frac{1}{4\pi}, \quad \text{for } \sqrt{x_\mu^2 + (n_\mu x_\mu)^2} \leq n_\mu x_\mu, \quad x_\mu^2 < 0, \quad \left. \vphantom{\int_0^{\infty}} \right\} \quad (\text{A} \cdot 6)$$

where we used the next formula:

$$\int_0^{\infty} \frac{dg}{g^2} \sin gX \sin gN = \frac{1}{2} \pi X, \quad \text{if } N \geq X, \\ = \frac{1}{2} \pi N, \quad \text{if } N \leq X.$$

If we operate ∂ to $\mathcal{D}(x)$ expressed by (A·6), it is easily proved that $\partial \mathcal{D}(x)$ satisfies the conditions (A·5) and next relation holds

$$(\partial_\mu + n_\mu \partial) \mathcal{D}(x) = 0$$

in the case of $n_\mu x_\mu = 0$.

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Some Remarks on the Reaction $K^- + d \rightleftharpoons \Sigma^- + p$

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Recently Lee has suggested that the reaction $K^- + d \rightleftharpoons \Sigma^- + p$ might lend itself to a determination of the spins of the negative K meson and negative Σ particle. Because of the mysterious degeneracy of the mass levels of heavy mesons, however, the application of the detailed balancing arguments is not so simple as was the case with the positive pion unless heavy mesons with different names represent the competing modes of decay of a single kind of heavy mesons.

In this paper, we propose a possible experimental method to determine the spins of the negative K meson and negative Σ particle from the observation of the process $K^- + d \rightleftharpoons \Sigma^- + p$ by using a K meson beam which is in general a mixture of distinct kinds of K mesons.

§ 1. Introduction

During the past few years there has accumulated an increasing amount of experimental informations on hyperons and heavy mesons. In the initiatory stage of investigation their nature appeared to be rather curious, but in turn their characteristic behaviour served to narrow the possible theoretical interpretations almost uniquely. At present, it is known that the principle of charge independence is very powerful in explaining and predicting some of the characteristic properties of these unstable particles qualitatively.¹⁾ Now it is felt that the next step towards the quantitative theory of these particles might be achieved by the determination of their spins and parities. Hence this paper is devoted to the investigation of this problem.

The Christian names of heavy mesons and hyperons are defined according to their characteristic modes of decay so that it is generally hard to distinguish between them unless we precisely observe their decay modes. This gives rise to a serious difficulty in investigating the nature of these particles, since we cannot settle what kind of hyperon or heavy meson it is if a hyperon or a heavy meson interacts with nuclear particles without exhibiting its characteristic mode of decay. The situation is especially serious for heavy mesons because of the mysterious degeneracy of their mass levels, *i.e.*, all kinds of heavy mesons now familiar seem to have the same mass eigenvalue within experimental errors. Bearing this in mind we shall discuss how to determine the spins of heavy mesons without observing their decays.

§ 2. The principle of detailed balancing

The spin of the positive pion was determined by comparing the forward and backward rates of the reaction²⁾

$$\pi^+ + d \rightleftharpoons p + p. \quad (2.1)$$

The forward and backward reactions are related by a detailed balancing argument, if one assumes initially unpolarized particles, so that

$$\frac{d\sigma(\rightarrow)}{d\Omega} \bigg/ \frac{d\sigma(\leftarrow)}{d\Omega} = \frac{4}{3} \frac{p^2}{q^2(2s+1)}, \quad (2.2)$$

where s is the spin of the positive pion, and p, q , are the momenta of the proton and pion in the centre of mass system.

Recently Lee has proposed to apply the same method to the reaction³⁾

$$K^- + d \rightleftharpoons \Sigma^- + p, \quad (2.3)$$

so that we can determine the spins of the K meson and Σ particle. If there were only one kind of K mesons, we would have

$$\frac{d\sigma(\rightarrow)}{d\Omega} \bigg/ \frac{d\sigma(\leftarrow)}{d\Omega} = \frac{2p^2}{3q^2} \cdot \frac{2S+1}{2s+1}, \quad (2.4)$$

where s, S are the spins of the K meson and Σ particle, and p, q , are the momenta of the proton and K meson in the centre of mass system. There is a preliminary evidence,¹⁾ however, that there are at least two distinct kinds of K mesons and hence we cannot directly apply the formula (2.4). In performing the experiments to measure the cross section of the reaction

$$K^- + d \rightarrow \Sigma^- + p, \quad (2.5)$$

we shall be obliged to use a K beam composed of a few distinct kinds of heavy mesons, say K_a, K_b, \dots . Then the observed cross section may be expressed

$$\sigma_{\text{obs}}(\rightarrow) = w_a \sigma_a(\rightarrow) + w_b \sigma_b(\rightarrow) + \dots, \quad (2.6)$$

where w_a, w_b, \dots are the proportions of the mesons K_a, K_b, \dots in the beam, and $\sigma_a(\rightarrow) = \sigma(K_a^- + d \rightarrow \Sigma^- + p)$ and so on. On the other hand, the observed cross section of the inverse process is given by

$$\sigma_{\text{obs}}(\leftarrow) = \sigma_a(\leftarrow) + \sigma_b(\leftarrow) + \dots, \quad (2.7)$$

where $\sigma_a(\leftarrow) = \sigma(\Sigma^- + p \rightarrow d + K_a^-)$, etc.

In the former reaction it is impossible to determine in each case what kind of K meson induced the reaction (2.5) since we can never observe the decay of the absorbed K meson. In the latter reaction, it is possible in principle to observe the K meson decays in fortunate cases, but it is experimentally difficult to observe both the production and decay of a K meson simultaneously. For this reason, we shall discuss an experimental method to determine the spins of K mesons and Σ particle in the next section.

§ 3. A proposed experimental method for the spin determination

Although it is impossible to determine what kind of K meson induced the reaction

(2.5) in each case, it is still possible to distinguish between them statistically. This is the point of our discussion. Suppose that a K beam is impinged upon deuterons in a pressure vessel, then the intensity of the beam will decrease with the distance travelled by the K mesons according to the exponential law

$$I = I_0 (w_a \exp(-x/\lambda_a) + w_b \exp(-x/\lambda_b)), \quad (3.1)$$

where λ 's are the mean free paths of the mesons K_a and K_b , and we have assumed without loss of generality that two kinds of heavy mesons are involved in the beam. The mean free path λ_a is given in terms of the total cross section σ_a for collisions between a K_a meson and a deuteron, and the density of the target deuterons ρ by

$$\lambda_a^{-1} = \rho \sigma_a, \quad (3.2a)$$

and similarly

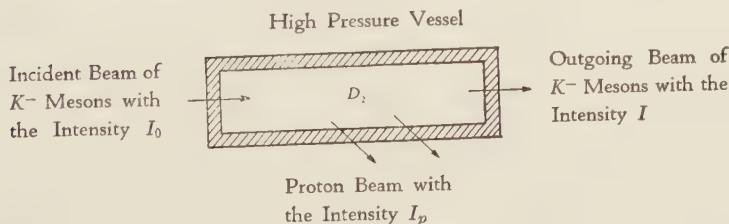
$$\lambda_b^{-1} = \rho \sigma_b. \quad (3.2b)$$

Hence the relation (3.1) is rewritten

$$I = I_0 (w_a e^{-\sigma_a X} + w_b e^{-\sigma_b X}), \quad (3.3)^*$$

where

$$X = \rho x.$$



Schematic Diagram of the Proposed Experiment

Fig. 1.

Then from the attenuation measurements of the beam intensity, I/I_0 can be determined, so that we can plot I/I_0 versus X . This curve provides us with a means to determine w_a , w_b , σ_a and σ_b if σ_a differs from σ_b by an appreciable amount. In order to vary X , it will be more favourable experimentally to change the density ρ of the deuterons in the pressure vessel than to do with the path length x of the K mesons. Suppose that the effective proportions w_a , w_b and total cross sections σ_a , σ_b are experimentally determined as described above, then we can also determine the differential cross sections $d\sigma_a(\rightarrow)/d\Omega$ and $d\sigma_b(\rightarrow)/d\Omega$.

* Rigorously speaking, the proportions w 's are expressed by $w_a = w_a^0 \exp(-d/\Lambda_a)$, where w_a^0 is the proportion of the K_a mesons in the incident beam, d is the thickness of the vessel wall, and Λ_a the mean free path of a K_a meson in the wall; but we need not know w^0 or $\exp(-d/\Lambda)$ separately. This enables us to eliminate the possible contributions of the background mesons produced at the walls. Hence in general one obtains $w_a + w_b \neq 1$.

Suppose that we measure the intensity I_p of the proton beam coming out of the vessel in a certain direction, then the proton intensity is given by

$$\frac{1}{I_0} \cdot \frac{dI_p}{d\Omega} = \frac{w_a}{\sigma_a} \cdot \frac{d\sigma_a(\rightarrow)}{d\Omega} (1 - e^{-\sigma_a X}) + \frac{w_b}{\sigma_b} \cdot \frac{d\sigma_b(\rightarrow)}{d\Omega} (1 - e^{-\sigma_b X}) + (\text{background corrections}). \quad (3.4)^*$$

Since (2.5) is a two body reaction, the outcoming protons should have a definite energy in a given direction. The background protons are produced at the vessel walls but their intensity is independent of X , hence the background corrections are easily distinguished from the protons produced by the reaction (2.5). By repeating similar measurements to the previous case, we can determine the differential cross sections separately. We shall denote them as

$$\left(\frac{d\sigma_a(\rightarrow)}{d\Omega} \right)_{\text{obs}} \text{ and } \left(\frac{d\sigma_b(\rightarrow)}{d\Omega} \right)_{\text{obs}}. \quad (3.5)$$

Then we must measure the cross section of the inverse reaction

$$\Sigma^- + p \rightarrow d + K^-. \quad (3.6)$$

The observed cross section is, according to (2.7), given by

$$\left(\frac{d\sigma(\leftarrow)}{d\Omega} \right)_{\text{obs}} = \frac{d\sigma_a(\leftarrow)}{d\Omega} + \frac{d\sigma_b(\leftarrow)}{d\Omega} \quad (3.7)$$

Inserting (2.4) into (3.7), we finally arrive at

$$\left(\frac{d\sigma(\leftarrow)}{d\Omega} \right)_{\text{obs}} = \frac{3q^2}{2p^2} \left[\frac{2s_a+1}{2S+1} \left(\frac{d\sigma_a(\rightarrow)}{d\Omega} \right)_{\text{obs}} + \frac{2s_b+1}{2S+1} \left(\frac{d\sigma_b(\rightarrow)}{d\Omega} \right)_{\text{obs}} \right], \quad (3.8)$$

where S , s_a , and s_b are the spins of the Σ^- particle, K_a^- meson, and K_b^- meson, respectively. If the measurements were performed in two or more directions the ratios $(2s_a+1)/(2S+1)$ and $(2s_b+1)/(2S+1)$ are uniquely determined. The identification of K_a and K_b mesons would be achieved by comparing w_a and w_b with the branching ratio of the free decays of the K mesons used in the first experiment.**)

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* In some cases, certain geometrical corrections to this formula might be required.

** When a heavy meson has two or more different Christian names, however, this identification would not be an easy task.

On the Interaction between Hyperons and Nucleons and the Hyperfragments

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Based on the principle of charge independence and the experimentally established evidences for the existence of the hyperfragment ${}^3H^*$, the nature of the force between a Λ^0 particle and a nucleon is investigated. It is shown that the range of the Λ - N force cannot be longer than half the pion Compton wave length and that its well depth parameter is larger than about 0.4. This result suggests us the possible existence of the so-called hyper di-nucleons. Next, the mechanism of the non-mesonic decays of a hyperfragment ${}^4He^*$ is investigated. A qualitative argument leads to the conclusion that the spin of the Λ^0 particle cannot be larger than $3/2$.

§ 1. Introduction

In studying the nature of heavy mesons, the most serious difficulty that we face is the mysterious degeneracy of their mass levels. We cannot determine what kind of K particle it is if a heavy meson interacts with nuclear particles before its decay.¹⁾ The situation is worse for neutral particles such as Λ^0 and θ^0 since they cannot be detected unless they undergo spontaneous disintegrations. Fortunately, however, it is an established fact that a Λ^0 particle can be bound to a nucleus to form a hyperfragment, and consequently it is possible to draw some conclusions on the nature of the Λ^0 particle through the investigation of hyperfragments. Along this line of reasoning some properties of the Λ^0 particle are investigated in this paper.

§ 2. Interaction between a Λ^0 particle and a nucleon

Since the first observation by Danysz and Pniewski²⁾ it is well established that a hyperfragment is such an excited state of a nucleus in which a Λ^0 particle is substituted for a nucleon. The rigidity of the Λ^0 particle in hyperfragments might be an indication of its elementarity. Besides there are some anomalous cases that this interpretation cannot be taken for granted³⁾, but they are distinguished from normal cases by their large Q values. In this paper, we shall confine ourselves to the discussion of normal hyperfragments. The existence of normal hyperfragments shows in a direct way that the Λ - N force is very strong, and this section will be devoted to the investigation of this force.

Let us first discuss some results obtained by applying the principle of charge symmetry to this force.*⁴⁾ The simplest but important result is the equality of the force between a

* We assume that the Λ - N interaction is charge independent and consequently charge symmetric. See ref. 4).

Λ^0 particle and a proton and that between a Λ^0 particle and a neutron, i.e.,

$$V_{\Lambda p} = V_{\Lambda n} \equiv V_{\Lambda N}. \quad (2.1)$$

Hence some consequences of charge symmetry known in the field of nuclear physics are also valid for hyperfragments. For instance, the existence of the hyperfragment ${}^4\text{He}^*$ requires that there must be its mirror hyperfragment ${}^4\text{H}^*$ and that the binding energies of Λ^0 in these two fragments should be approximately equal.

The principle of charge independence imposes more precise restrictions on the nature of Λ - N force⁴⁾. One of the most important conclusions is the deduction that the range of Λ - N force should not be longer than half the pion Compton wave length, i.e., the range of N - N force. This can be proved as follows:

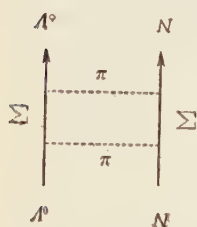


Fig. 1a

Examples of Feynman diagrams giving rise to the Λ - N force.

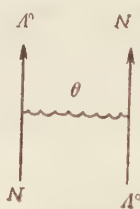


Fig. 1b

Let us consider Feynman diagrams which give rise to the Λ - N force. If a directed line starts with Λ^0 and ends again with Λ^0 , as in Fig. 1a, Λ^0 and N should exchange at least two mesons either pions or heavy mesons since $I=0$ for Λ^0 and there is no pure neutral meson ($I=0$) as light as a pion. On the contrary, if a directed line starts with Λ^0 but ends with N , as in Fig. 1b, Λ^0 and N should exchange at least one quantum of η -charge⁴⁾,

i.e., a heavy meson, as is obvious from the conservation of η -charge for the strong interactions. For both cases, the above statement is justified provided that the energy of the system is sufficiently low.

Next we shall study the strength of the Λ - N force based on the evidences for the existence of the hyperfragment ${}^3\text{H}^*$, the lightest one known at present.⁵⁾ The Hamiltonian for this system is given by

$$H_f = T_n + T_p + T_\Lambda + V_{np} + V_{\Lambda n} + V_{\Lambda p}, \quad (2.2)$$

where T 's are the kinetic energies of a neutron, a proton, and a Λ^0 particle constituting the fragment and V 's are the potentials between them. Then the wave function Ψ_f representing the fragment ${}^3\text{H}^*$ at rest satisfies the Schrödinger equation

$$H_f \Psi_f = -(B_d + B_\Lambda) \Psi_f, \quad (2.3)$$

where B_d is the binding energy of a deuteron and B_Λ the binding energy of the Λ^0 in this fragment defined by this equation.

Let us denote the lowest eigenvalue of an operator A as $\text{Min}(A)$, then we have an inequality

$$\text{Min}(A+B) \geq \text{Min}(A) + \text{Min}(B). \quad (2.4)$$

With this notation we have the following equations:

$$\text{Min}(H_f) = -(B_d + B_\Lambda), \quad (2.5)$$

$$\text{and} \quad \text{Min}(H_d) = \text{Min}(T_n + T_p + V_{np}) = -B_d.$$

Combining these relationships, we have

$$-(B_d + B_\Lambda) = \text{Min}(H_f) > \text{Min}(H_d) + \text{Min}(T_\Lambda + V_{\Lambda n} + V_{\Lambda p})$$

$$\text{or} \quad -B_\Lambda > \text{Min}(T_\Lambda + V_{\Lambda n} + V_{\Lambda p}) > \text{Min}((1/2)T_\Lambda + V_{\Lambda n}) + \text{Min}((1/2)T_\Lambda + V_{\Lambda p}).$$

With the help of (2.1), the above inequality reduces to

$$\text{Min}((1/2)T_\Lambda + V_{\Lambda N}) < -B_\Lambda/2, \quad (2.6)$$

$$\text{or} \quad \text{Min}(p^2/4M + V_{\Lambda N}) < -B_\Lambda/2,$$

where M is the rest mass of a Λ^0 particle.

The Hamiltonian for a system composed of a nucleon and a Λ^0 particle is given in the centre of mass system by

$$H = (m + M)p_\Lambda^2/2mM + V_{\Lambda N}, \quad (2.7)$$

where m is the nucleon rest mass. Comparing (2.7) with (2.6), we see that if the potential $V_{\Lambda N}$ is multiplied by $2(m + M)/m$, there always exists a bound state for this system. Hence we have

$$s > m/2(m + M) \sim 0.22, \quad (2.8)$$

where s is the well depth parameter⁽⁶⁾ of the potential $V_{\Lambda N}$. Rigorously speaking, it will be better to replace the inequality by a stronger one

$$s \geq 0.22, \quad (2.9)$$

as seen from the above derivation. So far as we know at present there is no definite evidence for the existence of hyper di-nucleons, and for the time being we shall assume the non-existence of such fragments, then we have an upper limit to the well depth parameter

$$1 > s. \quad (2.10)$$

In order to obtain a more intuitive information on the potential $V_{\Lambda N}$, we shall investigate the allowed regions of the well depths for square well and Yukawa potentials.

(1) Square well potential

In this case the well depth parameter is given in terms of the well depth V_0 and force range b by⁽⁷⁾

$$s = (4/\pi^2)\mu V_0 b^2, \quad (2.10)^*$$

where μ is the reduced mass $2mM/(m + M)$.

(2) Yukawa potential

We write the Yukawa potential in the form

$$V = -V_0 \frac{\exp(-r/b)}{(r/b)}, \quad (2.11)$$

* We employ the natural units $\hbar = c = 1$.

then the well depth parameter is given by⁷⁾

$$s = 0.59531 \mu V_0 b^2. \quad (2.12)$$

The inequalities (2.9) and (2.10) lead to inequalities for the well depths which are given in the Table I for two choices of the force range b . The range b is chosen very short in accordance with the principle of charge independence.

Table I
Allowed Regions of the Well Depths in Mev.

shape \ b	$0.5 \times 10^{-13} \text{cm}$	$0.4 \times 10^{-13} \text{cm}$
Square Well	$360 > V_0 \geq 80$	$625 > V_0 \geq 125$
Yukawa	$227 > V_0 \geq 54$	$419 > V_0 \geq 84$

It is very important to investigate more precisely the hyper triton problem by adopting a specific model. If we assume that θ is scalar and that Λ^0 has spin 1/2 and even parity, then the contribution from the Feynman diagram in the Fig. 1b leads to a Heisenberg exchange force⁴⁾

$$V_{\Lambda N} = (g^2/4\pi) P_x P_o e^{-m_\theta r}/r. \quad (2.13)$$

Hence in the 1S state, the well depth parameter V_0 defined by (2.11) is given by

$$V_0 = (g^2/4\pi) m_\theta = g^2/4\pi \times 493 \text{ Mev}. \quad (2.14)$$

Then noticing that $b = m_\theta^{-1} = 0.4 \times 10^{-13} \text{cm}$, we have with reference to the Table I the inequalities

$$0.85 > g^2/4\pi \geq 0.19. \quad (2.15)$$

By adopting the ΛN potential (2.13) and phenomenological neutron-proton potential, Iwao estimated the magnitude of the coupling constant $g^2/4\pi$ with a variational method so as to make the calculated binding energy of the Λ^0 in a hyper triton fit the experimental value 1.5 Mev and found $g^2/4\pi \sim 1.1$.⁸⁾ This value is large enough to allow the existence of hyper di-nucleons as is clear from (2.15), but this conclusion is not decisive for the following two reasons: (1) the variational method leads to a considerably larger value of $g^2/4\pi$ than the exact one unless the choice of the form of the trial wave function is excellent, and (2) we do not know the true shape of the potential. However, we may take it for granted that the well depth parameter is near unity and consequently the scattering cross section of a Λ^0 by a nucleon will be very large at low energies.

The rather small value of the lower limit of the well depth parameter in (2.9) can be increased by taking account of the short range character of the ΛN force. Roughly speaking, the distance between the proton and neutron in a hyper triton will be of the order of pion Compton wave length and the range of the ΛN force is only about half the neutron-proton distance, so that only one of the two bonds, i.e., the Λn bond and Λp bond, will effectively be operative. Assuming that it is really the case we can double the lower limit of the well depth parameter, i.e.,

$$s \gtrsim 0.4. \quad (2.16)$$

As seen in the above discussion, the existence of the hyper di-nucleons might be likely though not conclusive. A Λ^0 particle would probably be bound to a nucleon to form a hyper di-nucleon, and furthermore a Σ particle might also be bound to a nucleon, since the cosmotron experiments indicate that the ΣN force is comparably strong with the ΛN force.⁹⁾ In general, however, the Σ fragments are unstable against the processes which take place through strong interactions alone

$$\begin{aligned} \Sigma + \text{nucleons} &\rightarrow \Lambda^0 + \text{nucleons}, & \text{for } \Delta I = 0, \\ \Sigma + \text{nucleons} &\rightarrow \Lambda^0 + \text{nucleons} + \gamma, & \text{for } \Delta I = \pm 1. \end{aligned} \quad (2.17)$$

The hyper di-nucleons that might possibly exist would be of the types given in the Table II.

Table II
Possible Hyper Di-Nucleons

	$I_3 = 3/2$	$I_3 = 1/2$	$I_3 = -1/2$	$I_3 = -3/2$
$I = 3/2$	$(\Sigma^+ p)$	unstable		$(\Sigma^- n)$
$I = 1/2$		$(\Lambda^0 p)$	$(\Lambda^0 n)$	

The decay of $(\Sigma^+ p)$ or $(\Sigma^- n)$ through (2.17) or by a pion emission is forbidden by the conservation laws of charge and energy.

§ 3. Non-mesonic decay of the hyper helium ${}^4\text{He}^*$

The lightest hyperfragment ${}^3\text{H}^*$ is known to decay by emitting a pion as⁵⁾

$${}^3\text{H}^* \rightarrow {}^3\text{He} + \pi^-. \quad (3.1)$$

As the nuclear charge Z increases, the pion is often emitted only virtually but does not appear as a real particle.¹⁰⁾ This phenomenon is the so-called non-mesonic decay of a hyperfragment.

The next hyperfragment ${}^4\text{He}^*$ is known to decay either as¹¹⁾

$${}^4\text{He}^* \rightarrow {}^3\text{He} + p + \pi^-, \text{ (mesonic)} \quad (3.2a)$$

or as¹²⁾

$${}^4\text{He}^* \rightarrow p + p + n + n. \text{ (non-mesonic)} \quad (3.2b)^*$$

So far as we know, *these two competing modes are equally probable*, and we can deduce some information on the spin of the Λ^0 particle from this fact. This section is devoted to a qualitative discussion of the mechanism of the non-mesonic decay of the fragment ${}^4\text{He}^*$.

The essential point of our discussion consists in the fact that such a very light fragment can undergo the non-mesonic decay.

* This might be interpreted to represent the decay ${}^4\text{He}^* \rightarrow p + d + n$.

The radius of a ${}^4\text{He}^*$ is supposed to be of the order of pion Compton wave length or more precisely

$$R \sim 1.45 \times \sqrt[3]{4} \times 10^{-13} \text{ cm} = 2.5 \times 10^{-13} \text{ cm}. \quad (3.3)$$

Suppose that the Λ^0 particle in the fragment ${}^4\text{He}^*$ decayed virtually as

$$\Lambda^0 \rightarrow p + \pi^-, \quad (3.4)$$

and that the virtual pion were subsequently absorbed by nucleons. Let us first assume that the pion were emitted really with a kinetic energy of about 40 Mev and reabsorbed by "deuterons" in the nucleus. The cross section of the process¹³⁾

$$\pi^- + d \rightarrow n + n \quad (3.5)$$

is of the order of 6mb at this energy. Since the number of "deuterons" $NZ=3$ is comparable to the number of nucleons $N+Z=4$, the absorption mean free path for the pion is of the following order

$$\lambda_{\text{abs}} \sim 2.5 \times 10^{-12} \text{ cm}. \quad (3.6)$$

Hence the absorption probability by the above mechanism is given by

$$p_{\text{abs}} \sim R/\lambda_{\text{abs}} \sim 10^{-1}. \quad (3.7)$$

Rigorously speaking, we have estimated the "absorptive part" of the matrix element by the above discussion. The "reactive part" of the matrix element has been evaluated by Cheston and Primakoff¹⁴⁾ and has been shown to be much larger. Therefore we shall replace the cross section of (3.5) by the geometrical cross section ($\sim 60\text{mb}$) with the understanding that the reactive part of the matrix element is also included effectively. Although this estimation is very rough, our following reasoning is not sensitive to the magnitude of the assumed cross section. The revised mean free path is given by

$$\lambda_{\text{abs}} \sim R, \quad (3.8)$$

and hence the absorption probability will turn out to

$$p_{\text{abs}} \sim 1 - \exp(-R/\lambda_{\text{abs}}) \gtrsim 1/2. \quad (3.9)$$

At a glance, one might be inclined to equate p_{abs} to the probability of a non-mesonic decay defined by

$$P_{\text{non}} = \frac{(\text{non-mesonic})}{(\text{mesonic}) + (\text{non-mesonic})}, \quad (3.10)$$

but it is not true as seen from the following argument.

As is well known a nucleon or a deuteron absorbs a pion mainly from p - or s -state at low energies, so that we have

$$P_{\text{non}} = p_{\text{abs}} \cdot f, \quad (3.11)$$

where f represents the fraction of the p and s waves (from the sink) involved in the virtual pion wave emitted from the Λ^0 . Since we know experimentally

$$P_{\text{out}} \sim 1/2, \quad (3.12)$$

and theoretically

$$P_{\text{abs}} \sim 1/2, \quad (3.13)$$

we may conclude

$$f \sim 1. \quad (3.14)$$

Although the estimate of P_{abs} is not so trustworthy quantitatively, it is qualitatively certain that the emitted pion wave should consist mainly of p and s waves (from the sink).

Now suppose that the pion is emitted from the source Λ^0 with an angular momentum l_i and absorbed by the sink with an angular momentum l_f . Since ${}^4\text{He}^*$ is a light fragment, the distance between the sink and source is comparable to or even shorter than the impact parameter of the pion. The impact parameter for 40 Mev pion is equal to about $4/3$ times the pion Compton wave length. For a low energy pion and for small values of l_f , we have an order of magnitude selection rule from the requirement (3.14)

$$\Delta l = l_f - l_i = 0, +1, \quad (3.15)$$

based on an intuitive geometrical consideration about the impact parameter. The relative rarity of the transition $\Delta l = -1$ compared with $\Delta l = +1$ is due to a statistical reason. If we assume as suggested before

$$l_f = 0 \text{ or } 1, \quad (3.16)$$

we reach the following interesting conclusion

$$l_i = 0 \text{ or } 1. \quad (3.17)$$

This means that the pion emitted from Λ^0 by the process (3.4) is in the s - or p -state. Hence we may conclude that the spin of Λ^0 cannot be larger than $3/2$. This result serves to exclude certain higher spin models of hyperons.

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In the text we have assumed that the cross sections of the following two reactions are equal:

$$\pi^- + d \rightarrow n + n \quad \text{and} \quad \pi^+ + d \rightarrow p + p.$$

The latter cross section is given in the literature.

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The Relativistic and Exchange Current Corrections to the Deuteron Magnetic Moment and the Thermal N - P Capture Cross Section†

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Starting from the symmetrical ps meson theory, the relativistic and exchange current corrections to the thermal N - P capture cross section and the deuteron magnetic moment have been defined unambiguously and calculated using the T - D formalism up to the lowest order of the meson-nucleon coupling constant and the nucleon velocity. This work is a refinement and generalization of the previous paper on the exchange current correction to the deuteron moment, the result of which is also obtained here. With respect to the relativistic corrections, quite a different method is used. Numerical values of these corrections are then estimated by using the phenomenologically adjusted two nucleon wave functions with or without the hard core. The sum of these corrections is shown to be about $+1\%$ of the capture cross section and about -2% of the magnetic moment. The phenomenological thermal N - P capture cross section is also estimated and is shown to agree with the experimental one within much larger experimental and theoretical ambiguities than the above correction. As the exchange current correction is reduced very much if the D -state probability is chosen to be smaller and there may be in addition some small positive non-additivity correction, the deuteron D -state probability is finally estimated as $3 \pm 1\%$ or so, the same value as was reported previously.

§ 1. Introduction

Although many investigations have thus far been made with respect to the relativistic¹⁾ and exchange current²⁾ corrections to the electromagnetic properties and the radiative transitions of the deuteron, there exists a rather big ambiguity among them and there does not exist even an unambiguous separation of these effects from the phenomenological expressions. Previously the present author has done³⁾ a field theoretical work with respect to the exchange current correction to the deuteron moments, starting from the symmetrical ps meson theory and using the T - D formalism. In this paper a similar investigation has been made in order to refine and at the same time generalize it so that we may discuss on the same footing the relevant corrections to the deuteron moments and the radiative transitions, in the latter of which we are now mainly interested in the thermal N - P capture process.

The main improvements upon the previous one³⁾ consist of four parts: First of all we have done the calculations so that it can readily be applied to both cases mentioned above. Secondly we have also considered such terms as are due to the amplitudes containing two mesons and nucleon pairs as far as they are of the same order of the coupling

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constant as those calculated before. Thirdly, as regards the relativistic correction, a method quite different from the previous one has been used, which is quite natural in the present calculation and makes it possible to define the relativistic correction unambiguously. Finally, in order to get the numerical values of these corrections, more refined two nucleon wave functions than before, fitted to all the empirical facts, have been used with or without the hard core both in the singlet and triplet N - P states. In spite of these refinements, the same result as was obtained before⁽³⁾ for the exchange current correction to the deuteron moment has been obtained in this paper.

As regards the non-additivity question of nucleon moments, we have assumed throughout in this paper the rigorous additivity, which corresponds to introducing the Pauli-term. In the final section we shall refer to the existing literature about this problem.

We shall, therefore, start from the field theoretical Hamiltonian for nucleons interacting with the symmetrical ps meson field, both of which are interacting with the electromagnetic field. The general outline is explained in section 2. In section 3, in order to define the relevant corrections unambiguously, the phenomenological expressions are derived. The relativistic corrections are then defined and given in section 4 and those due to the exchange current in section 5, for both the deuteron moment and the thermal N - P capture matrix element. In section 6 are given the phenomenologically adjusted two nucleon wave functions with or without the hard core. In sections 7 and 8 are given the numerical values of the phenomenological capture cross section and the corrections to it and of the corrections to the deuteron moment, respectively. In the final section, the summary and the conclusion are given together with the discussions about the non-additivity question and the D -state probability.

§ 2. General formulation

As we are going to estimate these corrections up to the lowest order of the meson-nucleon coupling constant and the nucleon velocity, we shall start from the ps - $p\nu$ theory. The total Hamiltonian in the Schrödinger representation is then given by

$$H = H_0 + H' + H'', \quad (1)$$

where

$$H_0 = \phi^* \left[-i\hbar c \alpha \cdot \frac{\partial}{\partial \mathbf{x}} + \hbar c \kappa \beta \right] \phi + \frac{1}{2} \sum_{\alpha} \left[c^2 \pi_{\alpha}^2 + \left(\frac{\partial \phi_{\alpha}}{\partial \mathbf{x}} \right)^2 + \mu^2 \phi_{\alpha}^2 \right], \quad (2)$$

$$H' = \frac{g}{\mu} \phi^* \sigma \nabla \tau_3 \phi_{\alpha} \phi, \quad (3)$$

$$\begin{aligned} H'' = & -e \phi^* \frac{1 + \tau_3}{2} \alpha \phi A \\ & - \frac{e}{2\kappa} \phi^* \left[(\mu_p - 1) \frac{1 + \tau_3}{2} + \mu_N \frac{1 - \tau_3}{2} \right] \beta \sigma \phi \text{rot} A \\ & + \frac{e}{\hbar c} \left(\phi_1 \frac{\partial}{\partial \mathbf{x}} \phi_2 - \phi_2 \frac{\partial}{\partial \mathbf{x}} \phi_1 \right) A \end{aligned}$$

$$+\frac{eg}{\hbar c\mu}\psi^*\sigma(\tau_2\phi_1-\tau_1\phi_2)\psi A, \quad (4)$$

where the ordinary notations are used, A represents the transverse photon field and the Pauli-term expressing the anomalous magnetic moment has already been introduced, instead of which we drop afterwards all the self-meson processes throughout in the calculation.* It can be shown that the same expressions as those given above are obtained if we start from the ps - ps theory, apply the usual Dyson or Foldy transformations and neglect all the higher order terms with respect to the coupling constant and the nucleon velocity. It can also be shown that the characteristic processes in the ps - ps theory, in which virtual nucleon pairs are created, can be correctly accounted for by the last term of (4). The Pauli-term is shown to be quite independent of whether it is introduced before or after the canonical transformation in the present approximation.

The interaction term H'' can be treated as a small perturbation, while we shall use the Tamm-Dancoff method to solve the meson-nucleon system, the stationary states of which shall be denoted by $|i\rangle$ and $|f\rangle$ representing the initial and final states, respectively, in the relevant radiative transition. These states can be expanded in terms of $|P_1P_2\rangle$, the state in which two nucleons having positive energies and momenta P_1 , and P_2 and $|P_1P_2K\rangle$, the state where one meson of wave vector K is present besides two nucleons and so on. The expansion coefficients of these states shall be denoted by $C(P_1P_2)$, $C(P_1P_2K)$, etc. In these expansions, we shall neglect such states as containing nucleon pairs and more than two mesons, since they can contribute only to higher order corrections than what we are now interested in. The two meson amplitude $C(P_1P_2KK')$ shall be considered only if the mixed terms with the zero meson amplitude do not vanish.

In this stage of the calculation two essential improvements from the previous calculation⁽¹⁾ have been made; one is the inclusion of the last term of H'' in (4), in order to take account of the virtual nucleon pair creation and annihilation processes in the ps - ps theory, the other is the explicit inclusion of the two meson amplitude in the expansion of the state. The mixed terms between the zero and two meson amplitudes, however, belong to the so-called meson contribution, as is readily seen from (4). It is thus seen that both of these improvements give rise to the charge spin factor $\tau_x^{(1)}\tau_y^{(2)} - \tau_y^{(1)}\tau_x^{(2)}$, which means that the results of the previous calculation⁽¹⁾ of the exchange current correction to the deuteron magnetic moment need not be modified at all.

In order to remedy the ambiguity in the theory of nuclear force, we shall, as before, identify the zero meson amplitude $C(P_1P_2)$ with the Fourier transform of the phenomenologically adjusted two nucleon wave function, which shall be discussed later. Only when we calculate the matrix element of $\int H'' dx$ between the states in which mesons appear, shall we use the explicit Tamm-Dancoff solution of the Hamiltonian $H_0 + H'$, which can be obtained rigorously in the present approximation. It is to be added that the one meson amplitude need be calculated only up to the first power of the meson-nucleon coupling

* Although there is no justification for this manipulation, we assume here that it is allowed at least under the relevant approximations, only because there are no other better methods.

constant and not to the second power of it, although we must calculate also the two meson amplitude: This makes these solutions obtainable at once.

The transition probability of the radiative process of the meson-nucleon system is proportional to the square of the matrix element

$$\langle f | \int H'' d\mathbf{x} | i \rangle / \sqrt{\langle i | i \rangle \langle f | f \rangle}. \quad (5)$$

If we put both states $|i\rangle$ and $|f\rangle$ equal to the deuteron ground state, we get the corrections to the deuteron magnetic moments automatically.

§ 3. Phenomenological expressions

In this and the following sections, we can, without any loss of generality, restrict ourselves within the radiative transition from the deuteron ground state under the absorption of an incident photon of wave vector \mathbf{l} , the energy $h\nu$ and the direction of the polarization \mathbf{e}_λ to some state of two nucleons which shall not be specified from the beginning. In order to define the relevant corrections unambiguously, we shall first derive the phenomenological expressions for multipole transitions. For this purpose, we neglect all contributions to (5) due to the amplitudes containing mesons. Let us normalize the zero meson amplitude to unity, which makes it unnecessary to consider the denominator of (5). In the numerator we have only to consider the first two terms in (4), which makes it necessary to evaluate the matrix elements of α and $\beta\sigma$ with respect to the Dirac spinors $u_{\mathbf{p}}^*$ and $u_{\mathbf{p}'}$, belonging to the positive energies and momenta \mathbf{p} and \mathbf{p}' , respectively. The well-known procedure gives

$$\begin{aligned} (u_{\mathbf{p}}^* \alpha u_{\mathbf{p}'}) = & -\frac{1}{2Mc} \left[\{(\mathbf{p} + \mathbf{p}') + i\boldsymbol{\sigma} \times (\mathbf{p} - \mathbf{p}')\} \right. \\ & - \frac{1}{8M^2c^2} \{(\mathbf{p}^2 + \mathbf{p}'^2)(\mathbf{p} + \mathbf{p}') + 2(\mathbf{p}^2\mathbf{p} + \mathbf{p}'^2\mathbf{p}') \\ & \left. + i\boldsymbol{\sigma} \times [(\mathbf{p}^2 + \mathbf{p}'^2)(\mathbf{p} - \mathbf{p}') + 2(\mathbf{p}^2\mathbf{p} - \mathbf{p}'^2\mathbf{p}')]\} \right] \end{aligned} \quad (6)$$

and

$$(u_{\mathbf{p}}^* \beta \boldsymbol{\sigma} u_{\mathbf{p}'}) = \boldsymbol{\sigma} - \frac{1}{8M^2c^2} \left[(\mathbf{p}^2 + \mathbf{p}'^2)\boldsymbol{\sigma} + 2(\boldsymbol{\sigma}\mathbf{p})\boldsymbol{\sigma}(\boldsymbol{\sigma}\mathbf{p}') \right], \quad (7)$$

where we have retained only up to the second order quantities with respect to the nucleon velocity, M being the nucleon mass. It is to be mentioned that in (6) and (7) we are using a somewhat arbitrary convention as regards the matrix elements with respect to the Pauli spin operators.

If we make the usual non-relativistic approximation for these matrix elements, namely neglect the second terms in (6) and (7), we get

$$\begin{aligned} \langle f | \int H'' d\mathbf{x} | i \rangle = & \frac{\hbar c}{2\pi\sqrt{h\nu}} \iint \phi_f^*(\mathbf{x}_1, \mathbf{x}_2) \left[e^{i\mathbf{l}\cdot\mathbf{x}_1} \mathbf{e}_\lambda \left\{ -\frac{e}{Mc} \frac{1 + \tau_3^{(1)}}{2} \mathbf{p}_1 \right. \right. \\ & \left. \left. - \frac{e}{2\kappa} \left(\frac{1 + \tau_3^{(1)}}{2} \mu_P + \frac{1 - \tau_3^{(1)}}{2} \mu_N \right) i\boldsymbol{\sigma}^{(1)} \times \mathbf{l} \right\} + \text{term}(2) \right] \phi_i(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2, \end{aligned} \quad (8)$$

where we have introduced the ordinary two nucleon wave functions $\psi_i(\mathbf{x}_1, \mathbf{x}_2)$ and $\psi_f(\mathbf{x}_1, \mathbf{x}_2)$ instead of the zero meson amplitudes previously defined. This is just what would be predicted by the non-relativistic particle quantum mechanics and is nothing but the ordinary starting point of the phenomenological approach.

Further steps to define the conventional multipole transitions shall be outlined in the following. The first step is to introduce the centre of mass and the relative coordinates, the former of which does not contribute to the matrix element $\langle f | \int H'' d\mathbf{x} | i \rangle$ because of the transverse nature of the photon field. Then expanding the retardation factor $\exp(\pm i\mathbf{l} \cdot \mathbf{x}/2)$ in (8), \mathbf{x} being the relative coordinate, we get from the terms containing the relative momentum \mathbf{p} in (8) the electric dipole term first and then the combination of the electric quadrupole and the orbital magnetic dipole terms and so on. The separation into the latter two terms can be done by using the vector identity

$$(\mathbf{e}_\lambda \mathbf{p})(\mathbf{l}\mathbf{x}) = 1/2 \cdot \{[\mathbf{l} \times \mathbf{e}_\lambda] \cdot [\mathbf{x} \times \mathbf{p}] + (\mathbf{e}_\lambda \mathbf{p})(\mathbf{l}\mathbf{x}) + (\mathbf{e}_\lambda \mathbf{x})(\mathbf{l}\mathbf{p})\}, \quad (9)$$

where the first term on the right-hand side gives rise to the orbital magnetic dipole transition and the remaining terms the electric quadrupole transition. The other terms in (8) give first the spin magnetic dipole term and then relating higher moments.

We are now especially interested in the magnetic dipole transition matrix element, which can be summarized as

$$\langle f | \int H'' d\mathbf{x} | i \rangle = -\frac{i\hbar c}{2\pi\sqrt{h\nu}} \int \psi_f^*(\mathbf{x}) [\mathbf{l} \times \mathbf{e}_\lambda] \frac{e}{2\kappa} \mu \psi_i(\mathbf{x}) d\mathbf{x}, \quad (10)$$

$$\mu = (\mu_P + \mu_N) \frac{\sigma^{(1)} + \sigma^{(2)}}{2} + (\mu_P - \mu_N) \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} \frac{\sigma^{(1)} - \sigma^{(2)}}{2} + \frac{1}{2\hbar} \mathbf{x} \times \mathbf{p}, \quad (11)$$

where we have used the fact that the factor $\tau_3^{(1)} + \tau_3^{(2)}$ vanishes identically when it acts upon the deuteron ground state. It is well known that μ given by (11) gives the usual phenomenological expression for the deuteron magnetic moment if we make the expectation value of it with respect to the deuteron ground state and gives, on the other hand, the conventional magnetic dipole matrix element if $\psi_f(\mathbf{x})$ is identified with the singlet S -state.

Finally it is to be noted that in the phenomenological approaches even the starting point cannot be given uniquely; thus several authors have proposed various forms of the interaction of two nucleons with the electromagnetic field. The expressions for multipole transitions are, therefore, different for different approaches. Our definition (8) as a phenomenological starting point is, however, based upon the field theoretical consideration and may be considered as reasonable. It is also to be mentioned that the differences among various phenomenological approaches consist in the different expressions for the higher multipole moments than the electric and magnetic dipole ones and, therefore, have no practical importance.

§ 4. Relativistic corrections

It is reasonable to define as the relativistic corrections those contributions which are due to the second terms of (6) and (7). The correction to the magnetic dipole transition

matrix element (10) can be taken out according to the procedure outlined in the previous section. The additional term which must be added to μ in (10) is given by

$$\begin{aligned} \Delta\mu = & -\frac{p^2}{4M^2c^2} \frac{1}{\hbar} \mathbf{x} \times \mathbf{p} - \frac{p^2}{2M^2c^2} \left(\frac{1+\tau_3^{(1)}}{2} \sigma^{(1)} + \frac{1+\tau_3^{(2)}}{2} \sigma^{(2)} \right) \\ & - \frac{1}{4M^2c^2} \left\{ \frac{1+\tau_3^{(1)}}{2} [\sigma^{(1)} p^2 - \mathbf{p}(\mathbf{p}\sigma^{(1)})] \right. \\ & \left. + [(\mu_P-1) \frac{1+\tau_3^{(1)}}{2} + \mu_N \frac{1-\tau_3^{(1)}}{2}] 2\mathbf{p}(\mathbf{p}\sigma^{(1)}) + \text{term}(2) \right\}. \quad (12) \end{aligned}$$

The relativistic correction to the deuteron magnetic moment can then be calculated as

$$\begin{aligned} \Delta\mu_D = & \frac{\mu_P + \mu_N + 3}{6\kappa^2} \int_0^\infty u_g(r) u_g''(r) dr + \sqrt{2} \frac{(\mu_P + \mu_N - 3/2)}{6\kappa^2} \int_0^\infty w_g(r) \left[u_g''(r) \right. \\ & \left. - \frac{3}{r} u_g'(r) + \frac{3}{r^2} u_g(r) \right] dr, \quad (13) \end{aligned}$$

where $u_g(r)$ and $w_g(r)$ are the normalized S - and D -deuteron wave functions, $1/\kappa$ is the nucleon Compton wave length and dashes mean the derivatives with respect to r . In (13), we have neglected the D - D cross term for simplicity, since the S - D cross term is shown later to be already very small quantitatively.

On the other hand, the relativistic correction to the magnetic dipole matrix element is given by

$$\begin{aligned} \Delta M = & \frac{\mu_P - \mu_N + 3}{6\kappa^2(\mu_P - \mu_N)} \int_0^\infty u_g(r) u_g''(r) dr - \sqrt{2} \frac{(\mu_P - \mu_N - 3/2)}{6\kappa^2(\mu_P - \mu_N)} \int_0^\infty w_g(r) \left[u_g''(r) \right. \\ & \left. - \frac{3}{r} u_g'(r) + \frac{3}{r^2} u_g(r) \right] dr, \quad (14) \end{aligned}$$

which is exact and must be added to the phenomenological one

$$M = \int_0^\infty u_g(r) u_g(r) dr, \quad (15)$$

where $u_g(r)$ is the radial wave function of the singlet S -continuum.

It is to be added that these two corrections (13) and (14) are negative and the S - D cross terms will be shown to be much smaller than the S - S cross terms. Furthermore it is to be mentioned that the present estimation of the relativistic correction is quite independent of the special type of the meson field assumed, which is quite contrary to the method which has thus far been frequently used.¹⁾ The result (12) is by chance rather similar* to that obtained in the previous investigation³⁾ assuming the ps - ps meson theory. The result would, however, be quite different from (12) if one were to assume a different type of meson field and employ the quoted procedure. The present method based upon

* By remarking that $W+V = -p^2/M$ in formula (50) in reference 3, it is readily noted that the previous expression (50) is quite similar to the present one (12).

the field theoretical consideration may be considered to be more reliable and the negativeness of the relativistic corrections can be regarded as definite.

§ 5. Exchange current corrections

We shall call as the exchange current corrections all the contributions to the matrix element (5) due to the inclusion of the amplitudes containing one or two mesons. As these are already small corrections, we make the non-relativistic approximations for such matrix elements as (6) and (7) (neglecting the second terms) and neglect the nucleon kinetic energy compared with the meson energy in the energy denominators of the Tamm-Dancoff solutions, which allows us to go over analytically from the momentum to the configuration space representation. It is to be remarked that these non-relativistic approximations never mean the usual static approximation for nucleons.

Let us first consider the corrections in the numerator of (5), which shall be denoted by Δ' . The contributions due to the first two terms of (4) (nucleon contribution) can be evaluated in a manner similar to that used for the phenomenological ones given in section 3; the result can be written in the form of (10) where μ is replaced by

$$\Delta'\mu = \left(\frac{\alpha_g}{2\pi\mu} \right)^2 \int \tau_{\alpha}^{(1)}(\sigma^{(1)}\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}/2} \frac{\mu}{\omega_{\mathbf{k}}^3} e^{i\mathbf{k}\cdot\mathbf{x}/2} \tau_{\alpha}^{(2)}(\sigma^{(2)}\mathbf{k}) d\mathbf{k} \\ + (1 \cdot 2) \text{ exchange term} \quad (16)$$

with the same μ as (11) and $\alpha_g^2 = g^2/4\pi\hbar c$. As the cross terms between the zero and two meson amplitudes vanish for the nucleon contribution, the same result as was obtained previously follows from (16) as the exchange current correction to the deuteron magnetic moment if we make the expectation value of (16) with respect to the deuteron ground state;

$$\Delta'\mu_D = \frac{2}{\pi} \alpha_g^2 (\mu_P + \mu_N) \left\{ \int_0^\infty u_g^2(r) \left[K_0(x) - \frac{K_1(x)}{x} \right] dr \right. \\ + \sqrt{2} \int_0^\infty u_g(r) w_g(r) \left[K_0(x) + \frac{2K_1(x)}{x} \right] dr + \frac{1}{2} \int_0^\infty w_g^2(r) \left[K_0(x) + \frac{5K_1(x)}{x} \right] dr \Big\} \\ + \frac{2}{\pi} \alpha_g^2 \left\{ \frac{3\sqrt{2}}{2} \int_0^\infty u_g(r) w_g(r) \left[K_0(x) + \frac{2K_1(x)}{x} \right] dr \right. \\ \left. - \frac{3}{4} \int_0^\infty w_g^2(r) \left[K_0(x) + \frac{5K_1(x)}{x} \right] dr \right\}, \quad (17)$$

where $K_0(x)$ and $K_1(x)$ are the Hankel functions of imaginary arguments, $x = \mu r$ and the first term in (17) is due to the spin and the second term to the orbital motion of the nucleon. In (17), the S - D cross terms are dominant and make $\Delta'\mu_D$ positive, as was stressed before.²⁾ The effect upon the magnetic dipole matrix element can be evaluated as

$$\Delta'M = \frac{2}{\pi} \alpha_g^2 \int_0^\infty u_s(r) \left\{ \left[K_0(x) - \frac{K_1(x)}{x} \right] u_g(r) + \sqrt{2} \left[K_0(x) + \frac{2K_1(x)}{x} \right] w_g(r) \right\} dr, \quad (18)$$

where only the nucleon spin can contribute to $\Delta' M$, which is reflected in the similarity of (18) with the first term of (17).

The contribution due to the third term of (4) (meson contribution) to the matrix element $\langle f | \int H'' dx | i \rangle$ is given by

$$\frac{e\hbar c}{2\pi\sqrt{\hbar\nu}} \left(\frac{\alpha_g}{\pi\mu} \right)^2 \iint \phi_f^*(\mathbf{x}) Q(\mathbf{e}_\lambda \mathbf{k}) \times [(\sigma^{(1)} \mathbf{l}) (\sigma^{(2)} \mathbf{k}) - (\sigma^{(2)} \mathbf{l}) (\sigma^{(1)} \mathbf{k})] \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega_k^4} \phi_i(\mathbf{x}) d\mathbf{k} d\mathbf{x}, \quad (19)$$

with $Q = i(\tau_r^{(1)} \tau_y^{(2)} - \tau_y^{(1)} \tau_r^{(2)})/2$, where we have retained only the correction terms to the magnetic dipole matrix element by expanding not only the retardation factors but also other factors containing \mathbf{l} . The cross terms between the zero and two meson amplitudes give rise to just one half of (19), while the remaining half is due to the terms between one meson amplitudes. Although (19) has not yet the form of (10), it can be shown that it has this form by using the explicit form of the deuteron wave function. This term naturally cannot contribute to the deuteron magnetic moment owing to the factor Q . The correction to the magnetic dipole matrix element is finally given by

$$\Delta' M = \frac{(4\kappa/\mu)}{3(\mu_P - \mu_N)} \alpha_g^2 \int_0^\infty u_g(r) \left[\left(e^{-x} - \frac{2e^{-x}}{x} \right) u_g(r) + \frac{1}{\sqrt{2}} \left(e^{-x} + \frac{e^{-x}}{x} \right) w_g(r) \right] dr. \quad (20)$$

Also the last term of (4) (pair contribution) can be managed similarly. Its contribution to the numerator of (5) is

$$\frac{i e \hbar c}{\pi \sqrt{\hbar \nu}} \left(\frac{\alpha_g}{2\pi\mu} \right)^2 \iint \phi_f^*(\mathbf{x}) Q(\mathbf{l} \mathbf{x}) \times [(\sigma^{(1)} \mathbf{k}) (\sigma^{(2)} \mathbf{e}_\lambda) - (\sigma^{(1)} \mathbf{e}_\lambda) (\sigma^{(2)} \mathbf{k})] \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega_k^4} \phi_i(\mathbf{x}) d\mathbf{k} d\mathbf{x}, \quad (21)$$

which contains Q and again cannot contribute to the deuteron magnetic moment. Since (21) still contains corrections to the quadrupole transition matrix element, we must drop some terms before we can get the expression in the form of (10). The final result of $\Delta' M$ is

$$\Delta' M = \frac{(4\kappa/\mu)}{3(\mu_P - \mu_N)} \alpha_g^2 \int_0^\infty u_g(r) \left[e^{-x} + \frac{e^{-x}}{x} \right] \times \left[u_g(r) + \frac{1}{\sqrt{2}} w_g(r) \right] dr, \quad (22)$$

where the S - S term is apparently the dominant one, which means that this is positive. It will be shown that this final correction is large enough to make the whole correction to ΔM positive, although all other terms give negative contributions.

The normalization factor which appears in the denominator of (5) can easily be

evaluated. As the two meson amplitude naturally cannot contribute to $\langle i|i \rangle$ in this approximation, the result is the same as was reported previously:³⁾

$$\begin{aligned} \langle i|i \rangle &= 1 + \Delta N_g, \\ \Delta N_g &= \frac{2}{\pi} \alpha_g^2 \left\{ \int_0^\infty u_g^2(r) \left[K_0(x) - \frac{K_1(x)}{x} \right] dr \right. \\ &\quad + 4\sqrt{2} \int_0^\infty u_g(r) w_g(r) \left[K_0(x) + \frac{2K_1(x)}{x} \right] dr \\ &\quad \left. - \int_0^\infty w_g^2(r) \left[K_0(x) + \frac{5K_1(x)}{x} \right] dr \right\}, \end{aligned} \quad (23)$$

where the S - D cross term is the dominant one, as was stressed before,³⁾ which makes ΔN_g positive. In the radiative transition, the final state belongs to a continuum and the wave function extends to the whole space. Since the meson exchange takes place only when two nucleons are very close together, the effect of it upon the normalization can be neglected entirely.

Thus the over-all exchange current corrections are summarized as follows:

$$\Delta \mu_D = \Delta' \mu_D - \mu_D \Delta N_g, \quad (24)$$

with the same $\Delta' \mu_D$ as (17) and the same ΔN_g as (23) and

$$\begin{aligned} \Delta M &= \frac{2}{\pi} \alpha_g^2 \int_0^\infty u_s(r) \left[\left(K_0(x) - \frac{K_1(x)}{x} \right) u_g(r) + \sqrt{2} \left(K_0(x) + \frac{2K_1(x)}{x} \right) w_g(r) \right] dr \\ &\quad + \frac{(4\kappa/\mu)}{3(\mu_P - \mu_N)} \alpha_g^2 \int_0^\infty u_s(r) \left[\left(2e^{-x} - \frac{e^{-x}}{x} \right) u_g(r) + \sqrt{2} \left(e^{-x} + \frac{e^{-x}}{x} \right) w_g(r) \right] dr \\ &\quad - \frac{1}{2} M \Delta N_g, \end{aligned} \quad (25)$$

where M and ΔN_g are given by (15) and (23), respectively.

It is added finally that all results obtained in all the previous sections, namely (13), (14), (24) and (25) can be obtained more simply by defining the magnetic moment operator by assuming the constant external magnetic field \mathbf{H} by putting $\mathbf{A} = -\mathbf{x} \times \mathbf{H}/2$ instead of the transverse one in the interaction Hamiltonian (4) and then making the matrix element of this magnetic moment operator with respect to the relevant states. Indeed, in the previous investigation,³⁾ the exchange current corrections to the deuteron moments have been calculated along these lines. This shows also that our method of obtaining the corrections to the magnetic dipole transition matrix element is unambiguous.

It is also to be added that our result (23) agrees with those obtained by Sessler⁴⁾ and by Sato and Itabashi⁵⁾ and our result (17) with that obtained by Sato and Itabashi⁵⁾ in the same approximations as are employed in our paper. Our result concerning the correction of the deuteron quadrupole moment, which is given in our previous paper³⁾ but not in this paper, agrees with that obtained by Sessler⁴⁾ but does not agree with that

obtained by Sato and Itabashi.⁵⁾ The important improvements of our paper upon those previous works^{2) 1) 5)} are (i) we are calculating the corrections of the deuteron moments and the capture cross section at the same time in order to make a direct comparison with experiments possible, (ii) the relativistic correction (or the contribution of the nucleon kinetic energy, etc.) is now estimated by a direct use of the deuteron wave functions, while previous authors have introduced the potential energy (or the static nuclear potential) for this purpose, which, however, contains very many ambiguities and inaccessible singularities and (iii) the final estimation of the numerical values have now been done much more carefully than by previous authors.

§ 6. Two nucleon wave functions

In order to get the numerical values of the above corrections we assume the following deuteron wave functions, dropping the normalization factor :

$$\begin{aligned} u_g(x) &= \cos \epsilon [1 - e^{-\beta(x-x_c)}] e^{-x}, \quad x > x_c, \\ w_g(x) &= \sin \epsilon [1 - e^{-\gamma(x-x_c)}]^2 e^{-x} \\ &\quad \times \left[1 + \frac{3(1 - e^{-\gamma x})}{x} + \frac{3(1 - e^{-\gamma x})^2}{x^2} \right], \quad x > x_c, \\ u_g(x) = w_g(x) &= 0, \quad x < x_c, \end{aligned} \quad (26)$$

in units of $x = \alpha r$, $1/\alpha$ being the deuteron radius, x_c the hard core radius in the same unit and ϵ , β and γ adjustable parameters. The power of 2 in $w_g(r)$ is the simplest one that gives the correct boundary conditions at $x = x_c$ in both cases of $x_c = 0$ or $x_c \neq 0$. The value of $1/\alpha$ is chosen to be 4.3157×10^{-13} cm corresponding to the binding energy 2.226 Mev, while the meson Compton wave length $1/\mu$ is 1.414×10^{-13} cm, corresponding to the meson mass of 273 electronic mass. Since the asymptotic behaviours of the functions (26) are correct, they are fitted to the correct deuteron binding energy. According to the effective range theory, the deuteron effective range defined by

$$\rho(-\epsilon, -\epsilon) = 2 \left[\frac{1}{2\alpha} - \int_0^\infty (u_g^2(r) + w_g^2(r)) dr \right] \quad (27)$$

is shown to be very near to the empirically determined triplet effective range r_t . We then give $\rho(-\epsilon, -\epsilon)$ just the shape-independent value 1.704×10^{-13} cm of r_t .^{*} The triplet scattering length does not give any additional requirement upon u_g and w_g . The quadrupole moment Q and the D state probability P_D are, therefore, the only remaining requirements that must be imposed upon the phenomenologically adjusted deuteron wave functions. We assume $Q = 2.738 \times 10^{-27}$ cm² and $P_D = 3\%$ and 3.9% , since the present investigation predicts values of P_D somewhat smaller than 4% , as will be seen later. As regards x_c , we consider the cases $x_c = 0$ and $x_c = 0.13$, which corresponds to $r_c = 0.561 \times 10^{-13}$ cm = 0.397

* The effect of changing $\rho(-\epsilon, -\epsilon)$, which is due to the value of $P_t = 0.048$ (see § 7), can be shown to be quite negligible compared with those effects due to the ambiguities of x_c and P_D upon the deuteron wave functions.

$\times 1/\mu$. The three parameters are then determined as follows :

$$\begin{aligned} x_c=0 & \begin{cases} P_D=3\% : \beta=4.535, \gamma=2.48, \sin \epsilon=0.0326, \\ P_D=3.9\% : \beta=4.441, \gamma=2.92, \sin \epsilon=0.0292, \end{cases} \\ x_c=0.13 & \begin{cases} P_D=3\% : \beta=9.761, \gamma=3.42, \sin \epsilon=0.0289, \\ P_D=3.9\% : \beta=9.396, \gamma=4.06, \sin \epsilon=0.0263. \end{cases} \end{aligned} \quad (28)$$

The deuteron wave functions (26) with constants (28) are plotted in Fig. 1, which shows that the effect of changing P_D or x_c can modify the functions only in the inner region, while the outer behaviours are modified almost not at all, as it should be. The normalization constant of (26) is 3.3047. It is also to be noted that the change of P_D modifies $w_g(r)$ very much, while $u_g(r)$ is almost entirely uninfluenced.

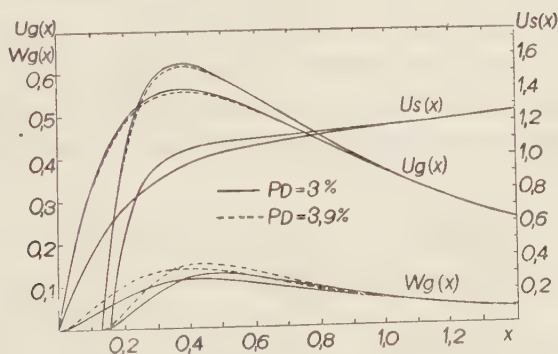


Fig. 1. Phenomenologically adjusted deuteron wave functions (26) with constants (28) in the text are plotted as a function of $x = ar$, $1/\alpha$ being the deuteron radius and the normalization constant being 3.3047, together with the singlet zero energy wave functions (29) with constants (31) in the text.

As the preliminary calculation has shown that the change of the analytic forms of (26) would be much less important than those of P_D or x_c , we shall not consider that effect in this paper.

As regards the singlet S -wave function, we need only the zero energy wave function, since we are going to calculate the thermal N - P capture cross section. The reasonable modification of the zero range approximation of the zero energy wave function would be

$$u_s(r) = 1 - r/a_s - \exp(-\xi r)$$

and

$$\begin{aligned} u_s(r) &= [1 - r/a_s][1 - \exp(-\xi(r - r_c))], \quad r > r_c, \\ u_s(r) &= 0, \quad r < r_c, \end{aligned} \quad (29)$$

in the case where the hard core of radius r_c is assumed in the singlet state. In (29), a_s is the singlet scattering length, which is known to be -23.69×10^{-13} cm and ξ is the adjustable parameter. The singlet effective range which is defined by

$$r_s = 2 \int_0^{\infty} [(1-r/a_s)^2 - u_s^2(r)] dr \quad (30)$$

determines ξ as follows :

$$\begin{aligned} r_c &= 0 ; & \xi &= 1.3040 \times 10^{-13} \text{ cm}^{-1}, \\ r_e &= 0.6787 \times 10^{-13} \text{ cm} ; & \xi &= 3.8557 \times 10^{-13} \text{ cm}^{-1}, \end{aligned} \quad (31)$$

both of which correspond to $r_s = 2.40 \times 10^{-13}$ cm. The singlet zero energy functions (29) with constants (31) are plotted in Fig. 1 as functions of $x = \alpha r$.

As one measure of the appropriateness of our wave functions given above, the low energy photodisintegration total cross section versus the incident photon energy in the laboratory system has been plotted in Fig. 2 together with the ratio of the photomagnetic σ_m to the photo-electric cross section σ_e . Here the wave functions without the hard core were used and the singlet S -function was assumed to be

$$u_s(r) = \sin kr \cot \hat{\delta}_0 + \cos kr [1 - e^{-\xi r}] \quad (32)$$

with the singlet S -phase shift $\hat{\delta}_0$ and the wave vector k in the c.m. system, which approaches (29) in the limit of vanishing k and, therefore, the same value of ξ as (31) was used.

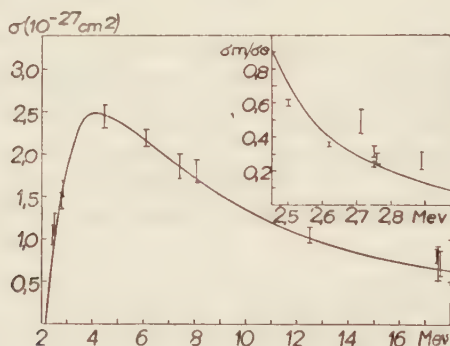


Fig. 2. Low energy photodisintegration total cross section versus the incident photon energy in the laboratory system, together with the ratio of the photomagnetic σ_m to the photo-electric cross section σ_e . The experimental data⁽¹⁾ are indicated by vertical lines.

§ 7. Thermal N - P capture cross section

Let us begin to discuss the phenomenological value of the matrix element M given by (15). In order to estimate the ambiguity of the theoretically predicted value, we must calculate the errors of the constants in the wave functions which are due to the experimental and theoretical ambiguities. Among many sources, the error of r_s , which is now assumed to be $(2.40 \pm 0.40) \times 10^{-13}$ cm, produces the biggest ambiguity which amounts to 2.3% of M , where we have considered the introduction of the hard core of radius up to 0.83×10^{-13} cm in both singlet and triplet states quite independently of each other. For other

quantities we have assumed the following values: $P_D = 3\% \pm 1\%$, $r_t = (1.704 \pm 0.0275) \times 10^{-13}$ cm and the triplet shape dependent coefficient $P_t = 0.048 \pm 0.089$ which means that $\rho(-\epsilon, -\epsilon) = r_t + 2\alpha^2 r_t^3 P_t$. We have neglected the errors of the remaining quantities, since they are quite negligible. Our final theoretical value of M is $4.0555 (1 \pm 0.023) \times 10^{-13}$ cm, which gives then

$$\sigma_{\text{theo}} = 0.3137 \pm 0.0160 \text{ barns} = 0.3137 (1 \pm 0.051) \text{ barns}, \quad (33)$$

where the error of 5.1% is mainly due to the error of r_s ; those due to r_t , P_D and P_t are of comparable order with each other. From the most recent experimental data,⁷⁾ we can estimate the experimental value as

$$\sigma_{\text{exp}} = 0.329 \pm 0.006 \text{ barns} \quad (34)$$

Thus it can be said that there is no definite discrepancy between theory and experiment.

It is to be noted here that the above conclusion is different from that of Austern,⁸⁾ who reported that there is definitely a difference between them of about 8 ± 5 percent. This discrepancy is partly due to his trivial numerical mistake and partly due to the larger error of r_s assumed by us (i.e., $\pm 0.4 \times 10^{-13}$ cm instead of $\pm 0.2 \times 10^{-13}$ cm adopted by him). It must be added furthermore that a larger value of r_s makes σ_{theo}^1 decrease. If we, therefore, assume $r_s = 2.60 \times 10^{-13}$ cm, for example, supported by the charge independence argument, then the small discrepancy may become apparent. We can then conclude that the theoretical value might be somewhat smaller (a few percent at most) than the experimental value, although the evidence is quite ambiguous yet, since the singlet effective range r_s is known only indefinitely.

The relativistic correction (14) can now be estimated by using the wave functions given in section 6. If we employ the wave functions for $P_D = 3\%$, we get after the numerical integration

$$\begin{aligned} \Delta M/M &= -0.160\% ; \text{ without hard cores,} \\ &= -0.344\% ; \text{ with hard cores for both singlet and triplet states.} \end{aligned} \quad (35)$$

The effect of the S - D cross term is 20% or negligible in the former or latter cases, respectively, which shows that the effect of the D -state admixture is small in the relativistic correction. The smallness of the above figure is due to the fact that the one state is the nearly zero energy state. It is also to be noted that the introduction of the hard core makes the relativistic correction increase very much, as it should be.

The exchange current correction (25) can be evaluated similarly. In the integrands of (25), (23) and (17), the factors multiplied by the two nucleon wave functions become infinite at $r=0$, which is, however, due to the fact that we have carried out the integration over the momenta of the exchanged mesons up to infinity. As we have used the non-relativistic approximations in evaluating these corrections, this method of integration is internally inconsistent. In order to remedy the false effects, we have introduced the straight cut-off method to these factors at $r=\pi/\kappa$ or the first zero point of $\sin kr/kr$ at $k=\kappa$. The effect of this cut-off is important only to the S - S terms and becomes negligible when the hard cores are introduced since the cut-off point is quite near to the hard core radii. The

numerical integrations give the following values as the exchange current correction (25) to the capture matrix element M for the coupling constant $\alpha_p^2 = 0.0553$ or 10 of the ps - ps coupling constant:

$$\left. \begin{aligned} \Delta M/M &= 0.597\% ; P_D = 3\% \\ &= 0.422\% ; P_D = 3.9\% \end{aligned} \right\} \text{without hard cores,}$$

$$\left. \begin{aligned} &= 1.079\% ; P_D = 3\% \\ &= 0.846\% ; P_D = 3.9\% \end{aligned} \right\} \text{with hard cores for both singlet and triplet states,} \quad (36)$$

where the increase of ΔM with hard cores is due to the reduction of ΔN_p , while $\Delta'M$ itself is almost unchanged.

If we assume that a reasonable estimate of P_D is 3%, then we get finally as the total correction to the theoretical capture cross section (33)

$$\Delta\sigma/\sigma = +0.875\% \text{ or } +1.27\% \quad (37)$$

without or with hard cores, respectively. The above figure is smaller than the theoretical and experimental ambiguities in (33) and (34), which makes it impossible to get a direct experimental proof. We may conclude, however, that the exchange current and the relativistic corrections to the capture cross section amount, in total, to only about +1% of that predicted by the phenomenological theory and this small positive correction may be considered to be consistent with the present theoretical and experimental values (33) and (34).

§ 8. Deuteron magnetic moment

The relativistic correction (13) and the exchange current correction (24) can also be estimated by using the wave functions in section 6 and employing the same cut-off method explained in the previous section. The numerical values for the same coupling constant as in the previous section are the following:

	without hard cores	with hard cores
Relativistic: $\Delta\mu_D/\mu_D =$	$-1.01\% (P_D = 3\%),$	$-1.88\% (P_D = 3\%),$
Exchange: $\Delta\mu_D/\mu_D =$	$-0.939\% (P_D = 3\%),$	$-0.561\% (P_D = 3\%),$
	$-1.174\% (P_D = 3.9\%),$	$-0.684\% (P_D = 3.9\%).$

(38)

The effect of the S D cross terms in the relativistic corrections given above is less than 9% or 2% of the values in (38) in the cases without or with the hard core, respectively. It is, therefore, legitimate for us to have neglected the D - D cross terms in (13). The introduction of the hard core makes the relativistic correction increase very much, while it makes the exchange current correction decrease, which makes the total effect rather insensitive to the hard core. The reduction of the latter effect is due to the fact that the exchange current correction comes mainly from the innermost region of the deuteron wave function and the introduction of the hard core removes much of the important regions from the integrals. This is contradictory to the similar calculation by Sessler.⁴⁾ This discrepancy seems to be due to his deuteron wave functions not being so well adjusted to

the empirical data as ours. It is to be noted that the relativistic correction is very important, which is contradictory to the statement in the previous paper.⁽¹⁾ This discrepancy is due to the fact that in the previous paper we have used another method to estimate the relativistic correction and have neglected the fact that the introduction of the hard core makes the relativistic correction increase very much.

If we assume again that a reasonable estimate of P_D is 3%, then we get as the sum of the relativistic and exchange current corrections

$$\Delta\mu_D/\mu_D = -1.95\% \text{ or } -2.44\% \quad (39)$$

without or with the hard core, respectively. The above figures are indeed very large compared with the empirically known value of the difference between the deuteron moment and the sum of the proton and neutron moments:

$$[\mu_D - (\mu_p + \mu_n)]/\mu_D = -2.60\% \quad (40)$$

It must be added finally that the exchange current correction is very sensitive to the D -state admixture and decreases very rapidly as the deuteron wave function is fitted to smaller and smaller values of P_D ; it practically vanishes in the limit of $P_D=0$, although the relativistic corrections are almost unchanged. The near equality of the figures in (39) and (40) does not, therefore, mean that P_D may be much smaller than 4%.

§ 9. Summary and conclusions

It has been shown that, as regards the corrections to the N - P capture cross section, the relativistic one is negative, but is a minor one since the initial state belongs to the nearly zero energy, while the exchange current effect is positive. Numerically the sum of these amounts to about +1% correction to the phenomenological N - P capture cross section, which has also been estimated and shown to agree with the experimental one within much larger theoretical and experimental errors than the above correction. As the theoretical value might be somewhat smaller than the experimental one for some reason, the above figure for the correction may be regarded as consistent with the present experimental data.

With respect to the deuteron magnetic moment, the same exchange current correction as was reported before⁽¹⁾ has been obtained. The present method of estimating the relativistic correction has, however, given much larger values of it than reported before⁽¹⁾ especially when the hard core is introduced, although the exchange current correction is very much reduced in this case. Quantitatively both effects are negative and their sum amounts to about -2% of the empirical deuteron moment.

It has been shown furthermore that the relativistic corrections are almost insensitive to the D -state probability of the deuteron, while the exchange current ones are very sensitive to it. The above figures are calculated for the D -state probability of 3% and the ps - ps coupling constant of 10. Although both these corrections are very sensitive to the introduction of the hard core, their sum has been shown to be rather insensitive to it.

As regards the non-additivity correction, the present author has made some estimation in a previous paper,⁽³⁾ which consists in estimating the effect of the appearance of the

deuteron binding energy in the energy denominator of the T - D solution of the meson-nucleon system and gave a rather definite value of $+0.26\%$ using a cut-off procedure and employing the same value of the coupling constant. In this investigation, however, all the essential processes which give rise to the non-additivity are not included, since the calculation has been done only up to the lowest order of the meson-nucleon coupling constant. Although this approximation would be sufficient in calculating only the exchange current effect, higher order effects may be important with respect to the non-additivity effect. Such effects have been estimated by Miyazawa,⁹⁾ who has shown that the non-additivity correction may amount to $+1 \pm 1\%$ of the empirical deuteron moment. Although his estimation is not yet quite satisfactory and more accurate calculations are necessary until we get a more definite value, we may reasonably expect a small positive non-additivity correction of the order of the figures given above.

Even if we neglect some possible positive non-additivity correction, our present investigation does not give a value of the D -state probability much smaller than 4% , as was remarked at the end of section 8. As the non-additivity correction would at most cancel out the negative correction obtained above, the over-all correction may remain negative, although its magnitude is very uncertain yet. According to these considerations, we may finally conclude that the over-all correction would amount to $-1 \pm 1\%$ or so of the deuteron moment and $+1\%$ or more of the thermal N - P capture cross section. The above figure gives rise to the estimate of the D -state probability of $3 \pm 1\%$ or so, which is the same value as was reported previously.

Finally the author should like to express his gratitude to Professor Lamek Hulthén for his hospitality and to Mr. Z. Sawa for helping him in the numerical work.

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Low Energy Limits of Photon-Nucleon and Pion-Nucleon Collisions

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Low energy limits of photon-nucleon and pion-nucleon collisions are investigated on the basis of the approximation which neglects higher order terms with respect to the pion-nucleon mass ratio but includes the radiative corrections to an arbitrary order in the coupling constants. It is shown that, near threshold, all the radiative corrections to the processes, $\pi + N \rightarrow N + 2\pi$ and $\gamma + N \rightarrow N + 3\pi$, can be concealed into the nucleon mass, the electric charge, the mesic charge and the Matthews' constant renormalized in accordance with the conventional prescription, provided that the renormalized Matthews' constant is of the same order of magnitude as or larger than the renormalized coupling constant for the pion-nucleon interaction (§§ 3 and 4). It is suggested that these processes may serve the experimental determination of the Matthews' constant and the test of validity of the $PS(PS)$ theory (§ 5). As preliminaries for these arguments, Thirring's and Kroll and Ruderman's theorems on the low energy Compton scattering and the threshold photomeson production are reviewed, taking account of those terms which have been omitted in the original works, and effective charges are introduced for pion-nucleon scattering and two-gamma decay of a neutral pion (§ 2). It is noted that the results are unaffected by virtual processes caused by heavy mesons obeying the "even-odd" rule proposed by Pais (§ 5).

§ 1. Introduction

It was first pointed out by Thirring¹⁾ that the observable radiative corrections to the Compton scattering by an electron disappear at the Thomson limit. In other words, the matrix element for the Compton scattering at this limit computed to an arbitrary order in $e^2/4\pi\hbar c$ is exactly equal to the weak coupling result obtained from second order perturbation theory, except that the renormalized mass and charge take the place of the unrenormalized ones. It is evident that a similar theorem should hold not only for the Compton scattering by an electron but also for the Compton scattering by a proton or by a charged pion, because the Thomson limit of the Compton scattering is a classical phenomenon irrelevant to the spin structure of the charged scattering center.

Kroll and Ruderman²⁾ have shown that an analogous theorem holds also for the threshold photomeson production from a nucleon. The content of Kroll-Ruderman's theorem is as follows: The matrix element for threshold photomeson production, which is rigorous except for the neglect of the real pion mass in comparison with the nucleon mass, is given by the same expression as the result obtained by means of the lowest order perturbation theory, provided that the renormalized electric and mesic charge, e and g , are substituted for the unrenormalized ones, e_u and g_u .^{*} This theorem is of great importance as it presents the most direct method of experimental determination of the value for g . Bernardini and Goldwasser³⁾ have obtained the value, $g^2/4\pi\hbar c = 11.8 \pm 0.14$, for the renormalized mesic charge from their measurement of the photomeson production from hydrogen near threshold.

No analogous theorem holds, however, for the pion-nucleon scattering at low energy limit, because there remain radiative corrections which can not be concealed in the (conventionally) renormalized mesic charge. But it is possible to express the matrix element for this process formally with the use of the lowest order formula of the perturbation theory, introducing the "effective mesic charge, g_s , for pion-nucleon scattering" to supersede g_u . Deser, Thirring and Goldberger⁴⁾ have tried to discuss the inconsistency of the $PS(PS)$ meson theory with experiment on the basis of the discrepancy between the large value 11.8 for $g^2/4\pi\hbar c$ and the small value 0.36 for $g_s^2/4\pi\hbar c$, interpreting the latter as an adequate weak coupling parameter. These authors have proposed a new prescription for renormalization with the intention of justifying the series expansion with respect to g_s . However, their prescription is open to criticism⁵⁾. Furthermore, as has been pointed out by Källén⁶⁾, one can not draw from the discrepancy mentioned above any decisive conclusion other than that the weak coupling approximation fails, before one can succeed in the field-theoretic evaluation of the ratio, g_s/g , by means of some reliable method without resort to the weak coupling perturbation theory. Such a non-perturbational method is too hard to work out at the present stage. Therefore, one must give up an attempt to test the consistency of the $PS(PS)$ meson theory with experiment by examining only the interrelation between the threshold photomeson production and the pion-nucleon scattering at low energy limit. This does not deny, however, that there may be left a possibility of testing the consistency of the theory with experiment, if one looks over the interrelations among various processes at low energy limit including multiple processes caused by photon-nucleon collision or by pion-nucleon collision. This possibility will be discussed in a part of the present paper.

Now we retrace the origin of Thirring's and Kroll-Ruderman's theorems. Let M denote the (renormalized) nucleon mass and μ' a quantity representing the order of magnitude of the four-momenta** carried by real bosons (photons and pions). Suppose we have expanded the matrix elements in an ascending power series in μ'/M . The assumption of low energy limits allows us to neglect all the higher order terms in μ'/M , which otherwise

* It is understood that the renormalization of the nucleon mass has already been performed.

** The natural units in which $\hbar=c=1$ are used throughout this paper.

must be computed by means of some non-perturbational method. On the other hand, the prescription for renormalization of vertex parts enables us to express the leading term in terms of the renormalized mass and coupling constants, formally with the use of the lowest order formula of the perturbation theory. This is the essential point for the two theorems to be proved. It will be shown in § 2·3 that a similar theorem holds also for the Møller scattering between two pions which may occur as a virtual process in the multiple processes caused by boson-nucleon collisions: so far as the leading term with respect to μ'/M is concerned, the matrix element for Møller scattering between two pions, including the radiative corrections to an arbitrary order in the coupling constants, is given by the result obtained from the lowest order perturbation theory, provided that the renormalized Matthews' constant⁷⁾, λ , is substituted for the divergent coefficient, λ_u . Here it seems worth while investigating whether there are any multiple processes for which the rigorous matrix elements can be expressed compactly in terms of M , e , g , λ , and g_s .

Of the processes caused by a photon-nucleon collision or by a pion-nucleon collision, the simplest ones are the Compton scattering by a proton, the photomeson production from a nucleon, and the pion-nucleon scattering. Although these processes have already been discussed by Thirring, by Kroll and Ruderman, and by Deser et al., we shall review the features of these processes in § 2 as preliminaries for later sections. In § 2 we shall also discuss the properties at low energy limit of those processes in which three or four real bosons but no nucleons participate. In § 3 we shall treat the double pion production from a nucleon bombarded by pion beams or by gamma rays. It will be shown that, near threshold, the rigorous matrix element for the former process, including the radiative corrections to an arbitrary order in the coupling constants, can be expressed in terms of g and λ . § 4 is concerned with the triple pion production from a nucleon bombarded by pion beams or gamma rays. For the latter process near threshold, the rigorous matrix element has been obtained as a function of e , g , λ , and M . The results obtained are unaffected by the presence of any heavy mesons that may participate in the virtual processes, so far as they are particles with an odd suffix in the sense of Pais' theory of V -particles⁸⁾ and do not violate the renormalizability of the theory. The possibility of testing the consistency of the current $PS(PS)$ meson theory with experiment will be discussed in § 5.

§ 2. Properties of propagation functions and vertex parts at low energy limit

It is convenient to use Schwinger's theory of Green's functions⁹⁾ for dealing with mutually interacting quantized fields in a closed form without resort to series expansion in the coupling constants. According to his theory, the matrix element for any process can be described in terms of Green's functions and their functional derivatives with respect to artificially introduced external source functions. For example, the matrix element for the Compton scattering by a nucleon can be written in the form

$$T^{(2,0,2)} = A_\mu(\xi) \bar{\psi} T_{\mu\nu}^{(2,0,2)}(\xi, \xi') \psi A_\nu(\xi') \quad (2\cdot1)$$

with $T_{\mu\nu}^{(2,0,2)}(\xi, \xi')$ defined by

$$T_{\mu\nu}^{(2,0,2)}(\xi, \xi') = D_{\mu\rho}^{-1}(\xi, \xi'') S^{-1} \partial^2 S / \partial J_\rho(\xi'') \partial J_\sigma(\xi''') S^{-1} D_{\sigma\nu}^{-1}(\xi''', \xi'), \quad (2.2)$$

where a summation or an integration is understood over a twice appearing tensor suffix such as μ, ν, ρ and σ or over a twice appearing photon coordinate such as ξ, ξ', ξ'' , and ξ''' . In the above expressions, S and $D_{\mu\nu}$ denote Green's functions for one nucleon and for one photon, respectively, in the presence of the external electric four-current, J_μ , and the artificially introduced pion source function, K_i . A_ν and ψ are the wave functions describing the incident photon and the nucleon in the initial state, while A_μ and $\bar{\psi}$ are those describing the scattering photon and the nucleon in the final state. The bracketed three integers, (n, π, γ) , attached to the right shoulder of T , T and I' to appear later on denote the numbers of external lines representing nucleon, pion, and photon, respectively, in the language of Feynman diagram.

Introducing the auxiliary operators describing various vertices

$$\begin{aligned} \Gamma_i^{(2,1,0)}(\xi) &\equiv -\partial S^{-1} / \partial \langle g_i \phi_i(\xi) \rangle, \\ \Gamma_\mu^{(2,0,1)}(\xi) &\equiv -\partial S^{-1} / \partial \langle e A_\mu(\xi) \rangle, \\ \Gamma_{\mu\nu}^{(2,0,2)}(\xi, \xi') &\equiv -\partial^2 S^{-1} / \partial \langle e A_\mu(\xi) \rangle \partial \langle e A_\nu(\xi') \rangle \end{aligned} \quad (2.3a)$$

and

$$\Gamma_{i\mu\nu}^{(0,1,2)}(\xi, \xi, \xi') \equiv -\partial D_{\mu\nu}^{-1}(\xi, \xi') / \partial \langle g_i \phi_i(\xi) \rangle,$$

we can write $T_{\mu\nu}^{(2,0,2)}$ as

$$\begin{aligned} T_{\mu\nu}^{(2,0,2)}(\xi, \xi') &= e^2 \{ \Gamma_\mu^{(2,0,1)}(\xi) S \Gamma_\nu^{(2,0,1)}(\xi') + \Gamma_\nu^{(2,0,1)}(\xi') S \Gamma_\mu^{(2,0,1)}(\xi) \\ &\quad + \Gamma_{\mu\nu}^{(2,0,2)}(\xi, \xi') \} + g_i g_j \Gamma_i^{(2,1,0)}(\xi) A_{ij}(\xi, \xi') \Gamma_{j\mu\nu}^{(0,1,2)}(\xi', \xi, \xi'), \end{aligned} \quad (2.4)$$

where $A_{ij}(i, j=1, 2, 3)$ denotes Green's function for one pion in the presence of J_μ and K_i , ξ and ξ' the meson coordinates. The pion is, of course, assumed to be described by a pseudoscalar field with three components, ϕ_1, ϕ_2 , and ϕ_3 , coupled with the nucleon field, ψ , through the pseudoscalar interactions with the coupling constants, $g_1=g_2$ and g_3 .* In the derivation of (2.4) a simplification has been brought in by the selection rule

$$\Gamma_{\mu\nu\lambda}^{(0,0,3)}(\xi, \xi', \xi'') \equiv -\partial D_{\mu\nu}^{-1}(\xi, \xi') / \partial \langle e A_\lambda(\xi'') \rangle = 0, \quad (2.5a)$$

which follows from the theorem of Fukuda, Miyamoto⁽¹⁰⁾, Pais and Jost⁽¹¹⁾. The presence of any kinds of mesons other than the pion in the intermediate states has been ignored. We shall show in § 5 that the results obtained in the present paper are practically unaffected by the participation of heavy mesons in the virtual processes, so far as their interactions with the other fields obey the "even-odd rule" of Pais' theory of V -particles⁽⁸⁾ and do not violate the renormalizability of the theory. It is convenient for later discussions to represent the equation (2.4) by a Feynman diagram as Fig. 1. Here it is to be stressed that the radiative corrections are wholly taken into account in the relations (2.1) to (2.5), and that these relations hold no matter whether all the quantities are renormalized or unrenormalized.

* For the sake of simplification, we have excluded the alternative neutral meson which is described by an invariant function under the rotation of the isotopic spin space.

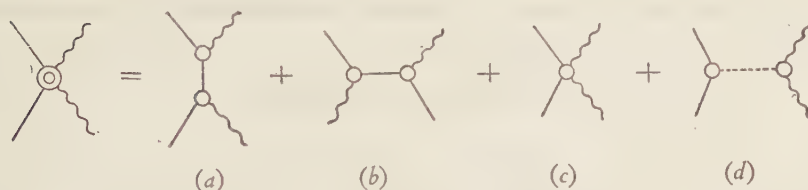


Fig. 1. Feynman diagrams representing the Compton scattering by a nucleon. Solid lines, broken lines and waved lines represent the propagation of nucleon wave, pion wave and photon wave, respectively. The double circle represents the transition matrix, while single circles stand for the vertex parts defined by the equation (2.3a).

From now on we shall assume that all the quantities have been renormalized.

The transition matrix element for the photo-pion production, which includes all the radiative corrections, is similarly given by

$$T^{(2,1,1)} = \phi_i(\zeta) \bar{\psi} T_{i\mu}^{(2,1,1)}(\zeta, \xi) \psi A_\mu(\xi) \quad (2.6)$$

with

$$\begin{aligned} T_{i\mu}^{(2,1,1)}(\zeta, \xi) &= \Delta_{ij}^{-1}(\zeta, \zeta') S^{-1} \partial^2 S / \partial K_j(\zeta') \partial J_\nu(\xi') S^{-1} D_{\nu\mu}^{-1}(\xi', \xi) \\ &= e g_j \{ \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_\mu^{(2,0,1)}(\xi) + I_\mu^{(2,0,1)}(\xi) S \Gamma_i^{(2,1,0)}(\zeta) + I_\mu^{(2,1,1)}(\zeta, \xi) \\ &\quad + I_k^{(2,1,0)}(\zeta'') J_{kj}(\zeta'', \zeta') \Gamma_{j\mu}^{(0,2,1)}(\zeta', \zeta, \xi) \\ &\quad + I_\lambda^{(2,0,1)}(\xi'') D_{\lambda\nu}(\xi'', \xi') \Gamma_{\nu\mu}^{(0,1,2)}(\zeta, \xi', \xi) \}, \end{aligned} \quad (2.7)$$

where ϕ_i denotes the wave function of the produced pion, and we have introduced the following notations of vertex parts

$$\begin{aligned} \Gamma_{i\mu}^{(2,1,1)}(\zeta, \xi) &= -\partial^2 S^{-1} / \partial \langle g \phi_i(\zeta) \rangle \partial \langle e A_\mu(\xi) \rangle, \\ \Gamma_{ij\mu}^{(0,2,1)}(\zeta, \zeta', \xi) &= -\partial \Delta_{ij}^{-1}(\zeta, \zeta') / \partial \langle e A_\mu(\xi) \rangle. \end{aligned} \quad (2.3b)$$

In the same way the transition matrix element for the pion-nucleon scattering, which includes all the radiative corrections, can be expressed in the form

$$T^{(2,2,0)} = \phi_i(\zeta) \bar{\psi} T_{ij}^{(2,2,0)}(\zeta, \zeta') \psi \phi_j(\zeta') \quad (2.8)$$

with

$$\begin{aligned} T_{ij}^{(2,2,0)}(\zeta, \zeta') &= \Delta_{ik}^{-1}(\zeta, \zeta'') S^{-1} \partial^2 S / \partial K_i(\zeta'') \partial K_l(\zeta''') S^{-1} \Delta_{lj}^{-1}(\zeta''', \zeta') \\ &= g_i g_j \{ \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_j^{(2,1,0)}(\zeta') + \Gamma_j^{(2,1,0)}(\zeta') S \Gamma_i^{(2,1,0)}(\zeta) + I_{ij}^{(2,2,0)}(\zeta, \zeta') \} \\ &\quad + e^2 \Gamma_\mu^{(2,0,1)}(\xi) D_{\mu\nu}(\xi, \xi') \Gamma_{\nu j}^{(0,2,1)}(\xi', \zeta, \zeta'), \end{aligned} \quad (2.9)$$

and

$$\Gamma_{ij}^{(2,2,0)}(\zeta, \zeta') = -\partial^2 S^{-1} / \partial \langle g_i \phi_i(\zeta) \rangle \partial \langle g_j \phi_j(\zeta') \rangle, \quad (2.3c)$$

where the external sources, J_μ and K_i , and thereby induced fields, $\langle e A_\mu \rangle$ and $\langle g_i \phi_i \rangle$,

should be put equal to zero after the functional differentiations. In the derivation of (2.9), we have taken account of the selection rule

$$\Gamma_{ij\bar{k}}^{(0,3,0)}(\xi, \xi', \xi'') \equiv -\partial A_{ij}^{-1}(\xi, \xi') / \partial \langle g_k \phi_k(\xi'') \rangle = 0 \quad (2.5b)$$

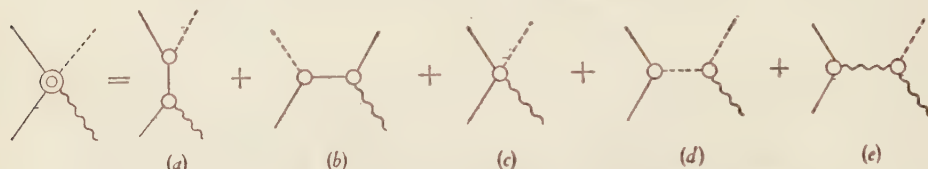


Fig. 2. Feynman diagrams representing the photopion production. Cf. equation (2.7).

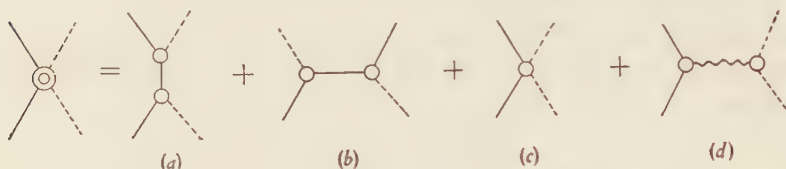


Fig. 3. Feynman diagrams representing the pion-nucleon scattering. Cf. equation (2.9).

which follows from the charge conservation law and the theorem of Fukuda-Miyamoto-Pais-Jost. The graphical representations of (2.7) and (2.9) are given in Figs. 2 and 3. We shall call the single internal boson line as $D_{\mu\nu}(\xi, \xi')$ in (2.9) the "bridge".

§ 2.1. Green's functions at low energy limit

Processes we are going to study in this paper are those in which a low energy pion or a low frequency photon collides with a nucleon initially at rest to result in scattering or pion production. The incident energy is assumed to be so low that it is insufficient to create nucleon pairs though it may be a little above the threshold energy for production of a few pions. Let μ' denote the order of magnitude of the incident energy. Then the magnitudes of the four-momenta carried by the outgoing bosons are at most of order μ' . Throughout this paper we shall assume that μ' is not larger than a small number times the pion mass, μ , and shall regard μ'/M as a small parameter with respect to which we can expand the matrix elements in a power series of good convergence. Remembering $\mu/M=0.15$, this assumption may be accepted in case where the number of pions produced is small. Since we are interested in the low energy limits, we assume that it suffices for the purpose of evaluation of the matrix element to retain only the lowest order term in μ'/M , neglecting all higher order terms. The estimation of the magnitude of the expansion coefficients has been performed carefully. In this subsection we shall study the behavior at low energy limit of the renormalized Green's functions and vertex parts for one nucleon.

First we shall begin with the study of properties of the renormalized Green's function for one nucleon at low energy limit. Confining ourselves to this limit, we are allowed to ignore the ξ and ζ dependence of $A_\mu(\xi)$'s and $\phi_i(\zeta)$'s in the ξ and ζ integrals of the transition matrix elements such as (2.1), (2.6) and (2.8). Accordingly, we may take $\langle eA_\mu \rangle$ and $\langle g_i \phi_i \rangle$ as constant parameters independent of ξ and ζ , as Deser et al.¹⁾ did.

This brings us a great simplification, because the functional derivatives when integrated over the whole space-time reduce to the ordinary derivatives, for example, as

$$\begin{aligned} \int d\xi \delta / \delta \langle e A_\mu(\xi) \rangle &= \partial / \partial \langle e A_\mu \rangle, \\ \int d\xi \delta / \delta \langle g_i \phi_i(\xi) \rangle &= \partial / \partial \langle g_i \phi_i \rangle. \end{aligned} \quad (2.10)$$

Let p_μ ($\mu=1, 2, 3, 4$) denote the operator representing the four-momentum of the nucleon, and M the renormalized nucleon mass which we shall not distinguish between proton and neutron for the sake of simplification. Further, let us assume the charge independence in order to avoid inessential complications.* Then, from the considerations of the invariance under the Lorentz and the charge conjugation transformations it follows that the renormalized inverse Green's function for one nucleon in the presence of the external pion source but in the absence of the electric four-current can be written in the form

$$\begin{aligned} S^{-1}(p, \langle g\phi \rangle) &= \gamma p + M - \sum_{i=1}^3 \langle g\phi_i \rangle \tau_i \gamma_5 + MF \\ &+ (\gamma p + M) G + \sum_{i=1}^3 \langle g\phi_i \rangle \tau_i \gamma_5 D \\ &+ (2M)^{-1} \{ \gamma p + M, \sum_{i=1}^3 \langle g\phi_i \rangle \tau_i \gamma_5 H \} \\ &+ (\gamma p + M) \sum_{i=1}^3 \langle g\phi_i \rangle \tau_i \gamma_5 J (\gamma p + M), \end{aligned} \quad (2.11)$$

where D, F, G, H and J are functions of $p^2 + M^2 = \sum_\mu p_\mu p_\mu + M^2$ and $\langle g\phi \rangle^2 = \sum_i \langle g\phi_i \rangle^2$, *e.g.*,

$$D = D(p^2 + M^2, \langle g\phi \rangle^2), \text{ etc.}$$

γ_μ ($\mu=1, 2, 3, 4$), γ_5 and τ_i ($i=1, 2, 3$) are the Dirac matrices** and the isotopic spin matrices, respectively. The functions, D, F and G , are subject to the conditions

$$F(0, 0) = 0, \quad G(0, 0) + 2M^2 F'(0, 0) = 0 \quad (2.12a)$$

and

$$D(0, 0) = 0, \quad (2.12b)$$

where the prime means the partial differentiation with respect to the first argument, $p^2 + M^2$. The arguments are, of course, to be put equal to zero after differentiation. The former condition (2.12a) follows from Dyson's prescription for renormalization of quantum electrodynamics¹²⁾, while the latter condition (2.12b) from Kroll-Ruderman's prescription for the pion-nucleon three-vertex parts²⁾.

Since γp is anticommutative with $\sum_i \langle g\phi_i \rangle \tau_i \gamma_5 H$ for ξ -independent $\langle \phi_i \rangle$, the γp within

* Thirring's theorem and Kroll-Ruderman's theorem can be proved without the assumption of the charge independence.

** Our γ matrices are the same as those used by Schwinger. Multiply our γ_μ and γ_5 by $-i$ and i , respectively, to get the conventional ones.

the anticommutator bracket of (2.11) may seem to be redundant. This is true so long as the application of (2.11) is restricted to the pion-nucleon two-field problems only. However, we can extend its applications to the photon-pion-nucleon three-field problems with the aid of the substitution

$$p_\mu \rightarrow \bar{p}_\mu \equiv p_\mu - \tau_P \langle e A_\mu \rangle \quad (2.13)$$

and a suitable symmetrization of the products to satisfy the requirement of the invariance under the charge conjugation transformation, where τ_P is the projection operator defined by

$$\tau_P \equiv \frac{1}{2}(1 + \tau_3) \quad (2.14)$$

which singles out the proton state. In this case, $\gamma \bar{p}$ no longer anticommutes with $\sum_i \langle g | \phi_i \rangle \tau_i \gamma \cdot H$ even if $\langle \phi_i \rangle$ is ζ -independent. This is the reason why we have reserved γp in the anticommutator bracket of (2.11).

Now, from the inverse Green's function given by (2.11) we can derive, remembering (2.3a), (2.3b), (2.3c) and (2.10), the expressions for the vertex parts relevant to boson-nucleon collision processes at low energy limit as

$$\begin{aligned} \Gamma_i^{(2,1,0)} &\equiv \int d\zeta \Gamma_i(\zeta) = -\partial S^{-1} / \partial \langle g \phi_i \rangle \\ &= \gamma_5 \tau_i (1 - D) - (2M)^{-1} \{ \gamma p + M, \gamma_5 \tau_i H \} \\ &\quad - (\gamma p + M) \gamma_5 \tau_i J (\gamma p + M), \end{aligned} \quad (2.15a)$$

$$\begin{aligned} \Gamma_{ij}^{(2,2,0)} &\equiv \int d\zeta \int d\zeta' I'_{ij}^{(2,2,0)}(\zeta, \zeta') = -\partial^2 S^{-1} / \partial \langle g \phi_i \rangle \partial \langle g \phi_j \rangle \\ &= -2\delta_{ij} \{ M \dot{F} + (\gamma p + M) \dot{G} \}, \end{aligned} \quad (2.15b)$$

$$\begin{aligned} \Gamma_\mu^{(2,0,1)} &\equiv \int d\hat{\zeta} \Gamma_\mu^{(2,0,1)}(\hat{\zeta}) = -\partial \bar{S}^{-1} / \partial \langle e A_\mu \rangle = \partial S^{-1} / \partial (\tau_P p_\mu) \\ &= \tau_P \gamma_\mu (1 + G) + 2\tau_P p_\mu \{ M F' + (\gamma p + M) G' \}, \end{aligned} \quad (2.15c)$$

$$\begin{aligned} \Gamma_{\mu\nu}^{(2,0,2)} &\equiv \int d\hat{\zeta} \int d\hat{\zeta}' I'_{\mu\nu}^{(2,0,2)}(\hat{\zeta}, \hat{\zeta}') = -\partial^2 \bar{S}^{-1} / \partial \langle e A_\mu \rangle \partial \langle e A_\nu \rangle \\ &= -\partial^2 S^{-1} / \partial (\tau_P p_\mu) \partial (\tau_P p_\nu) = -2\delta_{\mu\nu} \tau_P \{ M F' + (\gamma p + M) G' \} \\ &\quad - 2\tau_P (\gamma_\mu p_\nu + \gamma_\nu p_\mu) G' - 4\tau_P p_\mu p_\nu \{ M F'' + (\gamma p + M) G'' \}, \end{aligned} \quad (2.15d)$$

$$\begin{aligned} \Gamma_{i\mu}^{(2,1,1)} &\equiv \int d\zeta \int d\hat{\zeta} \Gamma_{i\mu}^{(2,1,1)}(\zeta, \hat{\zeta}) = -\partial^2 \bar{S}^{-1} / \partial \langle g \phi_i \rangle \partial \langle e A_\mu \rangle \\ &= -\partial^2 S^{-1} / \partial \langle g \phi_i \rangle \partial (\tau_P p_\mu) = \gamma_5 p_\mu \{ \tau_i, \tau_P \} D' + (2M)^{-1} \gamma_5 \gamma_\mu [\tau_i, \tau_P] H \\ &\quad + (2M)^{-1} \{ \gamma p + M, \gamma_5 \} \gamma_\mu \{ \tau_i, \tau_P \} H' \\ &\quad + \gamma_\mu \gamma_5 \cdot \tau_P \tau_i J (\gamma p + M) + (\gamma p + M) \gamma_5 \gamma_\mu \cdot \tau_i \tau_P J \\ &\quad + (\gamma p + M) p_\mu \gamma_5 \{ \tau_i, \tau_P \} J' (\gamma p + M), \end{aligned} \quad (2.15e)$$

where by the notation $\partial/\partial(\tau_i p_\mu)$ we mean that the differentiation should be performed after p_μ has been replaced by $\tau_P(\tau_P p_\mu) + \tau_n(\tau_n p_\mu)$. The bar over S denotes that p has been replaced by \bar{p} , and a dot over F and G stands for the partial differentiation with respect to $\langle g\phi \rangle^2$. $\langle g\phi_i \rangle$ and $\langle eA_\mu \rangle$ have already been put equal to zero.

It is to be noted here that the symmetrization of \bar{S} can not be performed without ambiguities. However, these ambiguities will not essentially affect the results, as may be explained in the following. For example, the simplest products, of which the symmetrization involves ambiguities, are (i) $\bar{p}_\mu \langle \phi_i \rangle^2$ and (ii) $\bar{p}_\mu \tau_i \langle \phi_i \rangle$. Either can be symmetrized in the following three ways:

$$\begin{aligned} \text{(i)} \quad & \tau_i \langle \phi_i \rangle \bar{p}_\mu \tau_i \langle \phi_i \rangle, & \frac{1}{2} \{ \bar{p}_\mu, \langle \phi_i \rangle^2 \}, & \frac{1}{4} \{ \tau_i \langle \phi_i \rangle, \{ \tau_i \langle \phi_i \rangle, \bar{p}_\mu \} \}; \\ \text{(ii)} \quad & \bar{p}_\mu \tau_i \langle \phi_i \rangle \bar{p}_\mu, & \frac{1}{2} \{ \tau_i \langle \phi_i \rangle, \bar{p}_\mu^2 \}, & \frac{1}{4} \{ \bar{p}_\mu, \{ \bar{p}_\mu, \tau_i \langle \phi_i \rangle \} \}. \end{aligned}$$

The vertex parts to which these three types of products give different contributions are (i) $\Gamma_{i\mu\nu}^{(2,2,1)} = -\partial^3 \bar{S}^{-1} / \partial \langle g\phi_i \rangle \partial \langle g\phi_j \rangle \partial \langle eA_\mu \rangle$ and (ii) $\Gamma_{i\mu\nu}^{(2,1,2)} = -\partial^3 \bar{S}^{-1} / \partial \langle g\phi_i \rangle \partial \langle eA_\mu \rangle \times \partial \langle eA_\nu \rangle$, respectively. More generally, the vertex parts which are affected by the ambiguities in the symmetrization of the products $\pi \tau_i \langle \phi_i \rangle$'s and $\gamma \bar{p}_\mu$'s are $\Gamma^{(2,\pi,\gamma)}$'s.* Hence, the vertex parts, $\Gamma^{(2,\pi,\gamma)}$ with $\pi + \gamma = 2$, given by (2.15) are free from ambiguities of this kind. Another example of ambiguity is (iii) that which is due to the difference between \bar{p}_μ^2 and $\tau_i \bar{p}_\mu \tau_i$. Of the five vertex parts of (2.15), the only one which may be accompanied by this ambiguity is $\Gamma_{\mu\nu}^{(2,0,2)}$ given by (2.15d). Here we should be reminded that this vertex part is derived from the inverse Green's function for one nucleon in the presence of the external electric four-current but in the absence of the external pion source, which must be arranged, according to Dyson's prescription for renormalization, in the form

$$S^{-1}(p, 0) = \gamma \bar{p} + M + (\gamma \bar{p} + M) L (\gamma \bar{p} + M), \quad (2.16)$$

where L is a function of $\bar{p}^2 + M^2$ and $\gamma \bar{p} + M$ with no singularities at $\bar{p}^2 + M^2 = 0$.** The ambiguity under consideration occurs only in L , and is sure to disappear by virtue of the factor, $(\gamma \bar{p} + M)$, standing on both sides, so far as the matrix elements containing $\Gamma_{\mu\nu}^{(2,0,2)}$ in the form $\bar{\psi}(p) \Gamma_{\mu\nu}^{(2,0,2)} \psi(p)$ are concerned, where $\bar{\psi}(p)$ and $\psi(p)$ are the Dirac spinors satisfying the equations

$$\bar{\psi}(p) (\gamma p + M) = 0 \quad \text{and} \quad (\gamma p + M) \psi(p) = 0. \quad (2.17)$$

(iv) The difference between \bar{p}_μ and $\tau_i \bar{p}_\mu \tau_i$, which seems at first sight to bring an ambiguity to $\Gamma_{\mu\nu}^{(2,0,1)}$, can be shown to be ineffective in the same way.

Now we shall discuss the behavior of the renormalized propagation function for one nucleon, which is obtained from (2.11) by putting $\langle \phi_i \rangle = 0$:

* It is understood that we have taken a special gauge in which the photon field is described by transverse waves referring to the rest system of the nucleon, i.e., $\sum_\mu p_\mu A_\mu = 0$.

** If we put $\mu = 0$ in the expression for L obtained from g^2 approximation of the perturbation theory, there appears a singularity of the form $(p^2 + M^2) \log(\bar{p}^2 + M^2)$. However, it leads to no difficulty. Hence, we shall not mind the singularities of this kind throughout this paper.

$$S^{-1}(p, 0) = (\gamma p + M) \{1 - (2M)^{-1}(\gamma p + M) 2M^2(p^2 + M^2)^{-1}F(p^2 + M^2, 0) \\ + G(p^2 + M^2, 0) + 2M^2(p^2 + M^2)^{-1}F(p^2 + M^2, 0)\}. \quad (2 \cdot 18a)$$

Since $S(p, 0)$ describes an internal nucleon line, p does not satisfy the condition

$$p^2 + M^2 = 0, \quad (2 \cdot 19)$$

which would hold if p were the four-momentum of a free nucleon. Let P denote the four-momentum of a free nucleon in the initial or final state, and q the deviation of p from P . In the applications to the boson-nucleon collisions at low energy limit, q is a linear combination of the four-momenta of the external bosons of order μ' . Hence,

$$p^2 + M^2 = 2Pq + q^2 = O(M\mu') \quad (2 \cdot 20)$$

is a small quantity as compared with p^2 and M^2 . Neglecting terms of higher order in μ'/M , and remembering (2.12a), we can write (2.18a) as

$$S^{-1}(p) = (\gamma p + M)[1 + (2M)^{-1}(\gamma p + M)G(0, 0) \\ + ((p^2 + M^2)/2M^2)\{2M^2G'(0, 0) + 2M^1F''(0, 0)\}], \quad (2 \cdot 18b)$$

in which we have suppressed the second argument 0 of S^{-1} . Operating $S(P+q)$ on $\gamma_5\psi(P)$, and neglecting terms of higher order in μ'/M , we obtain

$$S(P+q)\gamma_5\psi(P) = (2M)^{-1}[1 + G(0, 0)]^{-1}\gamma_5\psi(P), \quad (2 \cdot 21)$$

which is a reproduction of the result first found by Brueckner, Gell-Mann, and Goldberger¹³⁾, and $[1 + G(0, 0)]^{-1}$ is nothing but the so-called S-wave damping factor.

Next, we shall discuss the properties of the renormalized propagation function for one pion and that for one photon. According to the prescription for renormalization, these functions have the following form

$$J_{ij}^{-1}(q) = \delta_{ij}(q^2 + \mu^2) \{1 + Q((q^2 + \mu^2)/\kappa^2)\} \quad (2 \cdot 2a)$$

and

$$D_{\mu\nu}^{-1}(k) = \delta_{\mu\nu}k^2 \{1 + R(k^2/\kappa^2)\} \quad (2 \cdot 23a)$$

where κ denotes a quantity with the dimension of mass. $Q((q^2 + \mu^2)/\kappa^2)$ and $R(k^2/\kappa^2)$ are functions which vanish at the zero of their argument, representing the reactions of the fields in interaction with them. The main reaction on the propagation of the pion wave is considered to arise from the virtual nucleon pairs. In this case κ^2 may be taken as M^2 . Therefore, $(q^2 + \mu^2)/\kappa^2$ is a small quantity of the order of $(\mu'/M)^2$ for $q \simeq \mu'$. According to the result from q^2 approximation of the perturbation theory, the reaction term of (2.22a) is estimated to be

$$-(q^2/4\pi)(11/12\pi)(q^2 + \mu^2)M^{-2} \simeq -0.1$$

neglecting the terms of the higher order in μ'/M . On the other hand, for the case of the electromagnetic reaction κ is to be taken as μ , and the argument of Q becomes of the order of unity for $q \simeq \mu'$. According to the result obtained from e^2 approximation, Q is of the order of $e^2/4\pi$ for $(q^2 + \mu^2)/\mu^2 \simeq 1$. Thus it will be a fairly good approximation

to use

$$A_{ij}(q) = \delta_{ij}(q^2 + \mu^2)^{-1} \quad (2.22b)$$

instead of (2.22a), considering that the weak coupling approximation is apt to give overestimates. Now we shall turn to the reaction on the propagation of the light wave. In this case the weak coupling approximation is reliable. The result obtained from e^2 approximation informs us that the reaction term, $R(k^2/\kappa^2)$, is of the order of $(e^2/4\pi)(\mu'/M)^2$, or $(e^2/4\pi)$, or $(e^2/4\pi)\log(\mu'/m)$ for $k \simeq \mu'$, according as it is attributed to the virtual pairs of nucleons or pions or electrons, where m is the renormalized electron mass. Thus, we may safely use

$$D_{\mu\nu}(k^2) = \delta_{\mu\nu}(k^2)^{-1} \quad (2.23b)$$

for the renormalized propagation function for one photon.

The renormalized three-vertex part describing the meson current can be written in the momentum representation as

$$\Gamma_{i\mu}^{(0,2,1)}(q, q') = \epsilon_{ij}(q_\mu + q'_\mu) \{1 + U\}, \quad (2.24a)$$

where ϵ_{ij} is defined by

$$\epsilon_{ij} = 0 \quad \text{for } i, j = 1, 2, 3 \quad \text{except} \quad \epsilon_{12} = -\epsilon_{21} = i, \quad (2.25)$$

under the convention that q and q' stand for the four-momenta of positively charged pions or minus the four-momenta of negatively charged pions. U is a function of $(q^2 + \mu^2)/M^2$, $(q'^2 + \mu^2)/M^2$ and $(q - q')^2/M^2$, which vanishes at the simultaneous zero of the three arguments in accordance with prescription for charge renormalization. The magnitude of U is expected to be of the same order as $Q((q^2 + \mu^2)/\kappa^2)$ discussed in the preceding paragraph. Therefore, it seems that it is not a bad approximation to use

$$\Gamma_{i\mu}^{(0,2,1)}(q, q') = \epsilon_{ij}(q_\mu + q'_\mu) \quad (2.24b)$$

instead of (2.24a).

Finally, let us estimate the magnitude of the vertex part, $\Gamma_{i\mu}^{(0,1,2)}$. This is nothing but the transition matrix for the two-gamma decay of a neutral pion, which was first computed by Fukuda and Miyamoto⁽¹⁰⁾ using the weak-coupling approximation. From the consideration of the Lorentz covariance and the gauge invariance it follows that this vertex part can be written in the form

$$\Gamma_{i\mu\nu}^{(0,1,2)}(k, k') = i(2\pi)^{-1} \delta_{i3} (e^2/4\pi) (g_d/g) \epsilon_{\mu\nu\rho\sigma} k_\rho k'_\sigma M^{-1} \{1 + V\}, \quad (2.26)$$

where $\epsilon_{\mu\nu\rho\sigma}$ stands for 1 if $\mu\nu\rho\sigma$ is an even permutation of 1234 or for -1 if $\mu\nu\rho\sigma$ is an odd permutation of 1234 or for zero otherwise. V denotes a function of k^2/M^2 , k'^2/M^2 and $[(k - k')^2 + \mu^2]/M^2$ which tends to zero as the three arguments simultaneously approach zero. According to the results of experimental measurement of the mean life of the neutral pion⁽¹⁴⁾, the effective pion-nucleon coupling constant, $g_d^2/4\pi$, have a small value of order 10^{-1} to 10^{-2} . It will be observed from (2.24b) and (2.26) that $\Gamma_{i\mu\nu}^{(0,1,2)}$ is smaller than $\Gamma_{i\mu}^{(0,2,1)}$ by a factor of order

$$(e^2/4\pi) (g_d/g) (\mu'/M) \simeq 10^{-4}.$$

It is to be emphasized here that, in the approximation to the lowest order in $(\mu'/M)^2$ but including the radiative corrections to an arbitrary order in the coupling constants, the renormalized propagation function for one pion, that for one photon, and the renormalized pion-current three-vertex part take the same form as those without radiative corrections. This constitutes a basis of our arguments to be done in §§ 3 and 4.

§ 2.2. Thirring's and Kroll-Ruderman's theorems and pion-nucleon scattering at low energy limit

It was first shown by Thirring¹⁾ that the rigorous expression for the matrix element for the Compton scattering by an electron, including the radiative corrections to an arbitrary order in e^2 , reduces at Thomson limit to that obtained from the second order approximation of the perturbation theory, except for the renormalization of mass and charge. Thereafter, it has been pointed out by Kroll and Ruderman and Deser et al.⁴⁾ that the same theorem holds not only for the case where the scattering center is an electron ruled by the quantum electrodynamics but also for the case where the scattering center is a proton which suffers strong reaction from the meson field. The outline of their proof is as follows. Let $A_\nu(k)$ and $A_\mu(k')$ denote the wave functions in the momentum representation for the incident and the scattered photons, respectively. Then, substituting (2.15c) and (2.18a) for $\Gamma_\mu^{(2,0,1)}$'s and S in the first two terms of (2.4) [Fig. 1a and 1b], and neglecting terms of first and higher orders in k/M and k'/M with (2.12a) in mind, we obtain

$$-[1+G(0,0)](e^2/M) \sum_{\mu=1}^3 A_\mu^*(k') A_\mu(k) \quad (2.27a)$$

as a part of the matrix element. In the derivation of this result, the proton has been treated as being initially at rest, and such a gauge has been used as makes the fourth component of A_μ vanish. Furthermore, the terms containing a factor of the form, $\sum_{\mu=1}^4 p_\mu A_\mu = \sum_{\mu=1}^3 p_\mu A_\mu$, have been discarded. This is permitted, because the space component of p_μ , being a linear combination of k and k' , is of first order in k/M . Next, substituting (2.15d) for $\Gamma_{\mu\nu}^{(2,0,2)}$ in the third term of (2.4) [Fig. 1c], we get

$$G(e^2/M) \sum_{\mu=1}^3 A_\mu^*(k') A_\mu(k) \quad (2.27b)$$

as another part of the matrix element. Adding these two parts, we have

$$T^{(2,0,2)}(k', k) = \sum_\mu A_\mu^*(k') \bar{\psi} T_{\mu\nu}^{(2,0,2)} \psi A_\nu(k) \quad (2.28a)$$

with

$$T_{\mu\nu}^{(2,0,2)} = -(e^2/M) \delta_{\mu\nu}, \quad (2.28b)$$

which yields the Thomson formula

$$\sigma(\gamma + p \rightarrow p + \gamma) = (8\pi/3) (e^2/4\pi)^2 M^{-2} \quad (2.29)$$

for the total cross section of the Compton scattering by a proton. It must be pointed out here that the last term of (2.4) [Fig. 1d] has been left out of consideration in the above proof which is a reproduction of the works of the above authors. The contribution of this term to the matrix element can be evaluated with the substitution of (2.15a), (2.22b) and (2.26) for $\Gamma_i^{(2,1,0)}$, $A_{ij}(k'-k)$, and $\Gamma_{j\mu\nu}^{(0,1,2)}(k', k)$ as follows

$$\begin{aligned}
 & - (2\pi)^{-1} (e^2/M) (gg_d/4\pi) [(k'-k)^2 + \mu^2]^{-1} \times [E(k') H(k) \\
 & + H(k') E(k)] \psi^* \tau_3 \sum_{\mu=1}^3 \sigma_\mu (k'_\mu - k_\mu) M^{-1} \psi,
 \end{aligned} \tag{2.27c}$$

where E and H denote the electric and the magnetic field strengths describing the light waves, and σ_μ ($\mu=1, 2, 3$), ψ and ψ^* stand for Pauli's spin operator and non-relativistic wave functions. (2.27c) is small by a factor

$$(2\pi)^{-1} (g^2/4\pi) (g_d/g) (k/\mu)^2 (k/M) < 10^{-1} \text{ for } k < \mu \tag{2.30}$$

as compared with (2.28). Thus, the omission of the last term of (2.4) has been justified for the case where the scattering center is a proton. On the contrary, in the case where the scattering center is a neutron, the first three terms of (2.4) vanish owing to the projection operator τ_P , and (2.27c) becomes the main term.

The derivation of the Kroll-Ruderman's theorem on the threshold photopion production will be reviewed here. The substitution of (2.15a), (2.15c) and (2.18) into the first two terms of (2.7) [Fig. 2a and 2b] yields

$$[1 - H(0, 0)] (eg/2M) \sum_{\mu=1}^3 \phi_i^*(q) \bar{\psi} \{ \tau_P \gamma_\mu, \tau_i \gamma_5 \} \psi A_\mu(k) \tag{2.31a}$$

as a part of the matrix element, while the substitution of (2.15e) for $\Gamma_{i\mu}^{(2,1,1)}$ in the third term of (2.7) [Fig. 2c] gives a compensating contribution

$$H(0, 0) (eg/2M) \sum_{\mu=1}^3 \phi_i^*(q) \bar{\psi} \gamma_5 \gamma_\mu [\tau_i, \tau_P] \psi A_\mu(k), \tag{2.31b}$$

where $\phi_i(q)$ is the wave function for the produced pion in the momentum representation, and the terms of first and higher orders in μ'/M have been neglected. The condition (2.12b) has also been used. Adding these two terms, we have the resultant matrix element of the form

$$T^{(2,1,1)}(q, k') = \phi_i^*(q) \bar{\psi} T_{i\mu}^{(2,1,1)} \psi A_\mu(k) \tag{2.32a}$$

with

$$T_{i\mu}^{(2,1,1)} = (eg/2M) \{ \tau_P \gamma_\mu, \tau_i \gamma_5 \}, \tag{2.32b}$$

from which we can easily compute the total cross section for photo-production of a charged pion from a nucleon near threshold:

$$\begin{aligned}
 \sigma(\gamma + p \rightarrow n + \pi^+) &= \sigma(\gamma + n \rightarrow p + \pi^-) \\
 &= 2\pi (e^2/4\pi) (g^2/4\pi) M^{-2} v,
 \end{aligned} \tag{2.33}$$

where v denotes the velocity of the produced pion. (2.32) is the same as the result obtained from the lowest order approximation of perturbation theory, except that e, g and M are not the unrenormalized but the renormalized ones. Kroll and Ruderman have pointed out that the contribution of the fourth term of (2.7) [Fig. 2d] vanishes at threshold because it has a factor of the form $\sum_{\mu=1}^3 q_{\mu} A_{\mu}(k)$ and the space component of q_{μ} would be zero at threshold. The threshold is defined by $q_{\mu}=0$ ($\mu=1, 2, 3$) in reference to the center-of-mass system. But, in our coordinate system in which the nucleon is initially at rest, $q_{\mu}=0$ ($\mu=1, 2, 3$) corresponds to a little above the threshold. It can be derived from the conservation law of energy-momentum of the total system and the Lorentz transformation that

$$k_0 = |\mathbf{k}| = \mu(1 + \mu/2M), \quad \mathbf{q} = \mu(\mu/M) (1 - \mu/2(M + \mu)) (\mathbf{k}/k_0) \quad (2.34)$$

at threshold. The contribution of this term is obtained by substituting (2.15a), (2.22b) and (2.24b):

$$ieg\bar{\psi}\gamma_5\tau_i\psi\phi_i^*(q) [(k-q)^2 + \mu^2]^{-1} \sum_{\mu=1}^3 (2q_{\mu} - k_{\mu}) A_{\mu}(k) \quad (2.35)$$

which vanishes at threshold, since \mathbf{q} is parallel to \mathbf{k} , and $\mathbf{k} \cdot \mathbf{A}$ vanishes because of the transverse nature of the light wave. Hence, it has been confirmed that the pion-current term gives no contribution. However, there remains the last term of (2.7) to be estimated. It is evident from the consideration of the charge conservation law that this term is irrelevant to the charged pion production but gives contribution to the neutral production. The matrix element for the latter process can be obtained by substituting (2.15c), (2.23b) and (2.26) for $\Gamma_{\lambda}^{(2,0,1)}$, $D_{\lambda\nu}(q-k)$ and $\Gamma_{i\nu\mu}^{(0,1,2)}(q-k, k)$ of the last term of (2.7) as follows:

$$i(eg/M)\bar{\psi}\tau_F\gamma_{\nu}\psi(1/2\pi)(e^2/4\pi)(g_a/g)[(q-k)^2]^{-1} \\ \times \phi_3^*(q) \epsilon_{\mu\nu\rho\sigma} k_{\rho}(q_{\sigma} - k_{\sigma}) A_{\mu}(k), \quad (2.36)$$

which is smaller by a factor of order

$$(e^2/4\pi)(g_a/g)(\mu'/M) \simeq 10^{-4}$$

as compared with (2.32). Thus, it has turned out that this term can not account for the experimental results on the photo-production of a neutral pion that the cross section for this process is the same order of magnitude as that for the charged pion production. Therefore, the explanation of the fairly large cross section for the neutral pion production will have to be sought, as is well known, in the effects related to the anomalous magnetic moment of the nucleon, which belong to the radiative correction of order μ'/M that can not be taken into account merely by the substitution (2.13).

It is essential for Thirring's theorem or Kroll-Ruderman's theorem to hold that the unestimable parameters such as $G(0, 0)$ or $H(0, 0)$ have fortunately been cancelled between (2.27a) and (2.27b) or between (2.31a) and (2.31b). However, such a cancellation does not occur in the case of pion-nucleon scattering even at low energy limit. In fact, substituting (2.15a), (2.15b) and (2.18a) for $\Gamma_i^{(2,1,0)}$'s, $\Gamma_{ij}^{(2,2,0)}$ and S in the three terms

other than the last of (2.9), we obtain

$$T^{(2,2,0)}(q', q) = \phi_i^*(q') \bar{\psi} T_{ij}^{(2,2,0)} \psi \phi_j(q) \quad (2.37a)$$

with

$$T_{ij}^{(2,2,0)} = -(g_s^2/M) \delta_{ij} \quad (2.37b)$$

and

$$g_s^2 \equiv g^2 \{ [1 - H(0, 0)]^2 [1 + G(0, 0)]^{-1} + 2M^2 \dot{F}(0, 0) \} \quad (2.38)$$

for the transition matrix element to the lowest order in μ'/M , where q and q' denote the initial and the final four-momenta of the pion. The parameter, g_s , defined by (2.38) will be referred to in this paper as the effective coupling constant for pion-nucleon scattering. Its value can be determined by equating the cross section

$$\sigma(\pi^\pm + p \rightarrow p + \pi^\pm) = (g_s^2/4\pi)^2 (4\pi/M^2) \quad (2.39)$$

to the experimental result. The value cited by Deser et al.¹⁾ as determined from the experimental data of Fermi and Steinberger is

$$g_s^2/4\pi\hbar c \simeq 0.36 \text{ to } 0.37. \quad (2.40)$$

The last term of (2.9) is trivial, since it yields merely the Rutherford formula for the Coulomb scattering of a charged pion by a proton.

§ 2.3. Four boson processes

In this subsection we shall discuss the behavior at low energy limit of the S -matrices for those processes which are represented by Feynman diagrams with four external boson lines. These processes occur as virtual ones in those processes to be studied in the succeeding sections in which two or three bosons are simultaneously produced from a target nucleon bombarded by a boson with an energy near threshold.

First, let us discuss the Møller scattering between two pions. The matrix element can be written as

$$T^{(0,4,0)} = \phi_i(\zeta) \phi_j(\zeta') \phi_k(\zeta'') \phi_l(\zeta''') T_{ijkl}^{(0,4,0)}(\zeta, \zeta', \zeta'', \zeta''')$$

with

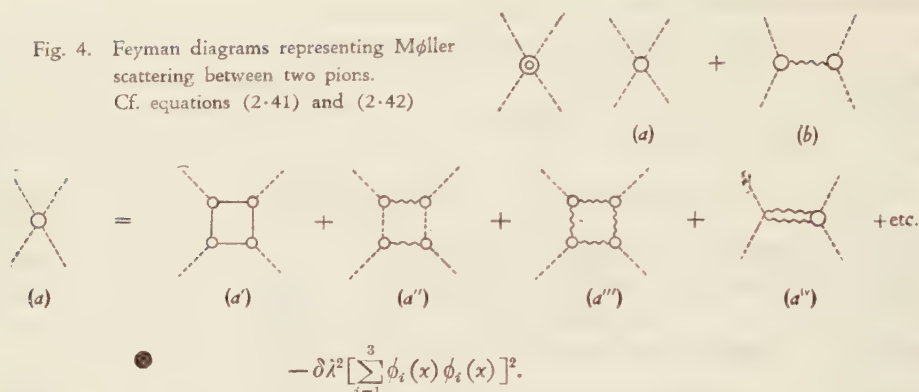
$$\begin{aligned} T_{ijkl}^{(0,4,0)}(\zeta, \zeta', \zeta'', \zeta''') &= g^2 \Gamma_{ijkl}^{(0,4,0)}(\zeta, \zeta', \zeta'', \zeta''') \\ &+ e^2 (2!)^{-3} \sum_{\text{perm}} \Gamma_{ij\mu}^{(0,2,1)}(\zeta, \zeta', \hat{\zeta}) D_{\mu\nu}(\hat{\zeta}, \hat{\zeta}') \Gamma_{kl\nu}^{(0,2,1)}(\zeta'', \zeta''', \hat{\zeta}') \end{aligned} \quad (2.41)$$

and

$$\Gamma_{ijkl}^{(0,4,0)}(\zeta, \zeta', \zeta'', \zeta''') \equiv -\delta^2 A_{ij}^{-1}(\zeta, \zeta') / \partial \langle g \phi_k(\zeta'') \rangle \partial \langle g \phi_l(\zeta''') \rangle, \quad (2.42)$$

where \sum_{perm} stands for summation over $4!$ permutations of $i\zeta$, $j\zeta'$, $k\zeta''$ and $l\zeta'''$. These expressions may conveniently be illustrated by Fig. 4. Without renormalization, the vertex part defined by (2.42) would be primitive divergent, since the skeletons constructing its graphs are square diagrams such as Figs. 4a', a'' and a''' or a closed photon loop like Fig. 4a^{iv}. According to Matthews⁷⁾, this divergence can be removed by starting from the Lagrangian provided with a counter-term of the form

Fig. 4. Feynman diagrams representing Møller scattering between two pions.
Cf. equations (2.41) and (2.42)



If we assume the charge independence as a good approximation, $\Gamma_{ijkl}^{(0,4,0)}$ thus renormalized may be written in the q representation as

$$\Gamma_{ijkl}^{(0,4,0)}(q, q', q'') = (\lambda^2/g^2) (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) (1+W), \quad (2.43)$$

where W is a function of four four-momenta of the external pion lines, and λ^2 is a coupling constant to be determined from experiment. For q 's of the order μ' , W may be neglected, because its magnitude is expected to be of the same order as that of $Q((q^2 + \mu^2)/\pi^2)$ discussed before. The value of λ^2 is as yet unknown. According to Ito and Minami¹⁵⁾, however, $\lambda^2/4\pi$ must be taken to be of the order of unity, if $\Gamma_{ijkl}^{(0,4,0)}$ should play a principal role in accounting for the experimental results obtained by Fowler et al.¹⁶⁾ and by Walker et al.¹⁷⁾ on the double pion production from a nucleon bombarded by 1.4 Bev pion beams. Therefore, throughout this paper, we shall assume that the magnitude of $\lambda^2/4\pi$ is of the order of unity. The second term of (2.41) [Fig. 4b], combined with those parts of the first term which are represented by Figs. 4a'', 4a''' and 4a'', describes a scattering through electromagnetic forces. Its magnitude can easily be estimated to be of the order of e^2 from (2.23b) and (2.24b). The ratio of the two terms of (2.41) being $\lambda^2:e^2 \simeq 10^2:1$, the second term may safely be omitted. Hence, in an admissible approximation, we may use

$$\begin{aligned} T_{ijkl}^{(0,4,0)}(q, q', q'') &= g^2 \Gamma_{ijkl}^{(0,1,0)}(q, q', q'') \\ &= \lambda^2 (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \quad (2.44)$$

for our later purposes.



Fig. 5. Feynman diagrams representing the process (0, 3, 1). Cf. (2.45).

Next, we shall consider a process which is represented by connected Feynman diagrams with three external pion lines and one external photon line, as shown in Fig. 5. The matrix element for this process can be written as

$$T^{(0,3,1)} = \phi_i(\zeta) \phi_j(\zeta') \phi_k(\zeta'') A_\mu(\hat{\zeta}) T_{ijk\mu}^{(0,3,1)}(\zeta, \zeta', \zeta'', \hat{\zeta})$$

with

$$\begin{aligned} T_{ijk\mu}^{(0,3,1)}(\zeta, \zeta', \zeta'', \hat{\zeta}) = & eg \Gamma_{ijk\mu}^{(0,3,1)}(\zeta, \zeta', \zeta'', \hat{\zeta}) \\ & + eg \sum_{\text{cycl}} \Gamma_{i\mu\nu}^{(0,1,2)}(\zeta, \hat{\zeta}, \xi') D_{\nu\lambda}(\hat{\zeta}', \xi'') \Gamma_{jk\lambda}^{(0,2,1)}(\zeta', \zeta'', \hat{\zeta}'') \end{aligned} \quad (2.45)$$

and

$$\Gamma_{ijk\mu}^{(0,3,1)} = -\delta^2 \Delta_{ij}^{-1}(\zeta, \zeta') / \partial \langle g \phi_k(\zeta'') \rangle \partial \langle e A_\mu(\hat{\zeta}) \rangle,$$

where \sum_{cycl} stands for a summation over cyclic permutations of $i\zeta$, $j\zeta'$ and $k\zeta''$. From the requirement of Lorentz covariance, the Fourier transform of $T_{ijk\mu}^{(0,3,1)}$ must have the form

$$T_{ijk\mu}^{(0,3,1)}(q, q', q'') = eg X_{ijk} M^{-3} q_\lambda q'_\nu q''_\rho \epsilon_{\lambda\nu\rho\mu}, \quad (2.46)$$

where X_{ijk} is a dimensionless scalar function of q , q' and q'' . The contribution of the second term of (2.45) to X_{ijk} can be evaluated with the use of (2.23b), (2.24b) and (2.26) as

$$X_{ijk}(b) \simeq (e^2/4\pi) (g_a/g) (\mu'/M)^{-2}.$$

On the other hand, from the considerations based on the weak coupling approximation, the contribution of $\Gamma_{ijk\mu}^{(0,3,1)}$ is expected to be of the order

$$X_{ijk}(a') \simeq (g^2/4\pi)$$

if it corresponds to Fig. 5a', or

$$X_{ijk}(a'') \simeq (e^2/4\pi)^2 (g_a/g)^2 (e/g) (\mu'/M)^{-1}$$

if it corresponds to Fig. 5a''. It is seen that the main contribution comes from the closed nucleon loop, Fig. 5a', as expected. Thus, the magnitude of $T_{ijk\mu}^{(0,3,1)}$ may be regarded to be of the order

$$T_{ijk\mu}^{(0,3,1)} \simeq eg (g^2/4\pi) (\mu'/M)^3 \quad (2.47)$$

which is smaller than $T_{ijk\mu}^{(0,1,0)}$ given by (2.44) by a factor of the order

$$(eg/\lambda^2) (g^2/4\pi) (\mu'/M)^3 \simeq 0.02.$$

Now we shall discuss the Compton scattering by a charged pion. Let the matrix element be written as

$$T^{(0,2,2)} = \phi_i(\zeta) A_\mu(\hat{\zeta}) T_{ij\mu\nu}^{(0,2,2)}(\zeta, \zeta', \xi, \xi') \phi(\zeta') A_\nu(\hat{\zeta}').$$

Then, taking into account that Thirring's theorem should hold also for this case, we can write the Fourier transform of $T_{ij\mu\nu}^{(0,2,2)}$ in the form

$$T_{ij\mu\nu}^{(0,2,2)}(q, k, k') = e^2 \partial_{ij} (1 - \delta_{i3}) \{ \delta_{\mu\nu} + Y_{\mu\nu} \}, \quad (2.48)$$

where k and k' denote the four-momenta of photons, and q and $q' = q + k - k'$ those of the pions. $Y_{\mu\nu}$ is a tensor function of q/μ , k/μ and k'/μ , which vanishes for $k = k' = 0$. For k 's and q 's of the order of μ , $Y_{\mu\nu}$ must be considered to be of the order of unity. If an analogy to the Compton scattering by an electron is allowed, $Y_{\mu\nu}$ is expected to vary in the direction to cancel $\partial_{\mu\nu}$ as k 's and q 's approaches μ' . Thus, it will be admitted to assume in the succeeding sections that $T_{ij\mu\nu}^{(0,2,2)}(q, k, k')$ is at most of the order of $e^2/4\pi$.

§ 3. $\pi + N \rightarrow N + 2\pi$ and $\gamma + N \rightarrow N + 2\pi$

The results obtained in the preceding sections will be applied to those processes which are represented by connected Feynman diagrams with two external nucleon lines and three external boson lines. Of these processes the most interesting is the inelastic pion-nucleon scattering in which an extra pion is produced. This process has recently been observed by Fowler et al.¹⁶⁾ and Walker et al.¹⁷⁾ in their experiments on the 1.4 BeV pion-nucleon collision.

§ 3.1. $\pi + N \rightarrow N + 2\pi$

The matrix element for this process is given by

$$T^{(2,3,0)} = \phi_i(\zeta) \phi_j(\zeta') \bar{\psi} T_{ijk}^{(2,3,0)}(\zeta, \zeta', \zeta'') \psi \phi_k(\zeta'') \quad (3.1)$$

with

$$T_{ijk}^{(2,3,0)}(\zeta, \zeta', \zeta'') = \mathcal{A}_{il}^{-1}(\zeta, \zeta''') \mathcal{A}_{jm}^{-1}(\zeta', \zeta''') \mathcal{A}_{kn}^{-1}(\zeta'', \zeta''') \times S^{-1} \delta^3 S / \delta K_l(\zeta''') \delta K_m(\zeta''') \delta K_n(\zeta''') \cdot S^{-1} \quad (3.2)$$

$$= g^2 (1/2!) \sum \{ \Gamma_{jk}^{(2,2,0)}(\zeta', \zeta) S \Gamma_i^{(2,1,0)}(\zeta) + \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_{jk}^{(2,2,0)}(\zeta', \zeta'') \} \quad (3.3a)$$

$$+ g^3 \sum \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_j^{(2,1,0)}(\zeta') S \Gamma_k^{(2,1,0)}(\zeta'') \quad (3.3b)$$

$$+ g^3 \Gamma_{ijk}^{(2,3,0)}(\zeta, \zeta', \zeta'') \quad (3.3c)$$

$$+ e (1/2!) \sum T'_{i\mu}^{(2,1,1)}(\zeta, \xi) D_{\mu\nu}(\xi, \xi') \Gamma_{jk\nu}^{(0,2,1)}(\zeta', \zeta'', \xi') \quad (3.3d)$$

$$+ e \Gamma_{\mu}^{(2,0,1)}(\xi) D_{\mu\nu}(\xi, \xi') T_{ijk\nu}^{(0,3,1)}(\zeta, \zeta', \zeta'', \xi') \quad (3.3e)$$

$$+ g \Gamma_m^{(2,1,0)}(\zeta''') \mathcal{A}_{ml}(\zeta''', \zeta''') T_{ijk}^{(0,4,0)}(\zeta, \zeta', \zeta'', \zeta''') \quad (3.3f)$$

and

$$\Gamma_{ijk}^{(2,3,0)}(\zeta, \zeta', \zeta'') = -\delta^3 S^{-1} / \delta \langle g \phi_i(\zeta) \rangle \delta \langle g \phi_j(\zeta') \rangle \delta \langle g \phi_k(\zeta'') \rangle, \quad (3.4)$$

where \sum stands for the summation over $3!$ permutations of $(i\zeta, j\zeta', k\zeta'')$, and the prime attached to $T'_{i\mu}^{(2,1,1)}$ of (3.3d) denotes to insert (2.7) with the last two terms omitted, which have already been included in (3.3e) and (3.3f). The graphical representation of (3.3a-f) is given in Fig. 6.

The insertion of (2.15a), (2.15b) and (2.18b) into the first term [Fig. 6a'] within the brackets of (3.3a) yields

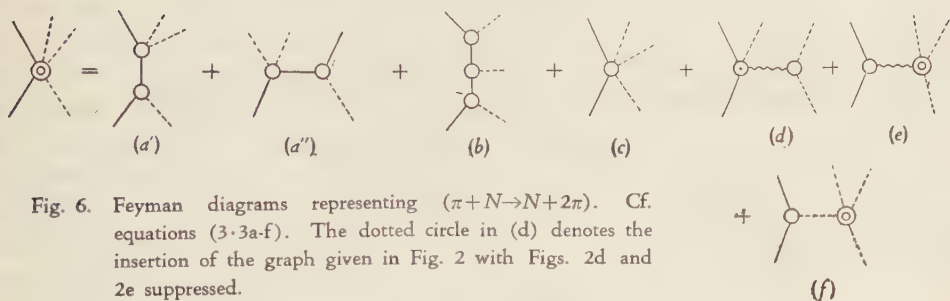


Fig. 6. Feynman diagrams representing $(\pi + N \rightarrow N + 2\pi)$. Cf. equations (3.3a-f). The dotted circle in (d) denotes the insertion of the graph given in Fig. 2 with Figs. 2d and 2e suppressed.

$$\begin{aligned} & g^3 \bar{\psi}(P+q+q'+q'') \Gamma_{jk}^{(2,2,0)} S(P+q) \Gamma_i^{(2,1,0)} \psi(P) \\ &= -g^3 \bar{\psi}(P+q+q'+q'') (\gamma q/q_0) \gamma_5 \tau_i \delta_{jk} \psi(P) \\ & \times [1-H(0,0)][1+G(0,0)^{-1}] \dot{F}(0,0) \end{aligned} \quad (3.5a')$$

to zeroth order in μ'/M , where q , q' and q'' denote the four-momenta carried by the external pion lines entering the graph with the suffixes, i , j and k , respectively, and q_0 the time-component of q . In the same way, the second term [Fig. 6a''] within the brackets of (3.3a) yields

$$\begin{aligned} & g^3 \bar{\psi}(P+q+q'+q'') \Gamma_i^{(2,1,0)} S(P+q'+q'') \Gamma_{jk}^{(2,2,0)} \psi(P) \\ &= -g^3 \bar{\psi}(P+q+q'+q'') \gamma_5 \tau_i \delta_{jk} (\gamma q/q_0) \psi(P) \\ & \times [1-H(0,0)][1+G(0,0)^{-1}] \dot{F}(0,0). \end{aligned} \quad (3.5a'')$$

Each of (3.5a') and (3.5a'') is of zeroth order in μ'/M . However, when added, they cancel each other. Hence, (3.3a) is considered to be of the order

$$(g^3/M^2) (\mu'/M) \quad (3.6a)$$

apart from a factor containing the parameters, $G(0,0)$, $H(0,0)$ and $M^2 \dot{F}(0,0)$.

Inserting (2.15a) and (2.18b) into a summand of (3.3b) [Fig. 6b], we obtain

$$\begin{aligned} & g^3 \bar{\psi}(P+q+q'+q'') \Gamma_i S(P+q'+q'') \Gamma_j S(P+q'') \Gamma_k \psi(P) \\ &= (g^3/4M^2) \bar{\psi}(P+q+q'+q'') \tau_i \tau_j \tau_k \{ -(\gamma q) q_0^{-1} \gamma_5 (\gamma q'') q_0''^{-1} \\ & \quad + \gamma_5 [(\gamma q) q_0^{-1} - (\gamma q'') q_0''^{-1}] H(0,0) \} \psi(P) \\ & \times [1-H(0,0)]^2 [1+G(0,0)]^{-2} \end{aligned} \quad (3.7)$$

to zeroth order in μ'/M . Performing the summation over 3! permutations of (iq, jq', kq'') , we find that the terms of zeroth order in μ'/M cancel each other, taking account of the relation

$$q_0 + q'_0 + q''_0 = \frac{3}{2} \mu (\mu/M) [1 + 2(\mu/M)]^{-1}, \quad (3.8)$$

which follows from the energy-momentum conservation law. Consequently, (3.3b) will be of the order

$$(g^3/M^2) (\mu'/M) \quad (3.6b)$$

apart from a factor containing $H(0, 0)$ and $G(0, 0)$.

(3.3c) [Fig. 6c] can be computed from (2.11) and (3.4) as follows:

$$\begin{aligned} g^3 \bar{\psi}(P+q+q'+q'') I_{ij}^{(2,3,0)} \psi(P) &= g^3 \bar{\psi}(P+q+q'+q'') \\ &\times [\partial_{ij} \tau_k + \partial_{ki} \tau_j + \partial_{jk} \tau_i] \gamma_5 \psi(P) \times 2\dot{D}(0, 0), \end{aligned} \quad (3.6c')$$

which is of the order

$$(g^3/M^2) (\mu'/M) \quad (3.6c'')$$

apart from a factor containing $M^2 \dot{D}(0, 0)$.

(3.3d) [Fig. 6d] can be estimated from (2.33b), (2.24b) and (2.32) to be of the order

$$(g^3/M^2) (e/g)^2 (\mu'/M)^{-1}. \quad (3.6d)$$

Similarly, using (2.15c), (2.23b) and (2.47), we can evaluate (3.3e) [Fig. 6e]. The result is of the order

$$(g^3/M^2) (e^2/4\pi) (\mu'/M)^2. \quad (3.6e)$$

Both (3.6d) and (3.6e) are negligibly small as compared with (3.6a), (3.6b) and (3.6c'') owing to the small factors $(e/g)^2 \simeq 10^{-3}$ and $e^2/4\pi \simeq 10^{-2}$, as expected, because (3.3d) and (3.3e) are electromagnetic corrections, which should have been omitted now that we have assumed the charge-independence as a good approximation.

Finally, inserting (2.15a), (2.22b) and (2.44), we obtain the following result for (3.3f) [Fig. 6f]:

$$\begin{aligned} g \bar{\psi}(P+q+q'+q'') I_m^{(2,1,0)} \psi(P) \mathcal{A}_{mt}(q+q'+q'') T_{ijkt}^{(0,4,0)}(q, q', q'') \\ = (g\lambda)^2 [(q+q'+q'')^2 + t^2]^{-1} \\ \times \bar{\psi}(P+q+q'+q'') [\partial_{ij} \tau_k + \partial_{jk} \tau_i + \partial_{ki} \tau_j] \gamma_5 \psi(P), \end{aligned} \quad (3.6f')$$

which is of the order

$$(g^3/M^2) (\lambda/g)^2 (\mu'/M)^{-1}. \quad (3.6f'')$$

The parameters, $G(0, 0)$, $H(0, 0)$, $M^2 \dot{D}(0, 0)$ and $M^2 \dot{F}(0, 0)$, appearing in (3.5a'), (3.5a''), (3.7) and (3.6c'), are estimated to be of the order of unity from the considerations based on the lowest order approximation of the perturbation theory. Though this approximation may be unreliable, we shall assume yet that these parameters are of the order of unity. Then the ratios of (3.6a), (3.6b) and (3.6c'') to (3.6f'') are of the order

$$(g/\lambda)^2 (\mu'/M)^2 \simeq 0.1 (g/\lambda)^2 \quad (3.9)$$

for

$$(\mu'/M) = 2 (\mu/M) \simeq 0.3. \quad (3.10)$$

Thus, it seems probable that, near threshold, (3.3f) [Fig. 6f] is predominant over the other terms, (3.3a) to (3.3e), if $g^2 \simeq \lambda^2$. This result will be interesting in connection with the results of recent analysis performed by Ito and Minami¹⁶⁾, according to whom, the observed angular correlations between two outgoing pions and the total cross sections can be accounted for by assuming phenomenologically that the Feynman graph [Fig. 6f] gives dominant contribution to the matrix element, and the coupling constant, $\lambda^2/4\pi$, is of the order of unity. It is to be emphasized that the matrix element given by (3.6f') contains no unknown parameters other than λ^2 . From (3.6f') we obtain the following results for the total cross section near threshold

$$\begin{aligned}
 (1/9) \sigma(\pi^0 + p \rightarrow p + 2\pi^0) &= (1/8) \sigma(\pi^+ + p \rightarrow n + 2\pi^+) \\
 &= (1/8) \sigma(\pi^- + p \rightarrow n + \pi^+ + \pi^-) = (1/2) \sigma(\pi^0 + p \rightarrow n + \pi^+ + \pi^0) \\
 &= (1/2) \sigma(\pi^- + p \rightarrow n + 2\pi^0) = \sigma(\pi^0 + p \rightarrow p + \pi^+ + \pi^-) \\
 &= \sigma(\pi^+ + p \rightarrow p + \pi^+ + \pi^0) = \sigma(\pi^- + p \rightarrow p + \pi^- + \pi^0) \\
 &= \frac{\sqrt{3}}{64} \pi \left(\frac{g^2}{4\pi} \right) \left(\frac{\lambda^2}{4\pi} \right)^2 \frac{1}{M^2} \left(\frac{\omega_0 - \omega_{\text{thr}}}{\mu} \right)^2
 \end{aligned} \tag{3.11}$$

for

$$\omega_{\text{thr}} \leq \omega_0 \leq 2\mu(1 + 0(\mu'/M)) \tag{3.12a}$$

with

$$\omega_{\text{thr}} = 2\mu \left(1 + \frac{3\mu}{4M} \right), \tag{3.12b}$$

where ω_{thr} and ω_0 are the threshold energy and the energy of the incident pion inclusive of the rest energy in the laboratory system.

§ 3.2. $\gamma + N \rightarrow N + 2\pi$

The matrix element for this process is given by

$$T^{(2,2,1)} = \sum_{\mu=1}^3 \phi_i(\zeta) \phi_j(\zeta') \bar{\phi} T_{ij\mu}^{(2,2,1)}(\zeta, \zeta', \xi) \phi A_\mu(\xi) \tag{3.13}$$

with

$$\begin{aligned}
 T_{ij\mu}^{(2,2,1)}(\zeta, \zeta', \xi) &\equiv A_{ik}^{-1}(\zeta, \zeta'') A_{jl}^{-1}(\zeta', \zeta''') D_{\mu\nu}^{-1}(\xi, \xi') \\
 &\times S^{-1} \delta^3 S / \delta K_k(\zeta'') \delta K_l(\zeta''') \delta J_\nu(\xi') \cdot S^{-1}
 \end{aligned} \tag{3.14}$$

$$\begin{aligned}
 &= e g^2 \sum \{ \Gamma_\mu^{(2,0,1)}(\xi) S \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_j^{(2,1,0)}(\zeta') + \Gamma_j^{(2,1,0)}(\zeta') S \Gamma_i^{(2,1,0)}(\zeta) \\
 &\times S \Gamma_\mu^{(2,0,1)}(\xi) + \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_\mu^{(2,0,1)}(\xi) S \Gamma_j^{(2,1,0)}(\zeta') \}
 \end{aligned} \tag{3.15a}$$

$$+ e g^2 \{ \Gamma_\mu^{(2,0,1)}(\xi) S \Gamma_{ij}^{(2,2,0)}(\zeta, \zeta') + \Gamma_{ij}^{(2,2,0)}(\zeta, \zeta') S \Gamma_\mu^{(2,0,1)}(\xi) \} \tag{3.15b}$$

$$+ e g^2 \sum \{ \Gamma_i^{(2,1,0)}(\zeta) S \Gamma_{j\mu}^{(2,1,1)}(\zeta', \xi) + \Gamma_{j\mu}^{(2,1,1)}(\zeta', \xi) S \Gamma_i^{(2,1,0)}(\zeta) \} \tag{3.15c}$$

$$+ e g^2 \Gamma_{ij\mu}^{(2,2,1)}(\zeta, \zeta', \xi) \tag{3.15d}$$

$$+g\sum T'_{i\lambda}{}^{(2,1,1)}(\zeta, \xi'')\dot{D}_{\lambda\nu}(\xi'', \xi')\Gamma_{j\nu\mu}^{(0,1,2)}(\zeta', \xi', \xi) \quad (3.15e)$$

$$+eT'_{\mu\nu}{}^{(2,0,2)}(\xi, \xi')D_{\nu\lambda}(\xi', \xi'')\Gamma_{ij\lambda}^{(0,2,1)}(\zeta, \zeta', \xi'') \quad (3.15f)$$

$$+e\Gamma_{\lambda}^{(2,0,1)}(\xi'')D_{\lambda\nu}(\xi'', \xi')T'_{ij\nu\mu}{}^{(0,2,2)}(\zeta, \zeta', \xi', \xi) \quad (3.15g)$$

$$+e\sum T'_{ik}{}^{(2,2,0)}(\zeta, \zeta'')\Delta_{kl}(\zeta'', \zeta''')\Gamma_{lj\mu}^{(0,2,1)}(\zeta''', \zeta', \xi) \quad (3.15h)$$

$$+g\Gamma_i^{(2,1,0)}(\zeta''')\Delta_{ik}(\zeta''', \zeta'')T'_{kj\mu}{}^{(0,3,1)}(\zeta'', \zeta', \zeta, \xi) \quad (3.15i)$$

and

$$\Gamma_{ij\mu}^{(2,2,1)}(\zeta, \zeta', \xi) \equiv -\delta^3 S^{-1}/\partial\langle g\phi_i(\zeta)\rangle\partial\langle g\phi_j(\zeta')\rangle\partial\langle eA_\mu(\xi)\rangle, \quad (3.16)$$

where $T'_{\mu\nu}{}^{(2,0,2)}$, $T'_{ij}{}^{(2,1,1)}$ and $T'_{ij}{}^{(2,2,0)}$ are defined by (2.4), (2.7) and (2.9) with those parts omitted, which contain either J_i or $D_{\lambda\nu}$, and \sum stands for the summation over 2! transpositions of $(i\zeta, j\zeta')$. The graphical representations for the above expressions are given in Fig. 7.

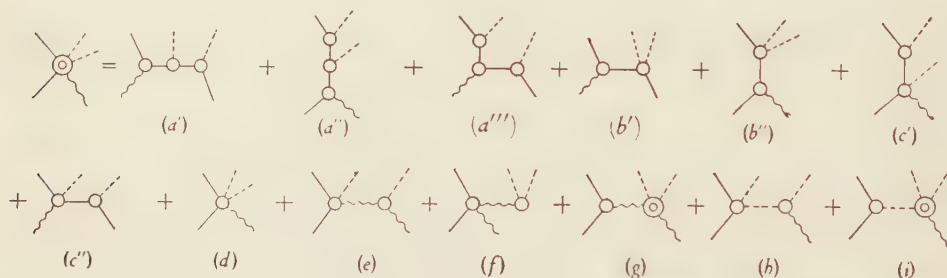


Fig. 7. Feynman diagrams representing the matrix element for $(\gamma + N \rightarrow N + 2\pi)$. Cf. equations (3.15a-i).

Let $k_\mu = (k, ik_0)$ denote the four-momentum of the incident photon, and $q_\mu = (q, iq_0)$ and $q'_\mu = (q', iq'_0)$ those of the two produced pions. In our reference system in which the nucleon is initially at rest, the threshold is defined by

$$k_0 = |\mathbf{k}| = 2\mu[1 + (\mu/M)],$$

$$q_0 = q'_0 = \mu[1 + 2\mu^2 M^{-1}(M + 2\mu)^{-1}]$$

and

$$\mathbf{q} = \mathbf{q}' = (\mu/M)[1 + 2(\mu/M)]^{-1}\mathbf{k}. \quad (3.17)$$

Inserting (2.15a-e) and (2.18b), and remembering

$$A_i = iA_0 = 0, \quad \text{and} \quad \sum_{\mu=1}^4 k_\mu A_\mu = 0, \quad (3.18)$$

we find that the contributions of (3.15a-c) [Figs. 7a'-c''] to the matrix element are all of the order

$$(eg^2/M^2)(\mu'/M) \quad (3.19a)$$

apart from factors containing the parameters, $G(0, 0)$, $H(0, 0)$ and $M^2\dot{F}(0, 0)$. Similarly, from the substitution of (2.11) into (3.16) it follows that (3.15d) [Fig. 7d] is of the same order as (3.19a) apart from a factor involving $M^2\dot{G}(0, 0)$.

The contributions of (3.15e) [Fig. 7e] and (3.15f) [Fig. 7f] to the matrix element can be evaluated to a good approximation by substituting (2.23b) for $D_{\nu\lambda}$, (2.26) for $I_{j\nu\mu}^{(0,1,2)}$, (2.24b) for $I_{ij\mu}^{(0,2,1)}$, and inserting (2.28b) and (2.32b). The results are given in terms of Fourier transforms by

$$(3.15e) = (eg^2/M^2) \gamma_\nu \gamma_5 (i/2) \{ [\tau_P, \tau_j] \delta_{i3} + [\tau_P, \tau_i] \delta_{j3} \} \\ \times (2\pi)^{-1} (e^2/4\pi) (g_d/g) \epsilon_{\mu\nu\rho\sigma} (q_\rho k_\sigma / 3\mu^2) \quad (3.20)$$

$$\simeq (eg^2/M^2) (e^2/4\pi) (g_d/g) \quad (3.19e)$$

and

$$(3.15f) = (eg^2/M^2) (e/g)^2 (\mu/M)^{-1} \epsilon_{ij} (q_\mu - q'_\mu) (4\mu)^{-1} \quad (3.21)$$

which vanishes at threshold where $q_\mu = q'_\mu$. Next, substituting (2.15c) for $I_{\lambda}^{(2,0,1)}$, (2.23b) for $D_{\lambda\lambda}$, and (2.48) for $T_{ij\nu\mu}^{(0,2,2)}$, we obtain the result that the Fourier transform of (3.15g) [Fig. 7g] is of the order

$$(eg^2/M^2) (e/g)^2 (\mu'/M)^{-1} \quad (3.19g)$$

remembering (3.18). It is observed that (3.19e) and (3.19g) are negligibly small as compared with (3.19a), owing to the small coupling coefficients, $(e^2/4\pi) (g_d/g) \simeq 10^{-3}$ and $(e/g)^2 \simeq 10^{-3}$, in spite of the fact that they are of lower order with respect to μ'/M . Here, it is to be noticed that (3.15e) [Fig. 7e], (3.15f) [Fig. 7f] and (3.15g) [Fig. 7g] have $T^{(2,1,1)}$ or $I^{(2,1,1)}$ and $T^{(0,1,1)}$ or $I^{(0,1,1)}$ linked with each other by a single photon-propagation function, which we shall call a "single photon-bridge". The Feynman graphs containing single photon-bridges, in general, yield small electromagnetic corrections, which may be neglected as compared with mesonic effects, unless the latter effects are non-existent from the beginning or they happen to be extraordinarily damped by accidental cancellations. Hence, such graphs will be left out of consideration in the arguments to be done in later sections.

(3.15h) [Fig. 7h] can be evaluated by substituting (2.22b) for Δ_{kl} , (2.24b) for $I_{ij\mu}^{(0,2,1)}$, and (2.37b) :

$$(3.15h) = - (eg^2/M^2) (g_s/g)^2 (M/4\mu^2) \epsilon_{ij} (2q'_\mu - k_\mu).$$

This contributes nothing to the matrix element at threshold, since q' is parallel to k , and kA vanishes because of the transverse nature of the light wave, (3.18).

Finally, inserting (2.15a), (2.22b) and (2.47), we find that (3.15i) [Fig. 7i] has the form

$$\partial_{ij} (eg^2/M^2) (g^2/4\pi) M^{-1} \gamma_5 [(k - q - q')^2 + \mu^2]^{-1} \epsilon_{\mu\nu\rho\sigma} k_\nu q_\rho q'_\sigma \quad (3.22)$$

which vanishes at threshold, because $q_\nu q'_\sigma = q_\nu q_\sigma$ is symmetric while $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric with respect to the interchange of the two suffixes, ρ and σ .

After all, equations (3.15a) to (3.15d) are expected to give main contribution to the matrix element at threshold, unless such a strong damping effect happens to occur as to reduce their resultant contribution far below (3.19e) or (3.19g). However, the expression thus obtained for the matrix element from equations (3.15a) to (3.15d) can not be used for comparison with experiment, because it not only involves unestimated parameters like $G(0,0)$, $H(0,0)$, $M^2\dot{F}(0,0)$, $M^2\dot{G}(0,0)$, etc., but also resorts to that Green's function, (2.11), which ignores terms of first order in μ'/M .

The processes, such as $(\gamma + N \rightarrow N + \pi + \gamma)$ (inner bremsstrahlung accompanying the photo-pion production from a nucleon) and $(\gamma + N \rightarrow N + 2\gamma)$, can be treated in the same way, and similar results to the above are attained.

§ 4. $\pi + N \rightarrow N + 3\pi$ and $\gamma + N \rightarrow N + 3\pi$

In this section, we shall discuss the low energy limits of those processes which are represented by Feynman graphs with two external nucleon lines and four external boson lines, especially, $(\pi + N \rightarrow N + 3\pi)$ and $(\gamma + N \rightarrow N + 3\pi)$. We shall not enter into details of the other processes, such as $(\gamma + N \rightarrow N + 2\pi + \gamma)$, $(\gamma + N \rightarrow N + \pi + 2\gamma)$ and $(\gamma + N \rightarrow N + 3\gamma)$, because the expressions obtained for their matrix elements are not available for the comparison of the theory with the experiment under the same circumstances as has been met with in § 3.2, and, furthermore, it seems to be hard to detect them experimentally.

§ 4.1. $\pi + N \rightarrow N + 3\pi$

The matrix element for this process is given by

$$T^{(2,4,0)} = \phi_i(\zeta) \phi_j(\zeta') \phi_k(\zeta'') \bar{\phi} T_{ijkl}^{(2,4,0)}(\zeta, \zeta', \zeta'', \zeta''') \phi_l(\zeta''') \quad (4.1)$$

with

$$\begin{aligned} T_{ijkl}^{(2,4,0)}(\zeta, \zeta', \zeta'', \zeta''') &= \Delta_{im}^{-1}(\zeta, \zeta^{iv}) \Delta_{jn}^{-1}(\zeta', \zeta^v) \Delta_{kp}^{-1}(\zeta'', \zeta^{vi}) \Delta_{lq}^{-1}(\zeta''', \zeta^{vii}) \\ &\times S^{-1} \delta^4 S / \delta K_m(\zeta^{iv}) \delta K_n(\zeta^v) \delta K_p(\zeta^{vi}) \delta K_q(\zeta^{vii}) S^{-1}. \end{aligned} \quad (4.2)$$

Omitting those terms which involve single photon-bridges, we have

$$\begin{aligned} T_{ijkl}^{(2,4,0)}(\zeta, \zeta', \zeta'', \zeta''') &= g^4 S^{-1} \delta^4 S / \delta \langle g \phi_i(\zeta) \rangle \delta \langle g \phi_j(\zeta') \rangle \delta \langle g \phi_k(\zeta'') \rangle \delta \langle g \phi_l(\zeta''') \rangle \cdot S^{-1} \\ &+ \sum T_{im}^{\prime(2,2,0)}(\zeta, \zeta^{iv}) \Delta_{mn}(\zeta^{iv}, \zeta^v) T_{njkl}^{(0,4,0)}(\zeta^v, \zeta', \zeta'', \zeta'''), \end{aligned} \quad (4.3a)$$

$$+ \sum T_{im}^{\prime(2,2,0)}(\zeta, \zeta^{iv}) \Delta_{mn}(\zeta^{iv}, \zeta^v) T_{njkl}^{(0,4,0)}(\zeta^v, \zeta', \zeta'', \zeta'''), \quad (4.3b)$$

where \sum stands for the summation over 4 cyclic permutations of (i, j, k, l) . The graphical representation is given in Fig. 8. Figs. 8a₁ to 8a₆ correspond to eq. (4.3a) while Fig. 8b to eq. (4.3b).

In the same way as we have derived (3.6a), (3.6b) and (3.6c) in the preceding sections, we can show that (4.3a) is of the order

$$(g^4/M^3) \quad (4.4a)$$

apart from a factor containing $G(0,0)$, $H(0,0)$, $M^2\dot{D}(0,0)$, $M^2\dot{F}(0,0)$, $M^2\dot{G}(0,0)$, etc.,

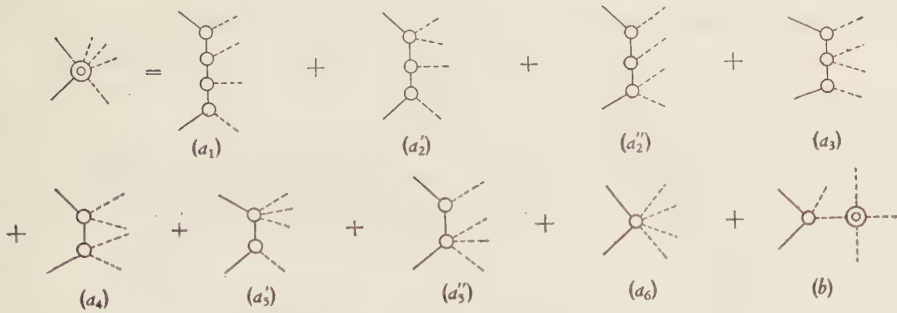


Fig. 8. Feynman diagrams representing the process, $(\pi + N \rightarrow N + 3\pi)$. Graphs involving single photon bridges have been omitted. Cf. eqs. (4.3a) and (4.3b).

though the summands of each sum over permutations corresponding to one of the graphs, Figs. $8a_1$, $8a_2'$, $8a_2''$, and $8a_4$ are individually of the order

$$(g^4/M^3) (\mu'/M)^{-1}.$$

(4.3b) can be evaluated by substituting (2.37b) for $T'_{im}^{(2,2,0)}$, (2.22b) for Δ_{mn} , and (2.44) for $T_{njk\ell}^{(0,4,0)}$ to a fairly good approximation as follows:

$$\begin{aligned} & \sum T'_{im}^{(2,2,0)}(q, q' + q'' + q''') \Delta_{mn}(q' + q'' + q''') T_{njk\ell}^{(0,4,0)}(q', q'', q''') \\ &= -(g_s^2 \lambda^2 / M) (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ & \times \{ [(q' + q'' + q''')^2 + \mu^2]^{-1} + [(q - q' - q'')^2 + \mu^2]^{-1} \\ & + [(q - q'' - q''')^2 + \mu^2]^{-1} + [(q - q''' - q')^2 + \mu^2]^{-1} \} \\ & \simeq -(g^4 / M^3) (g_s / g)^2 (\lambda / g)^2 (2\mu / M)^{-2} \\ & \times (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (4.4b)$$

where q is the four-momentum of the incident pion, and q' , q'' and q''' are those of the produced pions. The threshold values

$$q' = q'' = q''' = (\mu / M) [1 + 3(\mu / M)]^{-1} q$$

and

$$q_0 = 3\mu [1 + (4\mu / 3M)] \quad (4.5)$$

have been inserted to get the last from the second member of (4.4b).

On comparing (4.4a) with (4.4b), we observe that the latter is smaller than the former owing to the small factor, $(g_s / g)^2 \simeq 0.03$, notwithstanding that the latter is of lower order with respect to μ' / M . Hence, it can not be expected that (4.3b) predominates over (4.3a), unless a strong damping effect enough to reverse their orders happens to occur to (4.3a) due to virtual processes of higher order in the coupling constants. If such a strong damping effect should happen to occur, the cross section near threshold would be obtained from (4.4b) as follows:

$$\begin{aligned}
 (1/9) \sigma(\pi^0 + N \rightarrow N + 3\pi^0) &= (1/4) \sigma(\pi^\pm + N \rightarrow N + \pi^\pm + \pi^+ + \pi^-) \\
 &= \sigma(\pi^0 + N \rightarrow N + \pi^0 + \pi^+ + \pi^-) = \sigma(\pi^\pm + N \rightarrow N + \pi^\pm + 2\pi^0) \\
 &= (1/210) (1/M^2) (g_s^2/4\pi)^2 (\lambda^2/4\pi)^2 [(\omega_0 - \omega_{\text{thr}})/\mu]^{7/2},
 \end{aligned} \quad (4.6)$$

where ω_{thr} denotes the threshold energy of the incident pion and is defined by (4.5) as q_0 . Otherwise, we can not obtain any expression for the cross section available for the comparison with experiment on the same ground as mentioned in § 3.2.

§ 4.2. $\gamma + N \rightarrow N + 3\pi$

The matrix element for this process is given by

$$T^{(2,3,1)} = \phi_i(\xi) \phi_j(\xi') \phi_k(\xi'') \bar{\psi} T_{ijk\mu}^{(2,3,1)}(\xi, \xi', \xi'', \xi) \psi A_\mu(\xi) \quad (4.7)$$

with

$$\begin{aligned}
 T_{ijk\mu}^{(2,3,1)}(\xi, \xi', \xi'', \xi) &= J_{il}^{-1}(\xi, \xi''') J_{jm}^{-1}(\xi', \xi^{iv}) J_{kn}^{-1}(\xi'', \xi^v) D_{\mu\nu}^{-1}(\xi, \xi') \\
 &\times S^{-1} \delta^4 S / \partial K_l(\xi''') \partial K_m(\xi^{iv}) \partial K_n(\xi^v) \partial J_\nu(\xi) S^{-1}.
 \end{aligned} \quad (4.8)$$

Omitting those terms which contain single photon-bridges, we have

$$T_{ijk\mu}^{(2,3,1)}(\xi, \xi', \xi'', \xi) = e g^3 S^{-1} \delta^4 S / \partial \langle g \phi_i(\xi) \rangle \partial \langle g \phi_j(\xi') \rangle \partial \langle g \phi_k(\xi'') \rangle \partial \langle e A_\mu(\xi) \rangle \cdot S^{-1} \quad (4.9a)$$

$$+ \sum T_{il}^{(2,2,1)}(\xi, \xi''') J_{im}(\xi''', \xi^{iv}) T_{mjk\mu}^{(0,3,1)}(\xi^{iv}, \xi', \xi'', \xi) \quad (4.9b)$$

$$+ T_{l\mu}^{(2,1,1)}(\xi''', \xi) J_{lm}(\xi''', \xi^{iv}) T_{mijk}^{(0,4,0)}(\xi^{iv}, \xi, \xi', \xi''), \quad (4.9c)$$

where we have dropped the terms which vanish at threshold owing to the transverse nature of the light wave. \sum stands for the summation over cyclic permutations of i, j, k and ξ, ξ', ξ'' . The graphical representation is given in Fig. 9.

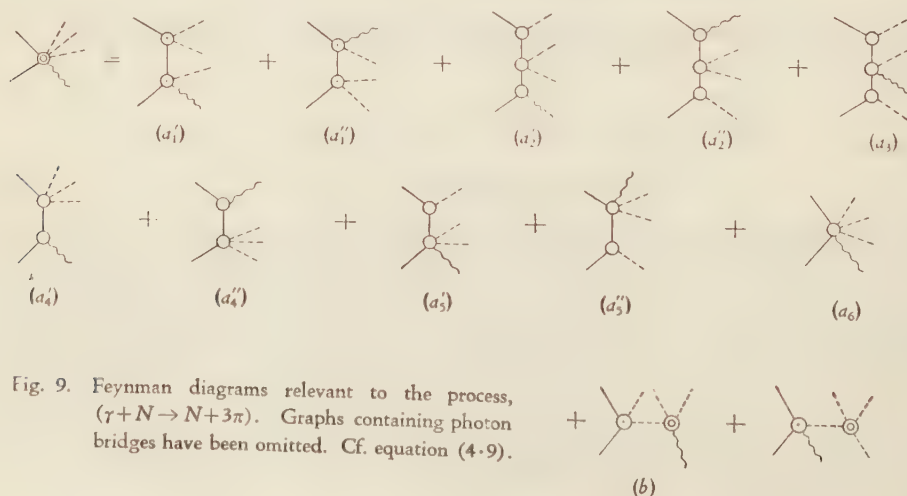


Fig. 9. Feynman diagrams relevant to the process, $(\gamma + N \rightarrow N + 3\pi)$. Graphs containing photon bridges have been omitted. Cf. equation (4.9).



Inserting (2.11) into (4.9a), we find that the contributions of Figs. 9a₂' to 9a₄ to (4.9a) are of the order

$$eg^3/M^3 \quad (4.10a)$$

apart from a factor containing the parameters, $G(0, 0)$, $H(0, 0)$, etc. On the other hand, each of Figs. 9a₁' and a₁'' individually gives contribution of the order

$$(eg^3/M^3) (\mu'/M)^{-1}.$$

However, their resultant contribution turns out to be of the same order as (4.10a) in consequence of the mutual cancellation similar to that we have met with in the case of Figs. 8a₁, a₂', a₂'' and a₄.

It can be shown in quite the same manner as the case of (3.15i) that, at threshold, (4.9b) [Fig. 9b] gives vanishing contribution to the matrix element.

Finally, (4.9c) [Fig. 9c] can be evaluated with the insertion of (2.32b), (2.22b) and (2.44) as

$$\begin{aligned} & T'_{i\mu}^{(2,1,1)}(q+q'+q'', k) A_{Im}(q+q'+q'') T_{mi\bar{j}\bar{k}}^{(0,4,0)}(q, q', q'') \\ &= (e\eta/2M) \{ \bar{\tau}_P \gamma_\mu, \bar{\tau}_i \gamma_5 \} [(q+q'+q'')^2 + \mu^2]^{-1} \lambda^2 (\partial_{ij} \partial_{kl} + \partial_{ik} \partial_{jl} + \partial_{il} \partial_{jk}) \\ &= (eg^3/M^3) (\lambda^2/g^2) (M/4\mu)^2 [\bar{\tau}_P, \partial_{ij} \bar{\tau}_k + \partial_{jk} \bar{\tau}_i + \partial_{kl} \bar{\tau}_j] \gamma_5 \gamma_\mu, \end{aligned} \quad (4.10c)$$

where k is the four-momentum of the incident photon, while q , q' , and q'' denote the four-momenta of the produced pions.

It is observed from (4.10a) and (4.10c) that the competition of (4.9a) with (4.9c) for preponderance is rather severe. This is due to the fact that the threshold energy, $k_{thr} = 3\mu(1 + 3\mu/2M)$, can no longer be regarded as sufficiently small in comparison with the rest energy of the nucleon. If $\lambda^2/4\pi$ may be taken to be comparable to or larger than $g^2/4\pi$, or if it happens that the contribution of (4.9a) to the matrix element is diminished to some extent below (4.10a) by virtue of the damping effect of the virtual processes of higher orders, then the main contribution to the matrix element comes from (4.9c), and the cross section is given by

$$\begin{aligned} \sigma(\gamma + p \rightarrow n + \pi^+ + 2\pi^0) &= (1/4) \sigma(\gamma + p \rightarrow n + \pi^- + 2\pi^+) \\ &= (1/1260 \sqrt{2} M^2) (e^2/4\pi) (g^2/4\pi) (\lambda^2/4\pi)^2 [(k - k_{thr})/\mu]^{7/2}, \end{aligned} \quad (4.11)$$

where k is the energy of the incident photon in the laboratory system. In the opposite case, (4.9c) will play second fiddle to (4.9a), and, on the same ground as mentioned in § 3.2, the result obtained for the cross section will not be available for comparison with the experiment.

§ 5. Summary and discussions

The results obtained in the preceding sections are summarized as follows: (i) it is probable that the main contribution to the matrix element for the process, $\pi + N \rightarrow N + 2\pi$, near threshold comes from the Feynman graph whose skeleton is constructed from a pion-

nucleon 3-vertex part and a pion-pion scattering part connected by a single pion line, provided that the renormalized Matthews' constant, λ , is comparable to or larger than the renormalized mesic charge, g ; (ii) in the same approximation and under the same assumption, the main contribution to the matrix element for the process, $\gamma + N \rightarrow N + 3\pi$, is expected to come from the Feynman graph whose skeleton is decomposed into a subgraph representing the photo-pion production from a nucleon, that representing the pion-pion scattering, and a single pion line connecting the two subgraphs, (iii) the expressions for the main terms of the two matrix elements including the radiative corrections to all orders in e and g can be obtained by applying the perturbation calculation of the lowest order to the skeleton graphs and by replacing the unrenormalized masses, charges and the divergent coefficient of the pion-pion scattering part with the renormalized ones.

Since explicit expressions for the matrix elements have been obtained for the above processes in terms of e , g , λ , M and μ , of which the values have already been known from experiment with the exception of λ , it is suggested that the renormalized Matthews' constant, λ , whose value is as yet unknown, may be determined from an experimental analysis of these processes near threshold. Furthermore, the possibility is also suggested of testing the consistency of the current PS(PS) meson theory examining whether the values determined from experiments on the two independent processes agree with each other.

Our results, (i) and (iii), seem to be interesting in the light of the results obtained by Ito and Minami from their phenomenological analysis on the process, $\pi + N \rightarrow N + 2\pi$. According to their results, the observed angular correlation between the two produced pions can be accounted for by assuming that the process occurs mainly through the ϕ^4 -interaction. However, it must be remembered that their analysis has been made on the experimental data for the incident energy near 1.5 Bev, while the approximations made in the derivation of our results hold good only at threshold, i.e., about 0.15 Bev.

It is worth-while remarking that our results remain unchanged, even if we allow for the participation of any V -particles in the virtual processes, so far as they obey the "even-odd" rule of Pais and do not violate the renormalizability of the theory. The reason for this is as follows. The radiative corrections due to virtual V -particles can be classified into the following two kinds: those of the first kind are such as to give contribution to the functions, D , F , G , H , J , K , Q , R , etc.; while those of the second kind are such as to be represented by a graph obtained from, e.g., Figs. 6(d, e, f) and Figs. 7(b, c), by replacing the single photon-line or the single pion-line connecting the two subgraphs with a single heavy-meson-line. The functions, D , F , etc., are irrelevant to the results, because the terms of the matrix elements containing these functions are of higher order in (μ'/M) as compared with the leading term. Hence the radiative corrections of the first kind may be ignored. As for the radiative corrections of the second kind, they are expected to be extraordinarily small, because each subgraph contains a single line representing an "odd" particle, consequently a "weak interaction", in accordance with the even-odd rule.

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Scattering Matrix in the Heisenberg Representation for a System with Bound States

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A definition is formulated of the scattering matrix for a closed physical system with bound states which makes use throughout only of the assumed observable properties of the system. A direct product space, X , is defined in which the ingredient factor space comprises the steady states—vacuum, one-particle, and bound states—of the physical system. It is argued that the boundary conditions for a scattering experiment are suitably expressed in terms of vectors in X and that these stand in unitary correspondence, U , to the Heisenberg states. Indeed, one defines two operators $U^{(\pm)}$ to express outgoing and ingoing wave boundary conditions, and the scattering matrix is constructed from these in the usual way. A suitable Yang-Feldman formalism is then developed in which the operators in the remote past and future also describe the bound states of the system. A representation of the framework thus constructed in terms of field operators for individual fields results in the well-known formulas for S -matrix elements in terms of covariant amplitudes.

§ 1. Introduction

The first attempts to formulate a definition of the scattering matrix in the Heisenberg representation¹⁾ were rather closely allied to the form of the equations of motion of conventional field theories and left open the question of how to include bound states. The relation of the Yang-Feldman formalism to the problem of bound states has since been clarified by Glaser and Zimmerman²⁾ and by Freese³⁾, but this work did not provide a detailed calculational technique for describing scattering processes which involved composite particles in the initial or final states. Nevertheless, the introduction of covariant wave functions⁴⁾ to describe bound states made it clear that the solutions of these equations should provide a suitable basis for such a description. The manner in which this is to be accomplished has been stated correctly by Nishijima⁵⁾, and the derivation from a definition of the scattering matrix has been essentially, if not completely given.

The primary purpose of the present note is to formulate an abstract definition of the scattering matrix for a closed system of interacting fields, including the possibility of bound states, by invoking throughout only the properties of the real physical states of the system. There have indeed been several recent attempts^{6,7)} to formulate and study an S -matrix theory without reference to the detailed structure of the field equations of conventional field theories. Our attempt is not unrelated to these in spirit, and may be considered as a slight generalization of the approach of these authors, at least in so far as the framework is concerned.

To effect a realization of the transition amplitudes in terms of covariant wave functions is the second purpose of this note. It is only here that a distinction between bound and continuum states becomes of practical significance. The results accord with those already given by Nishijima.⁵⁾

§ 2. Stationary state definition

We consider then a closed physical system consisting of interacting fields. We shall assume that the spectrum of all relevant constants of the motion is known. In particular the Hamiltonian may be supposed to describe a system with continuum, bound, and composite states. Thus, with an obvious notation, the Hamiltonian is given by the expression

$$H = \sum_C E_C \Psi_C \Psi_C^+ + \sum_{B,C} E_{BC} \Psi_{BC} \Psi_{BC}^+ + \sum_{BB',C} E_{BB'C} \Psi_{BB'C} \Psi_{BB'C}^+ + \dots, \quad (1)$$

where the vacuum state Ψ_0 is contained among the continuum states C and the simple bound states among the states BC , etc. In forming the decomposition (1), we assume, of course, that we have a criterion for distinguishing single-particle and bound states.

From the point of a scattering experiment, a more obviously pertinent decomposition of H may be made in accordance with the boundary conditions of the experiment. If we suppose these conditions to be specified in the remote past, for instance, then the real distinction is between states which initially contain at least two spatially separated parts and those states such as the vacuum, one-particle, and simple bound states which are described in terms of at most one spatially localized entity and thus suffer no real scattering. It is clear that in a real physical sense the *initial* condition for the former type of state should be expressible by a superposition of the latter steady states.

To express this supposition mathematically, we write H in the form

$$H = H_S + (H - H_S), \quad (2)$$

where

$$H_S = \sum_s E_s \Psi_s \Psi_s^+ \quad (3)$$

describes the spectrum of steady states. We then define the infinite direct product space, X , spanned by the eigenfunctions of the Hamiltonian H_X , with

$$H_X = +H_S^{(1)} + H_S^{(2)} + \dots, \quad (4)$$

where $H_S^{(1)}$ means H_S in subspace number 1 and unit operator in all others. Correspondingly an eigenvector of H_X is written

$$\Phi = \Psi_S^{(1)} \times \Psi_{S'}^{(2)} \times \dots, \quad (5)$$

and satisfies the equation

$$H_X \Phi_n = (E_S^{(1)} + E_{S'}^{(2)} + \dots) \Phi_n = E_n \Phi_n. \quad (6)$$

The two essential properties of the space X are that for a steady state we have $\Phi_S = \Psi_S$ and that the spectrum of H_X coincides with that of H . A similar statement obtains for the other suitably defined constants of the motion. We may therefore construct the unitary

correspondence, U , defined by the equation

$$U = \sum_n \Psi_n \Phi_n^+, \quad (7)$$

such that

$$\Psi_n = U \Phi_n = \sum_{n'} \Phi_{n'} U_{n'n}, \quad (8)$$

and

$$O = U O_X U^{-1}, \quad (9)$$

where O is any operator of the physical system and O_X the corresponding operator in the direct product (DP) space.

We shall think of the space X as uniquely defined. On the other hand there are alternative U matrices corresponding to different choices of boundary condition and consequently different sets of Heisenberg states. Let us note that if we write

$$H = H_X + H_1, \quad (10)$$

where $H_1 = 0$ for steady states, the Schrödinger equation is equivalent to the equations

$$U_{nn'}^{(\pm)} = \delta(E_n - E_{n'}) + (E_n \pm i\epsilon - E_{n'})^{-1} (H_1)_{n'n'} U_{n'n}^{(\pm)}, \quad (11)$$

or to the equations

$$\Psi_n^{(\pm)} = \Phi_n + (E_n \pm i\epsilon - H_X)^{-1} H_1 \Psi_n^{(\pm)}.$$

The boundary conditions satisfied by the two sets of Heisenberg states $\Psi_n^{(\pm)}$ are well-known and have recently³⁾ been stated rigorously as follows: Let Ψ_A be a proper element of the Hilbert space,

$$\Psi_A = \sum_n \Psi_n^{(+)} A_n, \quad \sum_n |A_n|^2 < \infty.$$

It is then a consequence of eq. (12) that

$$\lim_{t \rightarrow -\infty} \exp(iH_X t) \exp(-iHt) \Psi_A = \sum_n \Phi_n A_n \equiv \Phi_A, \quad (14)$$

or crudely speaking, $\Psi_n^{(+)}$ approaches Φ_n in the remote past. Similarly if

$${}_A \Psi = \sum_n \Psi_n^{(-)} A_n, \quad (15)$$

then

$$\lim_{t \rightarrow +\infty} \exp(iH_X t) \exp(-iHt) {}_A \Psi = \Phi_A, \quad (16)$$

or $\Psi_n^{(-)}$ approaches Φ_n in the remote future. If $\Psi_A, {}_A \Psi$ are superpositions of steady states only, then eqs. (14) and (16) hold by definition. Further, in the sense of these equations

$$U^{(\pm)} = \lim_{t \rightarrow \mp \infty} \exp(iHt) \exp(-iH_X t). \quad (17)$$

Since the transformations $U^{(\pm)}$ are unitary, it follows that there exists a transformation S relating $\Psi_n^{(-)}$ to $\Psi_n^{(+)}$,

$$\mathcal{V}_n^{(-)} = S \mathcal{V}_n^{(+)}, \quad (18)$$

where

$$S = U^{(-)} U^{(+)\dagger} \quad (19)$$

is the scattering matrix in the Heisenberg representation.

§ 3. Relation to Yang-Feldman formalism

To any Heisenberg operator $O(t)$ we may assign a one parameter family of operators $O^\tau(t)$ according to the equation

$$O^\tau(t) = \exp[iH_X(t-\tau)] O(\tau) \exp[-iH_X(t-\tau)]. \quad (20)$$

The general solution to the equation of motion

$$i\dot{O}(t) = [O(t), H] = [O(t), H_X + H_1] \quad (21)$$

can be obtained in the DP space, where H_X is diagonal, as

$$O_{nn'}(t) = O_{nn'}^\tau(t) - i \int_\tau^t dt' \exp[i(E_n - E_{n'})(t-t')] [O(t'), H_1(t')], \quad (22)$$

which is equivalent to the operator equation

$$\begin{aligned} \exp(-iH_X t) O(t) \exp(iH_X t) &= \exp(-iH_X \tau) O(\tau) \exp(iH_X \tau) \\ &\quad - i \int_\tau^t dt' \exp(-iH_X t') [O(t'), H_1(t')] \exp(iH_X t'). \end{aligned} \quad (23)$$

If we consider a matrix element of eq. (23) between two proper states ϕ_A , ϕ_B of the DP space and take the limit as $\tau \rightarrow -\infty$, we must evaluate

$$\begin{aligned} \lim(\tau \rightarrow -\infty) (\phi_A, \exp(-iH_X \tau) \exp(iH \tau) O(o) \exp(iH \tau) \exp(iH_X \tau) \phi_B) \\ = (\mathcal{V}_A^{(-)}, O(o) \mathcal{V}_B^{(-)}) \equiv (\phi_A, O_{in}(o) \phi_B). \end{aligned} \quad (24)$$

In obtaining (24), we have used the relation similar to (16),

$$\lim(t \rightarrow -\infty) \exp(-iHt) \exp(iH_X t) \phi_B = \lim(t \rightarrow +\infty) \exp(iHt) \exp(-iH_X t) \phi_B = \mathcal{V}_B^{(-)} \quad (25)$$

and defined the operator O_{in} by stating its matrix in the DP space,

$$O_{in}(o) = \sum_{nn'} \phi_n (\mathcal{V}_n^{(-)}, O(o) \mathcal{V}_{n'}^{(-)}) \phi_{n'}^\dagger, \quad (26)$$

and

$$O_{in}(t) = \exp(iH_X t) O_{in}(o) \exp(-iH_X t). \quad (27)$$

We have thus derived the equation

$$O(t) = O_{in}(t) - i \int_{-\infty}^t dt' \exp[iH_X(t-t')] [O(t'), H_1(t')]$$

$$\times \exp[-iH_X(t-t')], \quad (28)$$

and

$$\text{Lim}(t \rightarrow -\infty) (O(t) - O_{\text{in}}(t)) = 0 \quad (29)$$

in the sense of eq. (24). We note that eq. (26) is equivalent to the statement

$$O_{\text{in}} = U^{(-)\dagger} O U^{(-)}. \quad (30)$$

In an exactly similar fashion, we can achieve the equations

$$O(t) = O_{\text{out}}(t) + i \int_t^{\infty} dt' \exp[iH_X(t-t')] [O(t'), H_1(t')] \exp[-iH_X(t-t')], \quad (31)$$

where

$$O_{\text{out}}(t) = \exp(iH_X t) O_{\text{out}}(0) \exp(-iH_X t), \quad (32)$$

and

$$\begin{aligned} (\Phi_A, O_{\text{out}} \Phi_B) &= \text{Lim}(t \rightarrow +\infty) (\Phi_A, \exp(-iH_X t) \\ &\quad \exp(iHt) O(0) \exp(-iHt) \exp(iH_X t) \Phi_B) = (\Psi_A^{(+)}, O(0) \Psi_B^{(+)}), \end{aligned} \quad (33)$$

since

$$\text{Lim}(t \rightarrow -\infty) \exp(iHt) \exp(-iH_X t) \Phi_B = \Psi_B^{(-)}.$$

A comparison now of eq. (24), rewritten according to eq. (18) as

$$(\Phi_A, O_{\text{in}} \Phi_B) = (\Psi_A^{(-)} S^\dagger O S \Psi_B^{(-)}), \quad (35)$$

with eq. (33) establishes the relation

$$O_{\text{out}} = X^\dagger O_{\text{in}} X, \quad X = U^{(-)\dagger} U^{(+)}. \quad (36)$$

Corresponding to eq. (30), we have the expression

$$O_{\text{out}} = U^{(+)\dagger} O U^{(+)}. \quad (37)$$

From eqs. (30), (37), and the equation

$$H_X = U^{(+)\dagger} H U = U^{(-)\dagger} H U^{(-)}, \quad (38)$$

which expresses eq. (9) and the commutivity of S with H , we can conclude the fundamental relation

$$H_X(O) = H(O_{\text{out}}) = H(O_{\text{in}}). \quad (39)$$

As a further consequence, it follows that if we achieve a representation of $H_X(O)$ in terms of Heisenberg variables, the $H_X(O_{\text{in}})$ and $H_X(O_{\text{out}})$ will be a representation of the actual Hamiltonian.

§ 4. Realization of the scattering matrix

The observations of this section will be based on eq. (39) together with the significance

of the DP space. We consider first the case where there are no bound states. We can then obtain a representation of the DP space by introducing annihilation and creation operators $q^{(i)}$ and $q^{(i)\dagger}$, the Heisenberg variables for the i -th field, satisfying canonical commutation relations (anticommutation relations) and a vacuum state Ψ_0 ,

$$q^{(i)}\Psi_0=0. \quad (40)$$

We are free to choose

$$H_X(q^{(i)}, q^{(i)\dagger}) = H_0(q^{(i)}, q^{(i)\dagger}), \quad (41)$$

where $H_0(q^{(i)}, q^{(i)\dagger})$ is a superposition of Hamiltonians of free fields but with each field characterized by the experimental mass of its one-particle state. A complete set of states Ψ_i can be constructed then in the well-known way by the operation of the $q^{(i)\dagger}$ on Ψ_0 . Moreover it follows from eq. (39) and subsequent remarks that a complete set of states $\Psi_n^{(-)}$, $\Psi_n^{(+)}$ can be erected by the operation of $q_{in}^{(i)\dagger}$, $q_{out}^{(i)\dagger}$, respectively on Ψ_0 . Finally for the matrix element of S specifying the scattering of the set of free particles from the initial state α to the final state β we have

$$S_{\beta\alpha} = (\Psi_\beta^{(+)}, \Psi_\alpha^{(-)}) = (\Psi_0 q_{out}^{(\beta)} q_{in}^{(\alpha)\dagger} \Psi_0), \quad (42)$$

where $q^{(\alpha)\dagger}$ stands for the complex of operators that creates the state α . The matrix thus obtained is obviously unitary. For its evaluation we can relate its elements to the wave functions and Green's functions of the theory, obtaining these either by the general techniques of Lehmann *et al.*⁽⁶⁾ or from a specific conventional field theory plus renormalization.⁽³⁾

Consider next the existence of bound states. Again we introduce the canonical sets $q^{(i)}$, $q^{(i)\dagger}$, and (omitting henceforth superscriptions i) decompose q into two commuting (anticommuting parts),

$$q = q_C + q_B, \quad (43)$$

where in addition to

$$q\Psi_0=0, \quad (44)$$

we have also

$$q_C\Psi_B = q_C\Psi_B \times \Psi_{B'} = \dots = 0. \quad (45)$$

Equations (43)–(45) can be realized as follows: The matrix of q is such that if we write identically

$$q = \sum_{nn'} \Phi_n q_{nn'} \Phi_{n'}, \quad (46)$$

we require $E_n < E_{n'}$. Equation (44) is then true by definition. Moreover q_C is that part of eq. (46) in which Φ_n , $\Phi_{n'}$ are vectors describing the same bound states, though they differ otherwise. In each such subspace q_C has the same matrix elements as q itself has in the event of no bound states. Equation (45) then follows from the observation that $\Phi_{n'}$ cannot consist of a set of bound states only if a matrix element of q_C is in question.

The continuum states can now be constructed from q_C operating on Ψ_0 and the

composite states from q_c operating on the various bound states. The operators q_{in} and q_{out} may be decomposed similarly and a representation of the Heisenberg states $\Psi_n^{(\mp)}$ obtained.

Turning to the S matrix, eq. (42) still obtains if α and β are continuum states, but q_c must replace q on the right hand side. For scattering processes involving bound states, we must proceed differently merely because we do not have explicit representation for these states.

As an example consider the case of electron-deuteron scattering. Let $\psi(\mathbf{x}, t)$ with an appropriate superscript e, p , or n represent the field operator for electron, proton or neutron, respectively, and let D designate the Deuteron. Following Nishijima⁵⁾, we consider, instead of (42), the matrix element

$$\begin{aligned} & (\Psi_0 : \psi_{out}^e(\mathbf{x}_1, t) \psi_{out}^p(\mathbf{x}_2, t) \psi_{out}^n(\mathbf{x}_3, t) : \Psi_{in}^{eD}) \\ &= (\Psi_0, O_{out}^{epn}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \Psi_{in}^{eD}) \\ &= (\Psi_0, O_{in}^{epn}(t) S \Psi_{in}^{eD}) \\ &= \sum_{e'p'} (\Psi_0, O_{in}^{epn}(t) \Psi_{in}^{e'p'}) S_{e'p';eD} \\ &= \sum_{e'p'} \mu_{e'}(\mathbf{x}_1, t) (\Psi_0, O_{in}^{epn}(\mathbf{x}_2, \mathbf{x}_3, t) \Psi_{in}^{e'}) S_{e'p';eD}. \end{aligned} \quad (47)$$

Here $\mu_{e'}(\mathbf{x}_1, t)$ is the normalized plane wave state of the outgoing electron. For those β' which refer to inelastic scattering, eq. (47) may be further simplified by use of our knowledge of the state $\Psi_{in}^{e'}$. By constructing the scalar product of (47) with an appropriate plane wave state, we obtain for this case⁶⁾

$$\begin{aligned} S_{e'p'p';eD} &= \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \mu_{e'}^*(\mathbf{x}_1, t) \mu_{p'}^*(\mathbf{x}_2, t) \mu_{n'}^*(\mathbf{x}_3, t) \\ &\quad \times (\Psi_0, O_{in}^{epn}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \Psi_{in}^{eD}) \\ &= \text{Lim}(t \rightarrow +\infty) \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \mu_{e'}^*(\mathbf{x}_1, t) \mu_{p'}^*(\mathbf{x}_2, t) \mu_{n'}^*(\mathbf{x}_3, t) \\ &\quad \times \chi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t), \end{aligned} \quad (48)$$

where

$$\chi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = (\Psi_0 : \psi^e(\mathbf{x}_1, t) \psi^p(\mathbf{x}_2, t) \psi^n(\mathbf{x}_3, t) : \Psi_{in}^{eD}) \quad (49)$$

is the covariant amplitude for the electron-deuteron system evaluated at a common time.

Now suppose we are dealing with elastic scattering. We introduce the covariant amplitude for the deuteron $\chi_D(\mathbf{x}_1, \mathbf{x}_2, t)$, and obtain for the matrix element of S , the expression

$$\begin{aligned} S_{e'p'p';eD} &= \text{Lim}(t \rightarrow \infty) \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x} \mu_{e'}^*(\mathbf{x}_1, t) \chi_D^*(\mathbf{x}_2, \mathbf{x}_3, t) \\ &\quad \times \chi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \int d\mathbf{x}_2 d\mathbf{x}_3 \chi_D^*(\mathbf{x}_2, \mathbf{x}_3) \chi_D^*(\mathbf{x}_2, \mathbf{x}_3). \end{aligned} \quad (50)$$

Equation (50) clearly evidences the existence of a normalization problem. That problem indeed exists also for eq. (48) and will be the last matter accorded consideration in this paper.

§ 5. Normalization of covariant amplitudes

This problem has in fact been fully and adequately covered by Nishijima⁵⁾, and we wish merely to summarize his work for the sake of completeness. He has given two methods, both of which we shall illustrate by means of the deuteron problem. If $\psi(x)$ is now the nucleon operator, then in the first method we employ a conservation principle directly related to the normalization of the state vector, for example the conservation of heavy particles, which in the present example requires that

$$\int d\sigma_\mu (\Psi^D, : \bar{\psi}(x) \gamma_\mu \psi(x) : \Psi^D) = 2. \quad (51)$$

We must obtain an alternative expression for the left hand side of eq. (51) which contains only the deuteron amplitude χ_D , in addition to "known" kernels. For this purpose we consider the amplitude

$$\varphi(x_1, x_2, x_3; x_3) = (\Psi_0, T(\psi(x_1)\psi(x_2) : \psi(x_3)\bar{\psi}(x_3) :) \Psi^D). \quad (52)$$

By means of the field equations¹⁰⁾, eq. (52) can be exhibited in a form which depends on kernels and the amplitude for the two body problem only. Comparison of the resultant form, in the limit $t_1, t_2 > t_3$, with that which results from (52) directly,

$$\sum_n (\Psi_0, T(\psi(x_1)\psi(x_2)) \Psi_n) (\Psi_n, : \psi(x_3)\bar{\psi}(x_3) : \Psi^D), \quad (53)$$

allows us, by choosing $n=D$, to derive a form for (51) from which the normalization of χ_D may be inferred¹¹⁾.

Actually the second method, which employs external sources, is merely a more expeditious way of carrying out the program stated above. In the example at hand we introduce into the Lagrange density operator a coupling term

$$L'(x) = : \bar{\psi}(x) \gamma_\mu \psi(x) : A_\mu(x), \quad (54)$$

describing the coupling of a fictitious external vector field A_μ to the nucleon "particle current." Now consider the equation¹²⁾

$$(G_1^{-1} G_2^{-1} - I_{12}) G_{12} = I_{12} \quad (55)$$

for the two-nucleon Green's function G_{12} . By straight-forward functional differentiation and introduction of the definition of the vertex operator

$$\Gamma_{1\mu}(\hat{s}) = -\delta G_1^{-1} / \delta A_\mu(\hat{s}), \quad (56)$$

we obtain from (55) the knowledge that

$$\delta G_{12} / \delta A_\mu(\hat{s}) = -G_{12} \{ \Gamma_{1\mu}(\hat{s}) G_2^{-1} + G_1^{-1} \Gamma_{2\mu}(\hat{s}) + \delta I_{12} / \delta A_\mu(\hat{s}) \} G_{12}. \quad (57)$$

On the other hand as A_μ tends to zero, since

$$G_{12}(x_1, x_2; x'_1, x'_2) = -(\Psi_0, T(\psi(x_1)\psi(x_2)\bar{\psi}(x'_1)\bar{\psi}(x'_2)) \Psi_0), \quad (58)$$

we can see that

$$\delta G_{12} / \delta A_\mu(\hat{s}) = -i(\Psi_0, T(\psi(x_1)\psi(x_2)\bar{\psi}(x'_1)\bar{\psi}(x'_2) : \bar{\psi}(\hat{s}) \gamma_\mu \psi(\hat{s}) :) \Psi_0). \quad (59)$$

Introducing eq. (59) into (57), assuming the limits $t_1, t_2 \rightarrow +\infty$, $t_1', t_2' \rightarrow -\infty$, and exhibiting the resulting equation as a sum over states, we are led directly to the result true for any states n, n' of the two nucleon system,

$$i(\Psi_{n'} : \bar{\psi}(\xi) \gamma_\mu \psi(\xi) : \Psi_n) = - \int \bar{\chi}_n \{ \Gamma_{1\mu}(\xi) G_2^{-1} + G_1^{-1} \Gamma_{2\mu}(\xi) + \delta I_{12} / \delta A_\mu(\xi) \} \chi_{n'}. \quad (60)$$

Through eq. (60) may in principal be used to determine the normalization of χ_n , it suffers from the practical difficulty that it requires knowledge of the form of χ_n for unequal times. Nevertheless it can easily form the basis of an iterative procedure to determine the required norm.

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- 9) The limits taken below will exist again if we consider only proper states. This is the justification for the limiting procedure of M. Gell Mann and F. Low, Phys. Rev. **84** (1951), 350, which will be employed when needed without further comment.
- 10) It is clear that the work of this section is more closely tied to existing formalism.
- 11) Reference 5 should be consulted for further details and specific examples.
- 12) We use the notation and methods of J. Schwinger, reference 4.

Note added in proof :

- 1) The matrix defined in eqs. (18), (19) is the inverse of what is usually termed the S matrix.
- 2) The matrix X of eq. (36) is the S matrix in the DP representation.

Tables of U Coefficients

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Algebraic formulae and tables of the U coefficients

$$U \begin{pmatrix} l & l' & L \\ j & j' & J \\ 1/2 & 1/2 & S \end{pmatrix} \quad (l, l' : \text{integer})$$

are given.

§ 1. Introduction

The U coefficient arises in the relation between different ways of coupling the four vectors to give a resultant one¹⁾⁻⁵⁾. When the coupling of the angular momenta of two spin 1/2 particles comes into question, there often appear the U coefficients of the following form :

$$U \begin{pmatrix} l & l' & L \\ j & j' & J \\ 1/2 & 1/2 & S \end{pmatrix}, \quad (l, l' : \text{integer}). \quad (1)$$

The well-known cases of the appearance of such coefficients are the transformation coefficients between LS and jj coupling scheme of two equivalent particles¹⁾, β -decay correction factor⁶⁾⁻⁹⁾, nuclear matrix elements for β -decay¹⁰⁾¹¹⁾ and so on.

We have calculated the U coefficients of the above form, when L or J range between 0 and 4, and j and j' are confined between 1/2 and 7/2.

Tables of U coefficients of some other forms are given by Sharp et al.⁵⁾¹²⁾.

§ 2. Definition and some properties of U coefficient

In this section we shall summarize the definitions of the U and similar coefficient*¹⁾ which are useful to employ our tables.

* All these properties and some others are given in the paper by Arima, Horie and Tanabe¹⁾.

The U coefficient of Arima et al. is defined by¹⁾

$$\begin{aligned} & \langle (j_1 j_2) J_{12}, (j_3 j_4) J_{34} | JM | (j_1 j_3) J_{13}, (j_2 j_4) J_{24} | JM \rangle \\ & \equiv \sqrt{(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)} U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix}. \end{aligned} \quad (2)$$

This is the same with the $9j$ coefficient of Wigner²⁾ and the X coefficient of Fano⁴⁾, and is different from the χ coefficient of Hope¹⁾ and the S coefficient of Schwinger³⁾ by numerical factors :

$$\left\{ \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} \right\} = X \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} = U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix}, \quad (3)$$

$$\begin{aligned} & \chi(j_1 j_2 j_3 j_4, J_{12} J_{34}, J_{13} J_{24}, J) \\ & = \sqrt{(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)} U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix}, \end{aligned} \quad (4)$$

$$\begin{aligned} & S(j_1 j_2 j_3 j_4; J_{12} J_{34} J_{13} J_{24}; J) \\ & = (-1)^{j_1+j_4-J_{12}-J_{24}} U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix}. \end{aligned} \quad (5)$$

U coefficient can be represented in terms of Racah coefficients¹⁾⁻⁴⁾ :

$$\begin{aligned} & U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} \\ & = \sum_{\lambda} (2\lambda+1) W(J_{12} J_{24} j_1; J_{24} \lambda) W(J_{12} J_{34} j_1; J_{12} \lambda) W(J_{13} j_2 j_3 J_{12}; j_1 \lambda) \\ & = (-1)^{\sigma} \sum_{\lambda} (2\lambda+1) W(j_2 j_3 J_{12} J_{13}; \lambda j_1) W(j_2 j_3 J_{24} J_{34}; \lambda j_4) W(J_{12} J_{13} J_{34} J_{24}; \lambda J), \end{aligned} \quad (6)$$

where

$$\sigma = j_1 + j_2 + j_3 + j_4 + J_{12} + J_{34} + J_{13} + J_{24} + J = \text{integer}.$$

There are 72 symmetry relations of U coefficient¹⁾⁻⁴⁾ :

- i) An odd permutation of the rows or columns of U coefficient multiplies it by $(-1)^{\sigma}$;

$$U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} = (-1)^n U \begin{pmatrix} j_3 & j_4 & J_{34} \\ j_1 & j_2 & J_{12} \\ J_{13} & J_{24} & J \end{pmatrix} = (-1)^n U \begin{pmatrix} J_{13} & J_{24} & J \\ j_3 & j_4 & J_{34} \\ j_1 & j_2 & J_{12} \end{pmatrix} \text{ etc. (7)}$$

ii) A reflection of the coefficient in either of the two diagonals leaves it invariant ;

$$U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} = U \begin{pmatrix} j_1 & j_3 & J_{13} \\ j_2 & j_4 & J_{24} \\ J_{12} & J_{34} & J \end{pmatrix} \text{ etc. (8)}$$

From i), the following relations are given :

$$U \begin{pmatrix} j_1 & j_2 & J_{12} \\ j_1 & j_2 & J_{12} \\ J_{11} & J_{22} & J \end{pmatrix} = 0, \quad (J_{11} + J_{22} + J = \text{odd}) \text{ etc. (9)}$$

By making use of these symmetry relations and our tables, one can obtain the values of some of the U coefficients which contain two $1/2$ in the same column or in the same row but do not have the form of eq. (1).

§ 3. Algebraic formulae

The algebraic formulae of the U coefficients of the form of (1)^{*} can be obtained from eq. (6) and the algebraic formulae of Racah coefficients¹³⁾. They are :

(I) $S=0$

$$U \begin{pmatrix} j+1/2 & j'+1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left[\frac{(j+j'+J+2)(j+j'-J+1)}{2(2j+1)(2j+2)(2j'+1)(2j'+2)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j+1/2 & j'-1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left[\frac{(j-j'+J+1)(-j+j'+J)}{2(2j+1)(2j+2)(2j')(2j'+1)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j-1/2 & j'+1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 0 \end{pmatrix} = (-1) \left[\frac{(j-j'+J)(-j+j'+J+1)}{2(2j)(2j+1)(2j'+1)(2j'+2)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j-1/2 & j'-1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left[\frac{(j+j'+J+1)(j+j'-J)}{2(2j)(2j+1)(2j')(2j'+1)(2J+1)} \right]^{1/2},$$

* In the following formulae, j and j' can be integers or half-integers.

(II) $S=1$ i) $L=J+1$

$$\begin{aligned}
 & U \begin{pmatrix} j+1/2 & j'+1/2 & J+1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= \left[\frac{(j+j'+J+2)(j+j'+J+3)(j-j'+J+1)(-j+j'+J+1)}{3(2j+1)(2j+2)(2j'+1)(2j'+2)(2J+1)(2J+2)(2J+3)} \right]^{1/2}, \\
 & U \begin{pmatrix} j+1/2 & j'-1/2 & J+1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= (-1) \left[\frac{(j+j'+J+2)(j+j'-J)(j-j'+J+1)(j-j'+J+2)}{3(2j+1)(2j+2)(2j')(2j'+1)(2J+1)(2J+2)(2J+3)} \right]^{1/2}, \\
 & U \begin{pmatrix} j-1/2 & j'+1/2 & J+1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= \left[\frac{(j+j'+J+2)(j+j'-J)(-j+j'-J+1)(-j+j'+J+2)}{3(2j)(2j+1)(2j'+1)(2j'+2)(2J+1)(2J+2)(2J+3)} \right]^{1/2}, \\
 & U \begin{pmatrix} j-1/2 & j'-1/2 & J+1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= (-1) \left[\frac{(j+j'-J-1)(j+j'-J)(j-j'+J+1)(-j+j'+J+1)}{3(2j)(2j+1)(2j')(2j'+1)(2J+1)(2J+2)(2J+3)} \right]^{1/2},
 \end{aligned}$$

ii) $L=J$

$$\begin{aligned}
 & U \begin{pmatrix} j+1/2 & j'+1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= (j-j') \left[\frac{(j+j'+J+2)(j+j'-J+1)}{6(2j+1)(2j+2)(2j'+1)(2j'+2)J(J+1)(2J+1)} \right]^{1/2}, \\
 & U \begin{pmatrix} j+1/2 & j'-1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} \\
 &= (j+j'+1) \left[\frac{(j-j'+J+1)(-j+j'+J)}{6(2j+1)(2j+2)(2j')(2j'+1)J(J+1)(2J+1)} \right]^{1/2},
 \end{aligned}$$

$$U \begin{pmatrix} j-1/2 & j'+1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = (j+j'+1) \left[\frac{(j-j'+J)(-j+j'+J+1)}{6(2j)(2j+1)(2j'+1)(2j'+2)J(J+1)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j-1/2 & j'-1/2 & J \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = (-j+j') \left[\frac{(j+j'+J+1)(j+j'-J)}{6(2j)(2j+1)(2j')(2j'+1)J(J+1)(2J+1)} \right]^{1/2},$$

iii) $L=J-1$

$$U \begin{pmatrix} j+1/2 & j'+1/2 & J-1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = (-1) \left[\frac{(j+j'-J+1)(j+j'-J+2)(j-j'+J)(-j+j'+J)}{3(2j+1)(2j+2)(2j'+1)(2j'+2)(2J-1)(2J)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j+1/2 & j'-1/2 & J-1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = (-1) \left[\frac{(j+j'+J+1)(j+j'-J+1)(-j+j'+J-1)(-j+j'+J)}{3(2j+1)(2j+2)(2j')(2j'+1)(2J-1)(2J)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j-1/2 & j'+1/2 & J-1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = \left[\frac{(j+j'+J+1)(j+j'-J+1)(j-j'+J-1)(j-j'+J)}{3(2j)(2j+1)(2j'+1)(2j'+2)(2J-1)(2J)(2J+1)} \right]^{1/2},$$

$$U \begin{pmatrix} j-1/2 & j'-1/2 & J-1 \\ j & j' & J \\ 1/2 & 1/2 & 1 \end{pmatrix} = \left[\frac{(j+j'+J)(j+j'+J+1)(j-j'+J)(-j+j'+J)}{3(2j)(2j+1)(2j')(2j'+1)(2J-1)(2J)(2J+1)} \right]^{1/2}.$$

By these formulae, we have calculated the U coefficients of the form

$$U \begin{pmatrix} l & l' & L \\ j & j' & J \\ 1/2 & 1/2 & S \end{pmatrix}$$

for the following cases :

$L=J=0,$	$S=0,$	$j, j'=1/2, 3/2, 5/2, 7/2,$	Table I.
$L=J=1,$	"	"	Table II.
$L=J=2,$	"	"	Table III.
$L=J=3,$	"	"	Table IV.
$L=J=4,$	"	"	Table V.
$L=1, J=0,$	$S=1.$	"	Table VI.
$L=0, J=1,$	"	"	Table VII.
$L=1, J=1,$	"	"	Table VIII.
$L=2, J=1,$	"	"	Table IX.
$L=1, J=2,$	"	"	Table X.
$L=2, J=2,$	"	"	Table XI.
$L=3, J=2,$	"	"	Table XII.
$L=2, J=3,$	"	"	Table XIII.
$L=3, J=3,$	"	"	Table XIV.
$L=4, J=3,$	"	"	Table XV.
$L=3, J=4,$	"	"	Table XVI.
$L=4, J=4,$	"	"	Table XVII.
$L=5, J=4,$	"	"	Table XVIII.
$L=4, J=5,$	"	"	Table XIX.

As was done by Obi et al. in the tabulation of Racah coefficients¹⁾, we have tabulated the results as the products of the powers of prime numbers.

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Table I. $U \begin{pmatrix} l & l' & 0 \\ j & j' & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
0	1/2	0	1/2	1/2	2	5/2	2	5/2	$1/2\sqrt{3\cdot5}$
1	1/2	1	1/2	$1/2\sqrt{3}$	3	5/2	3	5/2	$1/2\sqrt{3\cdot7}$
1	3/2	1	3/2	$1/2\sqrt{2\cdot3}$	3	7/2	3	7/2	$1/4\sqrt{7}$
2	3/2	2	3/2	$1/2\sqrt{2\cdot5}$	4	7/2	4	7/2	$1/4\cdot3$

Table II. $U \begin{pmatrix} l & l' & 1 \\ j & j' & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
1	1/2	0	1/2	1/2·3	1	3/2	2	5/2	$1/2\sqrt{2\cdot3\cdot5}$
1	3/2	0	1/2	1/2·3	2	3/2	2	5/2	$1/2\cdot3\cdot5\sqrt{2}$
					2	5/2	2	5/2	$\sqrt{7}/3\cdot5\sqrt{2\cdot3}$
0	1/2	1	1/2	$-1/2\cdot3$	3	5/2	2	5/2	$1/2\cdot3\sqrt{3\cdot5\cdot7}$
1	1/2	1	1/2	$1/3\sqrt{2\cdot3}$	3	7/2	2	5/2	$1/2\cdot3\sqrt{7}$
1	3/2	1	1/2	$-1/2\cdot3\sqrt{2\cdot3}$					
2	3/2	1	1/2	$1/2\cdot3\sqrt{2}$	2	3/2	3	5/2	$1/2\cdot3\sqrt{5}$
					2	5/2	3	5/2	$-1/2\cdot3\sqrt{3\cdot5\cdot7}$
0	1/2	1	3/2	1/2·3	3	5/2	3	5/2	$\sqrt{5}/3\cdot7\sqrt{3}$
1	1/2	1	3/2	$1/2\cdot3\sqrt{2\cdot3}$	3	7/2	3	5/2	$-1/4\cdot3\cdot7$
1	3/2	1	3/2	$\sqrt{5}/4\cdot3\sqrt{3}$	4	7/2	3	5/2	$1/4\sqrt{3\cdot7}$
2	3/2	1	3/2	$1/4\cdot3\sqrt{5}$	2	5/2	3	7/2	$1/2\cdot3\sqrt{7}$
2	5/2	1	3/2	$1/2\sqrt{2\cdot3\cdot5}$	3	5/2	3	7/2	$1/4\cdot3\cdot7$
					3	7/2	3	7/2	$3/8\cdot7$
1	1/2	2	3/2	$1/2\cdot3\sqrt{2}$	4	7/2	3	7/2	$1/8\cdot3\sqrt{3\cdot7}$
1	3/2	2	3/2	$-1/4\cdot3\sqrt{5}$					
2	3/2	2	3/2	$\sqrt{3}/4\cdot5$	3	5/2	4	7/2	$1/4\sqrt{3\cdot7}$
2	5/2	2	3/2	$-1/2\cdot3\cdot5\sqrt{2}$	3	7/2	4	7/2	$-1/8\cdot3\sqrt{3\cdot7}$
3	5/2	2	3/2	$1/2\cdot3\sqrt{5}$	4	7/2	4	7/2	$\sqrt{5\cdot7}/8\cdot9\sqrt{3}$

Table III. $U \begin{pmatrix} l & l' & 2 \\ j & j' & 2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
2	3/2	0	1/2	$1/2 \cdot 5$	2	3/2	2	5/2	$1/2 \cdot 5\sqrt{2 \cdot 5}$
2	5/2	0	1/2	$1/2 \cdot 5$	2	5/2	2	5/2	$1/5\sqrt{3 \cdot 5}$
1	3/2	1	1/2	$-1/2\sqrt{2 \cdot 3 \cdot 5}$	3	5/2	2	5/2	$1/2 \cdot 5\sqrt{3 \cdot 7}$
2	3/2	1	1/2	$1/2 \cdot 5\sqrt{2}$	3	7/2	2	5/2	$\sqrt{3}/2 \cdot 5\sqrt{2 \cdot 7}$
2	5/2	1	1/2	$-1/3 \cdot 5\sqrt{2}$	4	7/2	2	5/2	$1/2 \cdot 3 \cdot 5\sqrt{2 \cdot 3}$
3	5/2	1	1/2	$1/2 \cdot 3\sqrt{5}$	1	1/2	3	5/2	$1/2 \cdot 3\sqrt{5}$
1	1/2	1	3/2	$1/2\sqrt{2 \cdot 3 \cdot 5}$	1	3/2	3	5/2	$-1/2 \cdot 3\sqrt{5 \cdot 7}$
1	3/2	1	3/2	$1/4\sqrt{3 \cdot 5}$	2	3/2	3	5/2	$1/5\sqrt{2 \cdot 7}$
2	3/2	1	3/2	$1/4 \cdot 5$	2	5/2	3	5/2	$-1/2 \cdot 5\sqrt{3 \cdot 7}$
2	5/2	1	3/2	$\sqrt{7}/2 \cdot 3 \cdot 5\sqrt{2}$	3	5/2	3	5/2	$1/7\sqrt{2 \cdot 5}$
3	5/2	1	3/2	$1/2 \cdot 3\sqrt{5 \cdot 7}$	3	7/2	3	5/2	$-1/4 \cdot 7\sqrt{5}$
3	7/2	1	3/2	$1/2\sqrt{2 \cdot 5 \cdot 7}$	4	7/2	3	5/2	$\sqrt{5}/4 \cdot 3\sqrt{3 \cdot 7}$
0	1/2	2	3/2	$-1/2 \cdot 5$	1	3/2	3	7/2	$1/2\sqrt{2 \cdot 5 \cdot 7}$
1	1/2	2	3/2	$1/2 \cdot 5\sqrt{2}$	2	3/2	3	7/2	$1/4 \cdot 5\sqrt{7}$
1	3/2	2	3/2	$-1/4 \cdot 5$	2	5/2	3	7/2	$\sqrt{3}/2 \cdot 5\sqrt{2 \cdot 7}$
2	3/2	2	3/2	$\sqrt{7}/4 \cdot 5\sqrt{5}$	3	5/2	3	7/2	$1/4 \cdot 7\sqrt{5}$
2	5/2	2	3/2	$-1/2 \cdot 5\sqrt{2 \cdot 5}$	3	7/2	3	7/2	$\sqrt{5}/8 \cdot 7$
3	5/2	2	3/2	$1/5\sqrt{2 \cdot 7}$	4	7/2	3	7/2	$1/8\sqrt{3 \cdot 5 \cdot 7}$
3	7/2	2	3/2	$-1/4 \cdot 5\sqrt{7}$	2	3/2	4	7/2	$1/4 \cdot 5$
4	7/2	2	3/2	$1/4 \cdot 5$	2	5/2	4	7/2	$-1/2 \cdot 3 \cdot 5\sqrt{2 \cdot 3}$
0	1/2	2	5/2	$1/2 \cdot 5$	3	5/2	4	7/2	$\sqrt{5}/4 \cdot 3\sqrt{3 \cdot 7}$
1	1/2	2	5/2	$1/3 \cdot 5\sqrt{2}$	3	7/2	4	7/2	$-1/8\sqrt{3 \cdot 5 \cdot 7}$
1	3/2	2	5/2	$\sqrt{7}/2 \cdot 3 \cdot 5\sqrt{2}$	4	7/2	4	7/2	$\sqrt{11}/8 \cdot 3\sqrt{3 \cdot 5}$

Table IV. $U \begin{pmatrix} l & l' & 3 \\ j & j' & 3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
3	5/2	0	1/2	$1/2 \cdot 7$	3	5/2	1	3/2	$\sqrt{5}/2 \cdot 3 \cdot 7\sqrt{2}$
3	7/2	0	1/2	$1/2 \cdot 7$	3	7/2	1	3/2	$\sqrt{3}/4 \cdot 7\sqrt{2}$
2	5/2	1	1/2	$-1/2 \cdot 3\sqrt{7}$	4	7/2	1	3/2	$1/4 \cdot 3\sqrt{2 \cdot 7}$
3	5/2	1	1/2	$1/3 \cdot 7$	1	3/2	2	3/2	$-1/2\sqrt{2 \cdot 5 \cdot 7}$
3	7/2	1	1/2	$-1/4 \cdot 7$	2	3/2	2	3/2	$1/2 \cdot 5\sqrt{7}$
4	7/2	1	1/2	$1/4\sqrt{3 \cdot 7}$	2	5/2	2	3/2	$-1/2 \cdot 5\sqrt{7}$
2	3/2	1	3/2	$1/2\sqrt{2 \cdot 5 \cdot 7}$	3	5/2	2	3/2	$\sqrt{3}/2 \cdot 7\sqrt{2 \cdot 5}$
2	5/2	1	3/2	$1/3\sqrt{2 \cdot 5 \cdot 7}$	3	7/2	2	3/2	$-1/4 \cdot 7\sqrt{2}$
					4	7/2	2	3/2	$1/4\sqrt{2 \cdot 3 \cdot 7}$

l	j	l'	j'	U	l	j	l'	j'	U
1	1/2	2	5/2	$1/2 \cdot 3\sqrt{7}$	0	1/2	3	7/2	$1/2 \cdot 7$
1	3/2	2	5/2	$1/3\sqrt{2 \cdot 5 \cdot 7}$	1	1/2	3	7/2	$1/4 \cdot 7$
2	3/2	2	5/2	$1/2 \cdot 5\sqrt{7}$	1	3/2	3	7/2	$\sqrt{3}/4 \cdot 7\sqrt{2}$
2	5/2	2	5/2	$1/2 \cdot 5\sqrt{7}$	2	3/2	3	7/2	$1/4 \cdot 7\sqrt{2}$
3	5/2	2	5/2	$1/7\sqrt{2 \cdot 3 \cdot 5}$	2	5/2	3	7/2	$1/4 \cdot 7$
3	7/2	2	5/2	$1/4 \cdot 7$	3	5/2	3	7/2	$1/2 \cdot 7\sqrt{2 \cdot 7}$
4	7/2	2	5/2	$1/4 \cdot 3\sqrt{3 \cdot 7}$	3	7/2	3	7/2	$\sqrt{11}/4 \cdot 7\sqrt{2 \cdot 7}$
0	1/2	3	5/2	$-1/2 \cdot 7$	4	7/2	3	7/2	$1/4 \cdot 7\sqrt{2 \cdot 3}$
1	1/2	3	5/2	$1/3 \cdot 7$	1	1/2	4	7/2	$1/4\sqrt{3 \cdot 7}$
1	3/2	3	5/2	$-\sqrt{5}/2 \cdot 3 \cdot 7\sqrt{2}$	1	3/2	4	7/2	$-1/4 \cdot 3\sqrt{2 \cdot 7}$
2	3/2	3	5/2	$\sqrt{3}/2 \cdot 7\sqrt{2 \cdot 5}$	2	3/2	4	7/2	$1/4\sqrt{2 \cdot 3 \cdot 7}$
2	5/2	3	5/2	$-1/7\sqrt{2 \cdot 3 \cdot 5}$	2	5/2	4	7/2	$-1/4 \cdot 3\sqrt{3 \cdot 7}$
3	5/2	3	5/2	$\sqrt{5}/2 \cdot 7\sqrt{3 \cdot 7}$	3	5/2	4	7/2	$\sqrt{11}/2 \cdot 3 \cdot 7\sqrt{2 \cdot 3}$
3	7/2	3	5/2	$-1/2 \cdot 7\sqrt{2 \cdot 7}$	3	7/2	4	7/2	$-1/4 \cdot 7\sqrt{2 \cdot 3}$
4	7/2	3	5/2	$\sqrt{11}/2 \cdot 3 \cdot 7\sqrt{2 \cdot 3}$	4	7/2	4	7/2	$\sqrt{5}/4 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$

Table V. $U \begin{pmatrix} l & l' & 4 \\ j & j' & 4 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
4	7/2	0	1/2	$1/2 \cdot 9$	2	5/2	3	5/2	$-1/9\sqrt{2 \cdot 7}$
3	7/2	1	1/2	$-1/4 \cdot 3\sqrt{3}$	3	5/2	3	5/2	$\sqrt{11}/2 \cdot 9 \cdot 7$
4	7/2	1	1/2	$\sqrt{5}/4 \cdot 9\sqrt{3}$	3	7/2	3	5/2	$-\sqrt{5}/2 \cdot 3 \cdot 7\sqrt{2 \cdot 3}$
3	5/2	1	3/2	$1/2 \cdot 3\sqrt{2 \cdot 7}$	4	7/2	3	5/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$
3	7/2	1	3/2	$\sqrt{5}/4 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$	1	1/2	3	7/2	$1/4 \cdot 3\sqrt{3}$
4	7/2	1	3/2	$\sqrt{7}/4 \cdot 9\sqrt{2 \cdot 3}$	1	3/2	3	7/2	$\sqrt{5}/4 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$
2	5/2	2	3/2	$-1/2 \cdot 3\sqrt{3 \cdot 5}$	2	3/2	3	7/2	$1/4\sqrt{2 \cdot 5 \cdot 7}$
3	5/2	2	3/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$	2	5/2	3	7/2	$\sqrt{11}/4 \cdot 3\sqrt{3 \cdot 5 \cdot 7}$
3	7/2	2	3/2	$-1/4\sqrt{2 \cdot 5 \cdot 7}$	3	5/2	3	7/2	$\sqrt{5}/2 \cdot 3 \cdot 7\sqrt{2 \cdot 3}$
4	7/2	2	3/2	$\sqrt{11}/4 \cdot 9\sqrt{2 \cdot 5}$	3	7/2	3	7/2	$1/4 \cdot 7\sqrt{2}$
2	3/2	2	5/2	$1/2 \cdot 3\sqrt{3 \cdot 5}$	4	7/2	3	7/2	$\sqrt{5}/4 \cdot 9\sqrt{2 \cdot 7}$
2	5/2	2	5/2	$1/2 \cdot 9\sqrt{5}$	0	1/2	4	7/2	$-1/2 \cdot 9$
3	5/2	2	5/2	$1/9\sqrt{2 \cdot 7}$	1	1/2	4	7/2	$\sqrt{5}/4 \cdot 9\sqrt{3}$
3	7/2	2	5/2	$\sqrt{11}/4 \cdot 3\sqrt{3 \cdot 5 \cdot 7}$	1	3/2	4	7/2	$-\sqrt{7}/4 \cdot 9\sqrt{2 \cdot 3}$
4	7/2	2	5/2	$1/4 \cdot 3\sqrt{3 \cdot 5}$	2	3/2	4	7/2	$\sqrt{11}/4 \cdot 9\sqrt{2 \cdot 5}$
1	3/2	3	5/2	$-1/2 \cdot 3\sqrt{2 \cdot 7}$	2	5/2	4	7/2	$-1/4 \cdot 3\sqrt{3 \cdot 5}$
2	3/2	3	5/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$	3	5/2	4	7/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$
					3	7/2	4	7/2	$-\sqrt{5}/4 \cdot 9\sqrt{2 \cdot 7}$
					4	7/2	4	7/2	$\sqrt{13}/4 \cdot 27\sqrt{2}$

Table VI. $U \begin{pmatrix} l & l' & 1 \\ j & j' & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
1	1/2	0	1/2	$-1/2 \cdot 3$	2	5/2	2	5/2	$-1.9\sqrt{2 \cdot 5}$
0	1/2	1	1/2	$1/2 \cdot 3$	3	5/2	2	5/2	$-1.2 \cdot 9$
1	1/2	1	1/2	$1/3\sqrt{2 \cdot 3}$	2	5/2	3	5/2	$1.2 \cdot 9$
1	3/2	1	3/2	$-1/4 \cdot 3\sqrt{3}$	3	5/2	3	5/2	$1.9\sqrt{7}$
2	3/2	1	3/2	$-1/4 \cdot 3$	3	7/2	3	7/2	$-1.8\sqrt{3 \cdot 7}$
1	3/2	2	3/2	$1/4 \cdot 3$	4	7/2	3	7/2	$-1.8 \cdot 3$
2	3/2	2	3/2	$1/4\sqrt{3 \cdot 5}$	3	7/2	4	7/2	$1.8 \cdot 3$
					4	7/2	4	7/2	$\sqrt{5}/8 \cdot 9$

Table VII. $U \begin{pmatrix} l & l' & 0 \\ j & j' & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
0	1/2	0	1/2	$1/2\sqrt{3}$	2	3/2	2	5/2	$-1.3 \cdot 5$
1	1/2	1	1/2	$-1/2 \cdot 9$	2	5/2	2	5/2	$\sqrt{7}/2 \cdot 3 \cdot 5\sqrt{3}$
1	3/2	1	1/2	$1/9$	3	5/2	3	5/2	$-\sqrt{5}/2 \cdot 3 \cdot 7\sqrt{3}$
1	1/2	1	3/2	$-1/9$	3	7/2	3	5/2	$1/3 \cdot 7$
1	3/2	1	3/2	$\sqrt{5}/2 \cdot 9\sqrt{2}$	3	5/2	3	7/2	$-1/3 \cdot 7$
2	3/2	2	3/2	$-1/2 \cdot 5\sqrt{2 \cdot 3}$	3	7/2	3	7/2	$1/4 \cdot 7$
2	5/2	2	3/2	$1/3 \cdot 5$	4	7/2	4	7/2	$-\sqrt{7}/4 \cdot 27$

Table VIII. $U \begin{pmatrix} l & l' & 1 \\ j & j' & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
1	1/2	0	1/2	$1/3\sqrt{2 \cdot 3}$	1	3/2	2	5/2	$1/4 \cdot 3\sqrt{5}$
1	3/2	0	1/2	$-1/2 \cdot 3\sqrt{2 \cdot 3}$	2	3/2	2	5/2	$1/4 \cdot 3\sqrt{3}$
0	1/2	1	1/2	$1/3\sqrt{2 \cdot 3}$	3	5/2	2	5/2	$1/3\sqrt{2 \cdot 5 \cdot 7}$
1	3/2	1	1/2	$1/4 \cdot 3$	3	7/2	2	5/2	$-1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$
2	3/2	1	1/2	$1/4 \cdot 3\sqrt{3}$	2	3/2	3	5/2	$-1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 5}$
0	1/2	1	3/2	$1/2 \cdot 3\sqrt{2 \cdot 3}$	2	5/2	3	5/2	$1/3\sqrt{2 \cdot 5 \cdot 7}$
1	1/2	1	3/2	$1/4 \cdot 3$	3	7/2	3	5/2	$1/4 \cdot 3\sqrt{2 \cdot 3}$
2	3/2	1	3/2	$1/3\sqrt{2 \cdot 3 \cdot 5}$	4	7/2	3	5/2	$1/4 \cdot 3\sqrt{2 \cdot 7}$
2	5/2	1	3/2	$-1/4 \cdot 3\sqrt{5}$	2	5/2	3	7/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 7}$
1	1/2	2	3/2	$-1/4 \cdot 3\sqrt{3}$	3	5/2	3	7/2	$1/4 \cdot 3\sqrt{2 \cdot 3}$
1	3/2	2	3/2	$1/3\sqrt{2 \cdot 3 \cdot 5}$	4	7/2	3	7/2	$1/9\sqrt{2 \cdot 7}$
2	5/2	2	3/2	$1/4 \cdot 3\sqrt{3}$	3	5/2	4	7/2	$-1/4 \cdot 3\sqrt{2 \cdot 7}$
3	5/2	2	3/2	$1/2 \cdot 3\sqrt{2 \cdot 3 \cdot 5}$	3	7/2	4	7/2	$1/9\sqrt{2 \cdot 7}$

Table IX. $U \begin{pmatrix} l & l' & 2 \\ j & j' & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
2	3/2	0	1/2	$-1/2\sqrt{2\cdot3\cdot5}$	2	5/2	2	5/2	$-1/3\cdot5\sqrt{3\cdot5}$
1	1/2	1	1/2	1/9	3	5/2	2	5/2	$-2/3\cdot5\sqrt{3\cdot7}$
1	3/2	1	1/2	1/4\cdot9	3	7/2	2	5/2	$-1/4\cdot3\sqrt{5\cdot7}$
2	3/2	1	1/2	$1/4\sqrt{3\cdot5}$	4	7/2	2	5/2	$-1/4\cdot3\sqrt{5}$
1	1/2	1	3/2	$-1/4\cdot9$	1	3/2	3	5/2	$1/2\cdot3\sqrt{2\cdot5}$
1	3/2	1	3/2	$-1/2\cdot9\sqrt{2\cdot5}$	2	3/2	3	5/2	$1/2\cdot3\cdot5$
2	3/2	1	3/2	$-1/2\cdot5\sqrt{2\cdot3}$	2	5/2	3	5/2	$2/3\cdot5\sqrt{3\cdot7}$
2	5/2	1	3/2	$-1/4\cdot3\cdot5$	3	5/2	3	5/2	$\sqrt{2}/3\cdot7\sqrt{5}$
3	5/2	1	3/2	$-1/2\cdot3\sqrt{2\cdot5}$	3	7/2	3	5/2	$1/4\cdot7\sqrt{2\cdot3}$
0	1/2	2	3/2	$1/2\sqrt{2\cdot3\cdot5}$	4	7/2	3	5/2	$1/4\cdot3\sqrt{2\cdot7}$
1	1/2	2	3/2	$1/4\sqrt{3\cdot5}$	2	5/2	3	7/2	$-1/4\cdot3\sqrt{5\cdot7}$
1	3/2	2	3/2	$1/2\cdot5\sqrt{2\cdot3}$	3	5/2	3	7/2	$-1/4\cdot7\sqrt{2\cdot3}$
2	3/2	2	3/2	$\sqrt{7}/2\cdot5\sqrt{2\cdot3\cdot5}$	3	7/2	3	7/2	$-1/4\cdot7\sqrt{2\cdot3}$
2	5/2	2	3/2	$\sqrt{7}/4\cdot3\cdot5\sqrt{5}$	4	7/2	3	7/2	$-1/4\cdot3\sqrt{2\cdot7}$
3	5/2	2	3/2	$1/2\cdot3\cdot5$	2	5/2	4	7/2	$1/4\cdot3\sqrt{5}$
1	3/2	2	5/2	$-1/4\cdot3\cdot5$	3	5/2	4	7/2	$1/4\cdot3\sqrt{2\cdot7}$
2	3/2	2	5/2	$-\sqrt{7}/4\cdot3\cdot5\sqrt{5}$	3	7/2	4	7/2	$1/4\cdot3\sqrt{2\cdot7}$
					4	7/2	4	7/2	$\sqrt{11}/4\cdot27\sqrt{2}$

Table X. $U \begin{pmatrix} l & l' & 1 \\ j & j' & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
1	3/2	0	1/2	$1/2\cdot3\sqrt{2}$	2	3/2	2	5/2	$-\sqrt{7}/4\cdot5\sqrt{3\cdot5}$
1	3/2	1	1/2	$1/4\cdot3\sqrt{3}$	2	5/2	2	5/2	$\sqrt{2\cdot7}/9\cdot5\sqrt{5}$
2	3/2	1	1/2	$-1/4\cdot3\cdot5$	3	5/2	2	5/2	$-2/9\cdot5\sqrt{7}$
2	5/2	1	1/2	$1/3\cdot5$	3	7/2	2	5/2	$1/2\cdot5\sqrt{2\cdot7}$
0	1/2	1	3/2	$1/2\cdot3\sqrt{2}$	2	3/2	3	5/2	$-1/2\cdot5\sqrt{2\cdot3\cdot7}$
1	1/2	1	3/2	$-1/4\cdot3\sqrt{3}$	2	5/2	3	5/2	$2/9\cdot5\sqrt{7}$
1	3/2	1	3/2	$1/2\cdot3\sqrt{2\cdot3}$	3	5/2	3	5/2	$-1/9\cdot7$
2	3/2	1	3/2	$-1/2\cdot3\cdot5\sqrt{2}$	3	7/2	3	5/2	$1/4\cdot7\sqrt{2}$
2	5/2	1	3/2	$\sqrt{7}/4\cdot3\cdot5$	4	7/2	3	5/2	$-1/4\cdot3\sqrt{2\cdot3\cdot7}$
1	1/2	2	3/2	$-1/4\cdot3\cdot5$	2	3/2	3	7/2	$-1/5\sqrt{3\cdot7}$
1	3/2	2	3/2	$1/2\cdot3\cdot5\sqrt{2}$	2	5/2	3	7/2	$1/2\cdot5\sqrt{2\cdot7}$
2	3/2	2	3/2	$-1/2\cdot5\sqrt{2\cdot3\cdot5}$	3	5/2	3	7/2	$-1/4\cdot7\sqrt{2}$
2	5/2	2	3/2	$\sqrt{7}/4\cdot5\sqrt{3\cdot5}$	3	7/2	3	7/2	$1/4\cdot7\sqrt{2}$
3	5/2	2	3/2	$-1/2\cdot5\sqrt{2\cdot3\cdot7}$	4	7/2	3	7/2	$-1/4\cdot3\sqrt{2\cdot3\cdot7}$
3	7/2	2	3/2	$1/5\sqrt{3\cdot7}$	3	5/2	4	7/2	$-1/4\cdot3\sqrt{2\cdot3\cdot7}$
1	1/2	2	5/2	$-1/3\cdot5$	3	7/2	4	7/2	$1\cdot4\cdot3\sqrt{2\cdot3\cdot7}$
1	3/2	2	5/2	$\sqrt{7}/4\cdot3\cdot5$	4	7/2	4	7/2	$-\sqrt{7}/4\cdot9\sqrt{2\cdot3\cdot5}$

Table XI. $U \begin{pmatrix} l & l' & 2 \\ j & j' & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
2	3/2	0	1/2	$1/2 \cdot 5\sqrt{2}$	1	3/2	2	5/2	$\sqrt{7}/4 \cdot 9 \cdot 5$
2	5/2	0	1/2	$-1/3 \cdot 5\sqrt{2}$	2	3/2	2	5/2	$1/4 \cdot 3\sqrt{5}$
1	3/2	1	1/2	$1/4\sqrt{3 \cdot 5}$	3	5/2	2	5/2	$1/5\sqrt{2 \cdot 3 \cdot 7}$
2	3/2	1	1/2	$1/4 \cdot 3 \cdot 5$	3	7/2	2	5/2	$-1/4 \cdot 5\sqrt{3 \cdot 7}$
2	5/2	1	1/2	$2/9 \cdot 5$	4	7/2	2	5/2	$7/4 \cdot 9 \cdot 5\sqrt{3}$
3	5/2	1	1/2	$1/9\sqrt{2 \cdot 5}$	1	1/2	3	5/2	$-1/9\sqrt{2 \cdot 5}$
1	1/2	1	3/2	$1/4\sqrt{3 \cdot 5}$	1	3/2	3	5/2	$\sqrt{5}/2 \cdot 9\sqrt{2 \cdot 7}$
2	3/2	1	3/2	$1/3 \cdot 5\sqrt{2}$	2	3/2	3	5/2	$-1/2 \cdot 3 \cdot 5\sqrt{7}$
2	5/2	1	3/2	$-\sqrt{7}/4 \cdot 9 \cdot 5$	2	5/2	3	5/2	$1/5\sqrt{2 \cdot 3 \cdot 7}$
3	5/2	1	3/2	$\sqrt{5}/2 \cdot 9\sqrt{2 \cdot 7}$	3	7/2	3	5/2	$1/4 \cdot 3\sqrt{2 \cdot 5}$
3	7/2	1	3/2	$-1/2 \cdot 3\sqrt{5 \cdot 7}$	4	7/2	3	5/2	$\sqrt{5}/4 \cdot 9\sqrt{2 \cdot 3 \cdot 7}$
0	1/2	2	3/2	$1/2 \cdot 5\sqrt{2}$	1	3/2	3	7/2	$1/2 \cdot 3\sqrt{5 \cdot 7}$
1	1/2	2	3/2	$-1/4 \cdot 3 \cdot 5$	2	3/2	3	7/2	$1/2 \cdot 5\sqrt{2 \cdot 7}$
1	3/2	2	3/2	$1/3 \cdot 5\sqrt{2}$	2	5/2	3	7/2	$1/4 \cdot 5\sqrt{3 \cdot 7}$
2	5/2	2	3/2	$1/4 \cdot 3\sqrt{5}$	3	5/2	3	7/2	$1/4 \cdot 3\sqrt{2 \cdot 5}$
3	5/2	2	3/2	$1/2 \cdot 3 \cdot 5\sqrt{7}$	4	7/2	3	7/2	$1/3\sqrt{2 \cdot 3 \cdot 5 \cdot 7}$
3	7/2	2	3/2	$1/2 \cdot 5\sqrt{2 \cdot 7}$	2	3/2	4	7/2	$-1/2 \cdot 3 \cdot 5\sqrt{2}$
4	7/2	2	3/2	$1/2 \cdot 3 \cdot 5\sqrt{2}$	2	5/2	4	7/2	$7 \cdot 4 \cdot 9 \cdot 5\sqrt{3}$
0	1/2	2	5/2	$1/3 \cdot 5\sqrt{2}$	3	5/2	4	7/2	$-\sqrt{5}/4 \cdot 9\sqrt{2 \cdot 3 \cdot 7}$
1	1/2	2	5/2	$2/9 \cdot 5$	3	7/2	4	7/2	$1/3\sqrt{2 \cdot 3 \cdot 5 \cdot 7}$

Table XII. $U \begin{pmatrix} l & l' & 3 \\ j & j' & 2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
3	5/2	0	1/2	$-1/2 \cdot 3\sqrt{7}$	2	5/2	2	3/2	$\sqrt{2}/3 \cdot 5\sqrt{5 \cdot 7}$
2	3/2	1	1/2	$1/3 \cdot 5$	3	5/2	2	3/2	$2/5 \cdot 7\sqrt{3}$
2	5/2	1	1/2	$1/2 \cdot 9 \cdot 5$	3	7/2	2	3/2	$\sqrt{3}/4 \cdot 5 \cdot 7\sqrt{2}$
3	5/2	1	1/2	$1/9\sqrt{7}$	4	7/2	2	3/2	$1/4 \cdot 3\sqrt{2 \cdot 7}$
2	3/2	1	3/2	$-1/2 \cdot 3 \cdot 5\sqrt{2}$	1	1/2	2	5/2	$-1/2 \cdot 9 \cdot 5$
2	5/2	1	3/2	$-1/9 \cdot 5\sqrt{7}$	1	3/2	2	5/2	$-1/9 \cdot 5\sqrt{7}$
3	5/2	1	3/2	$-2/9 \cdot 7$	2	3/2	2	5/2	$-\sqrt{2}/3 \cdot 5\sqrt{5 \cdot 7}$
3	7/2	1	3/2	$-1/4 \cdot 3 \cdot 7\sqrt{2}$	2	5/2	2	5/2	$-1/2 \cdot 5\sqrt{3 \cdot 5 \cdot 7}$
4	7/2	1	3/2	$-1/4\sqrt{2 \cdot 3 \cdot 7}$	3	5/2	2	5/2	$-1/5 \cdot 7\sqrt{2}$
1	1/2	2	3/2	$1/3 \cdot 5$	3	7/2	2	5/2	$-1/3 \cdot 5 \cdot 7$
1	3/2	2	3/2	$1/2 \cdot 3 \cdot 5\sqrt{2}$	4	7/2	2	5/2	$-1/9\sqrt{3 \cdot 7}$
2	3/2	2	3/2	$1/2 \cdot 5\sqrt{5}$	0	1/2	3	5/2	$1/2 \cdot 3\sqrt{7}$

l	j	l'	j'	U	l	j	l'	j'	U
1	1/2	3	5/2	$1/9\sqrt{7}$	3	5/2	3	7/2	$-1/3\cdot7\sqrt{2\cdot7}$
1	3/2	3	5/2	$2/9\cdot7$	3	7/2	3	7/2	$-1/4\cdot7\sqrt{2\cdot7}$
2	3/2	3	5/2	$2/5\cdot7\sqrt{3}$	4	7/2	3	7/2	$-\sqrt{11}/4\cdot3\cdot7\sqrt{2\cdot3}$
2	5/2	3	5/2	$1/5\cdot7\sqrt{2}$	1	3/2	4	7/2	$1/4\sqrt{2\cdot3\cdot7}$
3	5/2	3	5/2	$1/2\cdot7\sqrt{7}$	2	3/2	4	7/2	$1/4\cdot3\sqrt{2\cdot7}$
3	7/2	3	5/2	$1/3\cdot7\sqrt{2\cdot7}$	2	5/2	4	7/2	$1/9\sqrt{3\cdot7}$
4	7/2	3	5/2	$\sqrt{11}/9\cdot7\sqrt{2\cdot3}$	3	5/2	4	7/2	$\sqrt{11}/9\cdot7\sqrt{2\cdot3}$
1	3/2	3	7/2	$1/4\cdot3\cdot7\sqrt{2}$	3	7/2	4	7/2	$\sqrt{11}/4\cdot3\cdot7\sqrt{2\cdot3}$
2	3/2	3	7/2	$-\sqrt{3}/4\cdot5\cdot7\sqrt{2}$	4	7/2	4	7/2	$\sqrt{11}/4\cdot3\sqrt{2\cdot3\cdot5\cdot7}$
2	5/2	3	7/2	$-1/3\cdot5\cdot7$					

Table XIII. $U \begin{pmatrix} l & l' & 2 \\ j & j' & 3 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
2	5/2	0	1/2	$1/2\cdot3\sqrt{5}$	3	7/2	2	5/2	$1/7\sqrt{2\cdot3\cdot5}$
2	5/2	1	1/2	$1/9\sqrt{2\cdot5}$	4	7/2	2	5/2	$-1/9\sqrt{2\cdot3\cdot5\cdot7}$
3	5/2	1	1/2	$-1/2\cdot9\cdot7$	1	1/2	3	5/2	$-1/2\cdot9\cdot7$
3	7/2	1	1/2	$1/3\cdot7$	1	3/2	3	5/2	$\sqrt{2}/9\cdot7\sqrt{5}$
1	3/2	1	3/2	$1/4\sqrt{3\cdot5}$	2	3/2	3	5/2	$-1/3\cdot5\cdot7$
2	3/2	1	3/2	$-1/4\cdot3\cdot5$	2	5/2	3	5/2	$1/2\cdot5\cdot7$
2	5/2	1	3/2	$2/9\cdot5$	3	5/2	3	5/2	$-1/7\sqrt{2\cdot3\cdot5\cdot7}$
3	5/2	1	3/2	$-\sqrt{2}/9\cdot7\sqrt{5}$	3	7/2	3	5/2	$1/3\cdot7\sqrt{7}$
3	7/2	1	3/2	$1/2\cdot7\sqrt{2\cdot3}$	4	7/2	3	5/2	$-1/9\cdot7\sqrt{3}$
1	3/2	2	3/2	$1/4\cdot3\cdot5$	1	1/2	3	7/2	$-1/3\cdot7$
2	3/2	2	3/2	$-1/4\cdot5\sqrt{5\cdot7}$	1	3/2	3	7/2	$1/2\cdot7\sqrt{2\cdot3}$
2	5/2	2	3/2	$2/3\cdot5\sqrt{5\cdot7}$	2	3/2	3	7/2	$-\sqrt{3}/4\cdot7\sqrt{5}$
3	5/2	2	3/2	$-1/3\cdot5\cdot7$	2	5/2	3	7/2	$1/7\sqrt{2\cdot3\cdot5}$
3	7/2	2	3/2	$\sqrt{3}/4\cdot7\sqrt{5}$	3	5/2	3	7/2	$-1/3\cdot7\sqrt{7}$
4	7/2	2	3/2	$-1/4\cdot3\sqrt{3\cdot5\cdot7}$	3	7/2	3	7/2	$\sqrt{11}/8\cdot7\sqrt{7}$
0	1/2	2	5/2	$1/2\cdot3\sqrt{5}$	4	7/2	3	7/2	$-\sqrt{11}/8\cdot3\cdot7\sqrt{3}$
1	1/2	2	5/2	$-1/9\sqrt{2\cdot5}$	2	3/2	4	7/2	$-1/4\cdot3\sqrt{3\cdot5\cdot7}$
1	3/2	2	5/2	$2/9\cdot5$	2	5/2	4	7/2	$1/9\sqrt{2\cdot3\cdot5\cdot7}$
2	3/2	2	5/2	$-2/3\cdot5\sqrt{5\cdot7}$	3	5/2	4	7/2	$-1/9\cdot7\sqrt{3}$
2	5/2	2	5/2	$1/5\sqrt{5\cdot7}$	3	7/2	4	7/2	$\sqrt{11}/8\cdot3\cdot7\sqrt{3}$
3	5/2	2	5/2	$-1/2\cdot5\cdot7$	4	7/2	4	7/2	$-1/8\cdot3\sqrt{3\cdot7}$

Table XIV. $U \begin{pmatrix} l & l' & 3 \\ j & j' & 3 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
3	5/2	0	1/2	1/3·7	4	7/2	2	5/2	$\sqrt{7/8} \cdot 9\sqrt{3}$
3	7/2	0	1/2	-1/4·7	0	1/2	3	5/2	1/3·7
2	5/2	1	1/2	1/9 $\sqrt{7}$	1	1/2	3	5/2	-1/9·7
3	5/2	1	1/2	1/9·7	1	3/2	3	5/2	5 $\sqrt{5/4} \cdot 9 \cdot 7\sqrt{2}$
3	7/2	1	1/2	5/8·3·7	2	3/2	3	5/2	-1/4·7 $\sqrt{2 \cdot 3 \cdot 5}$
4	7/2	1	1/2	1/8 $\sqrt{3 \cdot 7}$	2	5/2	3	5/2	1/7 $\sqrt{2 \cdot 3 \cdot 5}$
2	3/2	1	3/2	1/3 $\sqrt{2 \cdot 5 \cdot 7}$	3	7/2	3	5/2	1/4·3 $\sqrt{2 \cdot 7}$
2	5/2	1	3/2	-1/2·9 $\sqrt{2 \cdot 5 \cdot 7}$	4	7/2	3	5/2	$\sqrt{11/4} \cdot 9 \cdot 7\sqrt{2 \cdot 3}$
3	5/2	1	3/2	5 $\sqrt{5/4} \cdot 9 \cdot 7\sqrt{2}$	0	1/2	3	7/2	1/4·7
3	7/2	1	3/2	-1/4·7 $\sqrt{2 \cdot 3}$	1	1/2	3	7/2	5/8·3·7
4	7/2	1	3/2	1/4·3 $\sqrt{2 \cdot 7}$	1	3/2	3	7/2	1/4·7 $\sqrt{2 \cdot 3}$
1	3/2	2	3/2	1/3 $\sqrt{2 \cdot 5 \cdot 7}$	2	3/2	3	7/2	1/4·7 $\sqrt{2}$
2	5/2	2	3/2	1/4·3 $\sqrt{7}$	2	5/2	3	7/2	1/8·3·7
3	5/2	2	3/2	1/4·7 $\sqrt{2 \cdot 3 \cdot 5}$	3	5/2	3	7/2	1/4·3 $\sqrt{2 \cdot 7}$
3	7/2	2	3/2	1/4·7 $\sqrt{2}$	4	7/2	3	7/2	1/3·7 $\sqrt{2 \cdot 3}$
4	7/2	2	3/2	1/4·3 $\sqrt{2 \cdot 3 \cdot 7}$	1	1/2	4	7/2	-1/8 $\sqrt{3 \cdot 7}$
1	1/2	2	5/2	1/9 $\sqrt{7}$	1	3/2	4	7/2	1/4·3 $\sqrt{2 \cdot 7}$
1	3/2	2	5/2	1/2·9 $\sqrt{2 \cdot 5 \cdot 7}$	2	3/2	4	7/2	-1/4·3 $\sqrt{2 \cdot 3 \cdot 7}$
2	3/2	2	5/2	1/4·3 $\sqrt{7}$	2	5/2	4	7/2	$\sqrt{7/8} \cdot 9\sqrt{3}$
3	5/2	2	5/2	1/7 $\sqrt{2 \cdot 3 \cdot 5}$	3	5/2	4	7/2	- $\sqrt{11/4} \cdot 9 \cdot 7\sqrt{2 \cdot 3}$
3	7/2	2	5/2	-1/8·3·7	3	7/2	4	7/2	1/3·7 $\sqrt{2 \cdot 3}$

Table XV. $U \begin{pmatrix} l & l' & 4 \\ j & j' & 3 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
4	7/2	0	1/2	-1/4·3 $\sqrt{3}$	3	7/2	2	3/2	1/4·3·7 $\sqrt{2}$
3	5/2	1	1/2	1/3·7	4	7/2	2	3/2	$\sqrt{11/4} \cdot 9\sqrt{2 \cdot 7}$
3	7/2	1	1/2	1/8·3·7	2	3/2	2	5/2	-1/4·5 $\sqrt{3 \cdot 7}$
4	7/2	1	1/2	$\sqrt{5/8} \cdot 9$	2	5/2	2	5/2	-1/9·5 $\sqrt{3 \cdot 7}$
3	5/2	1	3/2	- $\sqrt{5/4} \cdot 3 \cdot 7\sqrt{2}$	3	5/2	2	5/2	- $\sqrt{5/9} \cdot 9 \cdot 7\sqrt{2 \cdot 3}$
3	7/2	1	3/2	-1/4·3·7 $\sqrt{2 \cdot 3}$	3	7/2	2	5/2	-1/8·3·7
4	7/2	1	3/2	- $\sqrt{5/4} \cdot 9\sqrt{2 \cdot 3}$	4	7/2	2	5/2	- $\sqrt{11/8} \cdot 9\sqrt{7}$
2	3/2	2	3/2	1/5 $\sqrt{3 \cdot 7}$	1	1/2	3	5/2	1/3·7
2	5/2	2	3/2	1/4·5 $\sqrt{3 \cdot 7}$	1	3/2	3	5/2	$\sqrt{5/4} \cdot 3 \cdot 7\sqrt{2}$
3	5/2	2	3/2	$\sqrt{5/4} \cdot 9 \cdot 7\sqrt{2 \cdot 3}$	2	3/2	3	5/2	$\sqrt{5/4} \cdot 7\sqrt{2 \cdot 3}$
					2	5/2	3	5/2	$\sqrt{5/9} \cdot 9 \cdot 7\sqrt{2 \cdot 3}$

l	j	l'	j'	U	l	j	l'	j'	U
3	5/2	3	5/2	$\sqrt{5 \cdot 11/9 \cdot 7} \sqrt{3 \cdot 7}$	4	7/2	3	7/2	$-\sqrt{5/2 \cdot 9 \cdot 7} \sqrt{2}$
3	7/2	3	5/2	$\sqrt{11/4 \cdot 3 \cdot 7} \sqrt{2 \cdot 7}$	0	1/2	4	7/2	$1/4 \cdot 3 \sqrt{3}$
4	7/2	3	5/2	$\sqrt{5 \cdot 11/4 \cdot 9 \cdot 7} \sqrt{2}$	1	1/2	4	7/2	$\sqrt{5/8 \cdot 9}$
1	1/2	3	7/2	$-1/8 \cdot 3 \cdot 7$	1	3/2	4	7/2	$\sqrt{5/4 \cdot 9 \cdot 7} \sqrt{2 \cdot 3}$
1	3/2	3	7/2	$-1/4 \cdot 3 \cdot 7 \sqrt{2 \cdot 3}$	2	3/2	4	7/2	$\sqrt{11/4 \cdot 9 \cdot 7} \sqrt{2 \cdot 7}$
2	3/2	3	7/2	$-1/4 \cdot 3 \cdot 7 \sqrt{2}$	2	5/2	4	7/2	$\sqrt{11/8 \cdot 9 \cdot 7}$
2	5/2	3	7/2	$-1/8 \cdot 3 \cdot 7$	3	5/2	4	7/2	$\sqrt{5 \cdot 11/4 \cdot 9 \cdot 7} \sqrt{2}$
3	5/2	3	7/2	$-\sqrt{11/4 \cdot 3 \cdot 7} \sqrt{2 \cdot 7}$	3	7/2	4	7/2	$\sqrt{5/2 \cdot 9 \cdot 7} \sqrt{2}$
3	7/2	3	7/2	$-1/2 \cdot 3 \cdot 7 \sqrt{2 \cdot 7}$	4	7/2	4	7/2	$\sqrt{13/2 \cdot 27} \sqrt{2 \cdot 7}$

Table XVI. $U \begin{pmatrix} l & l' & 3 \\ j & j' & 4 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
3	7/2	0	1/2	$1/4 \sqrt{3 \cdot 7}$	2	3/2	3	5/2	$-1/4 \cdot 3 \cdot 7 \sqrt{2 \cdot 3}$
3	7/2	1	1/2	$1/8 \sqrt{3 \cdot 7}$	2	5/2	3	5/2	$1/9 \cdot 7 \sqrt{2}$
4	7/2	1	1/2	$-1/8 \cdot 27$	3	5/2	3	5/2	$-1/9 \cdot 7 \sqrt{7}$
2	5/2	1	3/2	$1/2 \cdot 3 \sqrt{2 \cdot 7}$	3	7/2	3	5/2	$\sqrt{5 \cdot 11/4 \cdot 3 \cdot 7} \sqrt{2 \cdot 3 \cdot 7}$
3	5/2	1	3/2	$-1/4 \cdot 3 \cdot 7 \sqrt{2}$	4	7/2	3	5/2	$-\sqrt{5/4 \cdot 9 \cdot 7} \sqrt{2}$
3	7/2	1	3/2	$\sqrt{5/4 \cdot 7} \sqrt{2 \cdot 3}$	0	1/2	3	7/2	$1/4 \sqrt{3 \cdot 7}$
4	7/2	1	3/2	$-\sqrt{5/4 \cdot 27} \sqrt{2 \cdot 7}$	1	1/2	3	7/2	$-1/8 \sqrt{3 \cdot 7}$
2	5/2	2	3/2	$1/4 \cdot 3 \sqrt{5 \cdot 7}$	1	3/2	3	7/2	$\sqrt{5/4 \cdot 7} \sqrt{2 \cdot 3}$
3	5/2	2	3/2	$-1/4 \cdot 3 \cdot 7 \sqrt{2 \cdot 3}$	2	3/2	3	7/2	$-\sqrt{5/4 \cdot 3 \cdot 7} \sqrt{2}$
3	7/2	2	3/2	$\sqrt{5/4 \cdot 3 \cdot 7} \sqrt{2}$	2	5/2	3	7/2	$\sqrt{5 \cdot 11/8 \cdot 3 \cdot 7} \sqrt{3}$
4	7/2	2	3/2	$-1/4 \cdot 3 \sqrt{2 \cdot 3 \cdot 5 \cdot 7}$	3	5/2	3	7/2	$-\sqrt{5 \cdot 11/4 \cdot 3 \cdot 7} \sqrt{2 \cdot 3 \cdot 7}$
1	3/2	2	5/2	$1/2 \cdot 3 \sqrt{2 \cdot 7}$	3	7/2	3	7/2	$\sqrt{11/2 \cdot 3 \cdot 7} \sqrt{2 \cdot 7}$
2	3/2	2	5/2	$-1/4 \cdot 3 \sqrt{5 \cdot 7}$	4	7/2	3	7/2	$-1/2 \cdot 3 \cdot 7 \sqrt{2 \cdot 3}$
2	5/2	2	5/2	$1/3 \sqrt{3 \cdot 5 \cdot 7}$	1	1/2	4	7/2	$-1/8 \cdot 27$
3	5/2	2	5/2	$-1/9 \cdot 7 \sqrt{2}$	1	3/2	4	7/2	$\sqrt{5/4 \cdot 27} \sqrt{2 \cdot 7}$
3	7/2	2	5/2	$\sqrt{5 \cdot 11/8 \cdot 3 \cdot 7} \sqrt{3}$	2	3/2	4	7/2	$-1/4 \cdot 3 \sqrt{2 \cdot 3 \cdot 5 \cdot 7}$
4	7/2	2	5/2	$-\sqrt{11/8 \cdot 9 \cdot 7} \sqrt{5 \cdot 7}$	2	5/2	4	7/2	$\sqrt{11/8 \cdot 9 \cdot 7} \sqrt{5 \cdot 7}$
1	3/2	3	5/2	$1/4 \cdot 3 \cdot 7 \sqrt{2}$	3	5/2	4	7/2	$-\sqrt{5/4 \cdot 9 \cdot 7} \sqrt{2}$
					3	7/2	4	7/2	$1/2 \cdot 3 \cdot 7 \sqrt{2 \cdot 3}$
					4	7/2	4	7/2	$-\sqrt{5/2 \cdot 27} \sqrt{2 \cdot 3 \cdot 7}$

Table XVII. $U \begin{pmatrix} l & l' & 4 \\ j & j' & 4 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
4	7/2	0	1/2	$\sqrt{5/4 \cdot 9\sqrt{3}}$	2	3/2	3	5/2	$-1/4 \cdot 9\sqrt{2 \cdot 5 \cdot 7}$
3	7/2	1	1/2	$\sqrt{5/8 \cdot 9}$	2	5/2	3	5/2	$1/3\sqrt{2 \cdot 3 \cdot 5 \cdot 7}$
4	7/2	1	1/2	$1/8 \cdot 9$	3	7/2	3	5/2	$1/4 \cdot 9\sqrt{2}$
3	5/2	1	3/2	$\sqrt{5/4 \cdot 3\sqrt{2 \cdot 3 \cdot 7}}$	4	7/2	3	5/2	$1/4 \cdot 9\sqrt{2 \cdot 5 \cdot 7}$
3	7/2	1	3/2	$-1/4 \cdot 9\sqrt{2 \cdot 7}$	1	1/2	3	7/2	$\sqrt{5/8 \cdot 9}$
4	7/2	1	3/2	$\sqrt{7/4 \cdot 9\sqrt{2 \cdot 5}}$	1	3/2	3	7/2	$1/4 \cdot 9\sqrt{2 \cdot 7}$
2	5/2	2	3/2	$1/4 \cdot 9$	2	3/2	3	7/2	$\sqrt{3/4 \cdot 5\sqrt{2 \cdot 7}}$
3	5/2	2	3/2	$1/4 \cdot 9\sqrt{2 \cdot 5 \cdot 7}$	2	5/2	3	7/2	$\sqrt{11/8 \cdot 9 \cdot 5\sqrt{7}}$
3	7/2	2	3/2	$\sqrt{3/4 \cdot 5\sqrt{2 \cdot 7}}$	3	5/2	3	7/2	$1/4 \cdot 9\sqrt{2}$
4	7/2	2	3/2	$\sqrt{11/4 \cdot 9 \cdot 5\sqrt{2 \cdot 3}}$	4	7/2	3	7/2	$1/9\sqrt{2 \cdot 3 \cdot 7}$
2	3/2	2	5/2	$1/4 \cdot 9$	0	1/2	4	7/2	$\sqrt{5/4 \cdot 9\sqrt{3}}$
3	5/2	2	5/2	$1/3\sqrt{2 \cdot 3 \cdot 5 \cdot 7}$	1	1/2	4	7/2	$-1/8 \cdot 9$
3	7/2	2	5/2	$-\sqrt{11/8 \cdot 9 \cdot 5\sqrt{7}}$	1	3/2	4	7/2	$\sqrt{7/4 \cdot 9\sqrt{2 \cdot 5}}$
4	7/2	2	5/2	$7/8 \cdot 9 \cdot 5$	2	3/2	4	7/2	$-\sqrt{11/4 \cdot 9 \cdot 5\sqrt{2 \cdot 3}}$
1	3/2	3	5/2	$\sqrt{5/4 \cdot 3\sqrt{2 \cdot 3 \cdot 7}}$	2	5/2	4	7/2	$7/8 \cdot 9 \cdot 5$
					3	5/2	4	7/2	$-1/4 \cdot 9\sqrt{2 \cdot 5 \cdot 7}$
					3	7/2	4	7/2	$1/9\sqrt{2 \cdot 3 \cdot 7}$

Table XVIII. $U \begin{pmatrix} l & l' & 5 \\ j & j' & 4 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
4	7/2	1	1/2	$1/27$	4	7/2	3	5/2	$\sqrt{13/9\sqrt{5 \cdot 7 \cdot 11}}$
4	7/2	1	3/2	$-\sqrt{7/2 \cdot 27\sqrt{2 \cdot 5}}$	2	3/2	3	7/2	$-1/4 \cdot 3 \cdot 5\sqrt{7}$
3	5/2	2	3/2	$1/3\sqrt{3 \cdot 5 \cdot 7}$	2	5/2	3	7/2	$-1/3 \cdot 5\sqrt{2 \cdot 3 \cdot 7 \cdot 11}$
3	7/2	2	3/2	$1/4 \cdot 3 \cdot 5\sqrt{7}$	3	5/2	3	7/2	$-1/3 \cdot 7\sqrt{3 \cdot 11}$
4	7/2	2	3/2	$\sqrt{7/4 \cdot 3 \cdot 5\sqrt{3}}$	3	7/2	3	7/2	$-\sqrt{5/8 \cdot 3 \cdot 7\sqrt{11}}$
3	5/2	2	5/2	$-1/2 \cdot 9\sqrt{5 \cdot 7}$	4	7/2	3	7/2	$-\sqrt{13/8 \cdot 3\sqrt{3 \cdot 7 \cdot 11}}$
3	7/2	2	5/2	$-1/3 \cdot 5\sqrt{2 \cdot 3 \cdot 7 \cdot 11}$	1	1/2	4	7/2	$1/27$
4	7/2	2	5/2	$-\sqrt{7/9 \cdot 5\sqrt{2 \cdot 11}}$	1	3/2	4	7/2	$\sqrt{7/2 \cdot 27\sqrt{2 \cdot 5}}$
2	3/2	3	5/2	$1/3\sqrt{3 \cdot 5 \cdot 7}$	2	3/2	4	7/2	$\sqrt{7/4 \cdot 3 \cdot 5\sqrt{3}}$
2	5/2	3	5/2	$1/2 \cdot 9\sqrt{5 \cdot 7}$	2	5/2	4	7/2	$\sqrt{7/9 \cdot 5\sqrt{2 \cdot 11}}$
3	5/2	3	5/2	$\sqrt{5/9 \cdot 7\sqrt{2}}$	3	5/2	4	7/2	$\sqrt{13/9\sqrt{5 \cdot 7 \cdot 11}}$
3	7/2	3	5/2	$1/3 \cdot 7\sqrt{3 \cdot 11}$	3	7/2	4	7/2	$\sqrt{13/8 \cdot 3\sqrt{3 \cdot 7 \cdot 11}}$
					4	7/2	4	7/2	$\sqrt{5 \cdot 7 \cdot 13/8 \cdot 27\sqrt{3 \cdot 11}}$

Table XIX. $U \begin{pmatrix} l & l' & 4 \\ j & j' & 5 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

l	j	l'	j'	U	l	j	l'	j'	U
3	7/2	1	3/2	$1/4 \cdot 3 \sqrt{2 \cdot 3}$	1	3/2	3	7/2	$1/4 \cdot 3 \sqrt{2 \cdot 3}$
4	7/2	1	3/2	$-1/4 \cdot 9 \sqrt{2 \cdot 3 \cdot 5}$	2	3/2	3	7/2	$-1/4 \cdot 3 \cdot 5 \sqrt{2}$
3	7/2	2	3/2	$1/4 \cdot 3 \cdot 5 \sqrt{2}$	2	5/2	3	7/2	$1/3 \cdot 5 \sqrt{7}$
4	7/2	2	3/2	$-\sqrt{7/4} \cdot 9 \cdot 5 \sqrt{2 \cdot 11}$	3	5/2	3	7/2	$-1/3 \cdot 7 \sqrt{2 \cdot 11}$
2	5/2	2	5/2	$1/2 \cdot 9 \sqrt{3}$	3	7/2	3	7/2	$\sqrt{5 \cdot 13/4} \cdot 3 \cdot 7 \sqrt{2 \cdot 11}$
3	5/2	2	5/2	$-1/9 \sqrt{2 \cdot 3 \cdot 5 \cdot 7}$	4	7/2	3	7/2	$-\sqrt{13/4} \cdot 9 \sqrt{2 \cdot 7 \cdot 11}$
3	7/2	2	5/2	$1/3 \cdot 5 \sqrt{7}$	1	3/2	4	7/2	$1/4 \cdot 9 \sqrt{2 \cdot 3 \cdot 5}$
4	7/2	2	5/2	$-1/9 \cdot 5 \sqrt{11}$	2	3/2	4	7/2	$-\sqrt{7/4} \cdot 9 \cdot 5 \sqrt{2 \cdot 11}$
2	5/2	3	5/2	$1/9 \sqrt{2 \cdot 3 \cdot 5 \cdot 7}$	2	5/2	4	7/2	$1/9 \cdot 5 \sqrt{11}$
3	5/2	3	5/2	$-\sqrt{5/2} \cdot 9 \cdot 7 \sqrt{3 \cdot 11}$	3	5/2	4	7/2	$-1/9 \sqrt{2 \cdot 5 \cdot 7 \cdot 11}$
3	7/2	3	5/2	$1/3 \cdot 7 \sqrt{2 \cdot 11}$	3	7/2	4	7/2	$\sqrt{13/4} \cdot 9 \sqrt{2 \cdot 7 \cdot 11}$
4	7/2	3	5/2	$-1/9 \sqrt{2 \cdot 5 \cdot 7 \cdot 11}$	4	7/2	4	7/2	$-\sqrt{5/4} \cdot 27 \sqrt{2 \cdot 11}$

Letters to the Editor

Meson-theoretical Potentials in N^{14}

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November 14, 1955

There have so far been various calculations about static properties of N^{14} based on the shell model: D. R. Inglis¹⁾ calculated the energy levels by the intermediate coupling method assuming a phenomenological central potential satisfying the saturation condition, and determined the spin-orbit coupling parameter a and matrix element K of the exchange integral, so as to reproduce the ground state and the first two $T=1$ states. According to his result, N^{14} is closer to the jj -limit than Li^6 . T. Regge²⁾ studied the influence of the tensor forces by the Inglis' method assuming a potential of Feshbach-Schwinger type, and found that its effect is not negligible. According to his result, a , the spin-orbit coupling parameter required in order to explain the first two levels of N^{14} , is too large for Li^6 , and furthermore the level $I=2, T=0$ becomes so low that for small values of a/K this level represents the ground state. G. E. Tauber and Ta-You Wu³⁾ reproduced up to the first excited state assuming a Yukawa type central potential by intermediate coupling method, and calculated the magnetic moment. W. J. Robinson⁴⁾ applied the J. P. Elliot's method⁵⁾ to N^{14} and showed that the total angular momentum, μ and Q , may be explained under some conditions, which are not inconsistent with the deuteron data. Although these calculations can explain the level-ordering, level-spacing, magnetic moment and quadrupole moment of N^{14} to some extent, no work has been done to reproduce all of them at one time using the same model and the same parameters.

In our previous paper,⁶⁾ using the intermediate coupling shell model, we found that the symmetrical ps meson-theoretical potential is very favorable to Li^6 . Following the shell model, both Li^6 and N^{14} can

be treated simultaneously, because the former belongs to a p^2 configuration and the latter to a p^{-2} configuration, so the relative energy levels are solutions of the same secular equation, only the sign of a being opposite. Therefore, it is very interesting to study the properties of N^{14} using the same method of calculation and the same potential as those for Li^6 , though it may be an over-simplification in the case of N^{14} .

The range $1/\sqrt{\nu}$ of the p -shell nucleon wave function of N^{14} would be somewhat larger than that of Li^6 . So we use $1/\sqrt{\nu}=3.0$ and 2.4×10^{-13} cm. The results are not so sensitive to $1/\sqrt{\nu}$ in this region. For Li^6 , $1/\sqrt{\nu}=2.4 \times 10^{-13}$ cm gives the best fit and this is just the value that is expected from the Coulomb energy difference of Li^7 - Be^7 . If we assume smaller values for $1/\sqrt{\nu}$, the level spacing becomes larger.

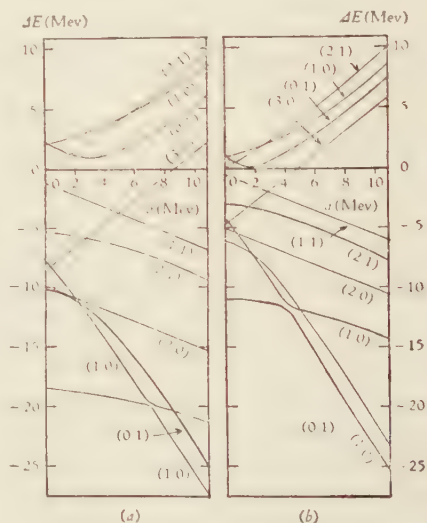


Fig. 1. Theoretical curves of N^{14} level scheme for (a) $1/\sqrt{\nu}=2.4 \times 10^{-13}$ cm and (b) $1/\sqrt{\nu}=3.0 \times 10^{-13}$ cm.

Fig. 1(a) and (b) show the calculated curves of N^{14} level scheme for $1/\sqrt{\nu}=2.4$ and 3.0×10^{-13} cm, respectively. The level ordering is the same in both cases. The level spacing is larger for $1/\sqrt{\nu}=2.4 \times 10^{-13}$ cm than for $1/\sqrt{\nu}=3.0 \times 10^{-13}$ cm. Experimental

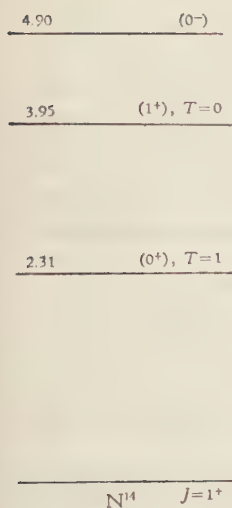


Fig. 2. Energy levels of N^{14}

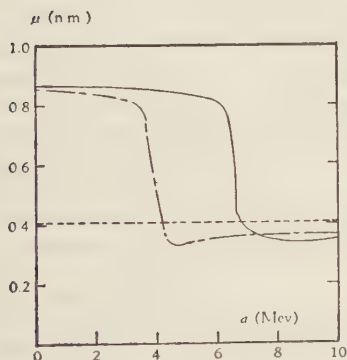


Fig. 3. Magnetic moment

—: $1/\sqrt{\nu} = 2.4 \times 10^{-13}$ cm, - - - -: $1/\sqrt{\nu} = 3.0 \times 10^{-13}$ cm.

data are taken from F. Ajzenberg and T. Lauritsen^{b)} (see Fig. 2). We get the correct level ordering of the first three low states at the region $a \sim 4.8$ Mev for $1/\sqrt{\nu} = 3.0 \times 10^{-13}$ cm but somewhat smaller than the experimental spacing, and at $a \sim 9.2$ Mev for $1/\sqrt{\nu} = 2.4 \times 10^{-13}$ cm somewhat larger. Therefore, it is possible to get the correct level spacing of these states for some a and for some $1/\sqrt{\nu}$ between these two values. The second state of $T=1$ is much higher than the first excited state of

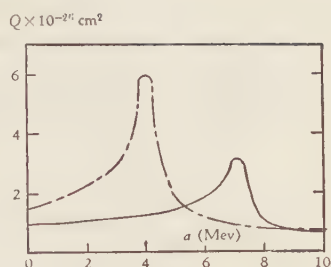


Fig. 4. Quadrupole moment

—: $1/\sqrt{\nu} = 2.4 \times 10^{-13}$ cm, - - - -: $1/\sqrt{\nu} = 3.0 \times 10^{-13}$ cm.

a which gives the $Q_{exp} = +0.007$ b⁹⁾ also agrees with that reproducing the correct level spacing. Thus the level-ordering and level-spacing up to the second excited state, μ and Q of N^{14} can be reproduced consistently when $6 < a < 9$ Mev is assumed.

Summarizing the above results, we can conclude that the meson-theoretical potential is favorable qualitatively for N^{14} as well as Li^6 , if we take into account the over-simplification of the model and some possible change in the potential parameters.

We are very grateful to Professors M. Kobayasi, S. Nakamura, S. Takagi and other colleagues of the laboratory for their discussions and encouragement.

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C^{14} . The level ordering agrees with that of Inglis. Results for μ and Q are plotted in Fig. 3 and Fig. 4. The sudden decrease in μ and the peak in Q are due to the abrupt change in the ground level (see Fig. 1(a) and (b)). For jj coupling limit we get $\mu \approx 0.36$ n.m. and for LS coupling limit $\mu \approx 0.86$ n.m., μ_{exp} is 0.4037 n.m.. At the value of a which shows the best fit to the level spacing we get for both $1/\sqrt{\nu} = 2.4$ and 3.0×10^{-13} cm, $|C_0|^2 \approx 2\%$, $|C_1|^2 \approx 13\%$ and $|C_2|^2 \approx 85\%$ and hence $\mu \approx 0.35$ n.m.. The value of

On the Relation between Hill-Wheeler's and Bohr-Mottelson's Descriptions of the Nuclear Collective Model

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In the description of the nuclear collective model, Hill and Wheeler (H. W.) have adopted a wave function¹⁾;

$$\Phi(x_1, \dots, x_A) = \int \exp\{-i(M/\hbar) \sum_{i=1}^A \phi(x_i)\} \\ \times \psi(x_1, \dots, x_A; \alpha) h_n(\alpha) d\alpha,$$

where ϕ is the velocity potential of an irrotational and incompressible carrier fluid, ψ a wave function of the individual particles in a potential well of a given shape, and $h_n(\alpha)$ is an oscillator function describing the potential wall motion. This procedure of incorporating the collective motion both through the exponential factor involving the velocity potential and in the oscillator function $h_n(\alpha)$ seems difficult to interpret at first sight, as pointed out by Bohr and Mottelson and others²⁾³⁾.

In this note, however, we shall show that the difference between H.W.'s and Bohr-Mottelson's (B.M.) descriptions¹⁾ of the nuclear collective model is nothing but that between the representations in the description of the nuclear states except at several trivial points.

Our original Schrödinger equation is

$$H\Phi \equiv \left\{ \sum_{i=1}^A p_i^2/2M + V(x_1, \dots, x_A) \right\} \Phi(x_1, \dots, x_A) \\ = E\Phi(x_1, \dots, x_A), \quad (1)$$

where M is the nucleon mass and V is the interaction potential between nucleons. Now we introduce the collective coordinates α_{lm} ^{*}) and start from the following equation:

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* α_{lm} are the expansion parameters with regard to the nuclear surface defined by

$$R(\theta, \varphi) = R_0(1 + \sum_{lm} \alpha_{lm} Y_{lm}(\theta, \varphi)),$$

where R_0 is the equilibrium radius.

$$\left\{ \sum_{i=1}^A p_i^2/2M + V(x_1, \dots, x_A) \right\} \Phi'(x_1, \dots, x_A, \alpha) \\ = E\Phi'(x_1, \dots, x_A, \alpha), \quad (2)$$

which is equivalent to (1) under the condition:

$$\alpha_{lm} \Phi'(x_1, \dots, x_A, \alpha) = 0. \quad (2')$$

We transform (2) and (2') into the "collective representation"⁵⁾ by the unitary transformation:⁶⁾

$$U = \exp\{i(M/\hbar) \int \rho(x) \phi(x) dx\} \\ = \exp\{i(M/\hbar) \int \sum_{i=1}^A \delta(x - x_i) \phi(x) dx\} \\ = \exp\{i(M/\hbar) \sum_{i=1}^A \phi(x_i)\}. \quad (3)$$

Here $\phi(x)$ is the velocity potential of an irrotational and incompressible carrier fluid; $\phi(x) = \sum_{lm} \beta_{lm} r^l \times Y_{lm}(\theta, \varphi)$. One should note that, as is well known in the liquid drop model of the nucleus, β_{lm} are related to the canonical conjugate quantities π_{lm} of the coordinates α_{lm} through the relation:

$$\pi_{lm} = B_l \cdot l R_0^{l-2} \beta_{lm}^*, \quad B_l = l^{-1} \cdot (3/4\pi) A M R_0^2.$$

In this "collective representation", (2) and (2') are expressed as

$$\left\{ \sum_{i=1}^A p_i^2/2M + M/2 \cdot \int \rho(x) (\text{grad } \phi(x))^2 dx \right. \\ \left. + V(x_1, \dots, x_A, \alpha)^{**} \right\} \\ + M \int \mathbf{v}(x) \nabla \phi(x) dx \} \Psi(x_1, \dots, x_A, \alpha) \\ = E\Psi(x_1, \dots, x_A, \alpha), \quad (4)$$

$$\{ \alpha_{lm} - \sum_{i=1}^A F_{lm}(x_i) \} \Psi(x_1, \dots, x_A, \alpha) = 0, \quad (4')$$

where

$$\sum_{i=1}^A F_{lm}(x_i) = (4\pi/3A) \sum_{i=1}^A (r_i/R_0)^l Y_{lm}^*(\theta_i, \varphi_i),^{***}$$

and

** Strictly speaking, the α -dependence of V is not uniquely defined by this transformation, but this dependence can be given approximately by Tomonaga's procedure⁷⁾ or Miyazima's transformation⁸⁾.

*** This function is just the expression for the collective parameters in terms of the coordinates of the individual particles. (cf. 2). p. 10)

$$\mathbf{v}(\mathbf{x}) = (1/2M) \sum_{i=1}^A \{ \mathbf{p}_i \delta(\mathbf{x} - \mathbf{x}_i) + \delta(\mathbf{x} - \mathbf{x}_i) \mathbf{p}_i \},$$

which is the velocity operator for the individual particles.

The eq. (4) with (4') is equivalent to B.M.'s except at several trivial points*), and we can easily ascertain that this representation is used in B.M.'s model, by calculating any physical operators in this representation.

Now we shall solve the eq. (4) with (4') using the adiabatic approximation. In this case Ψ is approximately represented by

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_A, \alpha) = \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) h_n(\alpha), \quad (5)$$

where $\psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha)$ is the solution of

$$\begin{aligned} \left\{ \sum_{i=1}^A \mathbf{p}_i^2 / 2M + V(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \right\} \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \\ = E_n(\alpha) \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \end{aligned}$$

and must satisfy approximately

$$\begin{aligned} \int \psi^*(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \rho(\mathbf{x}) \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) d\mathbf{x}_1 \dots d\mathbf{x}_A \\ = \begin{cases} \rho_0 (= 3A/4\pi R_0^3) & \text{inside the deformed nucleus,} \\ 0 & \text{outside the deformed nucleus,} \end{cases} \end{aligned}$$

because of the condition (4')**). In the 0-th order approximation, ψ may be represented by the wave function of the individual particles in a potential well of a given shape.

In these approximations we transform (5) into the original representation. Then $\Phi'(\mathbf{x}_1, \dots, \mathbf{x}_A, \alpha)$ which satisfies the eq. (2) is

$$\begin{aligned} \Phi'(\mathbf{x}_1, \dots, \mathbf{x}_A, \alpha) &= U^{-1} \Psi \\ &= \exp \left\{ -i(M/\hbar) \sum_{i=1}^A \phi(\mathbf{x}_i) \right\} \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) h_n(\alpha). \end{aligned}$$

* It is clear that Tolhoek's⁹⁾ and Coester's¹⁰⁾ analyses of B.M.'s model are equivalent to solving the eq. (4) with (4') by use of some assumptions; if we assume that $\Phi'(\mathbf{x}_1, \dots, \mathbf{x}_A, \alpha)$ has a form, $\delta(\alpha - \sum F(\mathbf{x}_i)) \psi(\mathbf{x}_1, \dots, \mathbf{x}_A) g(\alpha)$, and calculate the equation which $g(\alpha)$ satisfies, we obtain the same results as Tolhoek's and Coester's.

Here one should note that Φ' is of the form:

$$\Phi'(\mathbf{x}_1, \dots, \mathbf{x}_A, \alpha) = \delta(\alpha) \Phi(\mathbf{x}_1, \dots, \mathbf{x}_A)$$

because of the condition (2'). Therefore, $\Phi(\mathbf{x}_1, \dots, \mathbf{x}_A)$ which satisfies the eq. (1) is represented by

$$\begin{aligned} \Phi(\mathbf{x}_1, \dots, \mathbf{x}_A) &= \int \exp \left\{ -i(M/\hbar) \sum_{i=1}^A \phi(\mathbf{x}_i) \right\} \\ &\quad \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) h_n(\alpha) d\alpha. \end{aligned}$$

This is nothing but the wave function proposed by H.W.

The details will soon be submitted to this journal.

The author wishes to express his thanks to Prof. S. Sakata for his encouragements, and also he is indebted to Drs. H. A. Tolhoek and F. Coester for sending him the preprints before publication.

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** When this relation is satisfied, then

$$\begin{aligned} \int \psi^*(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \sum_{i=1}^A F_{lm}(\mathbf{x}_i) \psi(\mathbf{x}_1, \dots, \mathbf{x}_A; \alpha) \\ d\mathbf{x}_1 \dots d\mathbf{x}_A = \alpha_{lm}. \end{aligned}$$

Errata

The Properties of the Meson Theoretical Potential in Li^6

S. OTSUKI, T. SAWADA and S. SUEKANE

(Prog. Theor. Phys. **13** (1955), 79)

Eq. (8) and Fig. 9 are not correct. Fig. 9 should be replaced by Fig. 9', here and the both figures should be corrected as: $Q = \frac{e}{20\nu} (-7|C_2|^2 + 10|C_1|^2 - 8\sqrt{5} C_0 C_2)$.

At $a \sim -6$ Mev, the new estimation gives $|Q_{exp}| < 0.001b$ and the level ordering and spacing are reproduced up to the second excited state. $a \sim -9.5$ Mev is not true.

Recent experiment has shown that there exists a new $(I, T) = (2, 0)$ level at 4.5 Mev. We can reproduce the level ordering up to the fourth excited level including this new level at $-4.8 \text{ Mev} < a < -2.4$ Mev, though the level spacing is not so satisfactory.

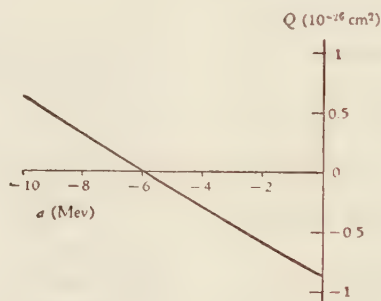


Fig. 9'. Theoretical curve for quadrupole moment of Li^6 .

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